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An adaptive least squares boundary element method for elliptic boundary value problems

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Abstract

In this paper we formulate and analyze a least squares boundary element method for the weakly singular boundary integral equation which is related to the solution of a Dirichlet boundary value problem for a second order partial differential equation, with the Laplacian as model problem. In particular we may assume less regular boundary data $g \notin H^{1/2}(\Gamma)$ but $g \in L^2(\Gamma)$. For this we consider the single layer boundary integral operator $V : H^{-1}(\Gamma) \to L^2(\Gamma)$, i.e., we will solve the boundary integral equation $Vw = f$ by minimizing $\frac{1}{2}\, \|Vw - f\|^2_{L^2(\Gamma)}$. This results in a mixed variational formulation where we use piecewise constant approximations to discretize both the primal unknown $w \in H^{-1}(\Gamma)$ and the adjoint $p := f - Vw \in L^2(\Gamma)$. Using nested boundary element spaces $S^0_H(\Gamma) \subseteq S^0_h(\Gamma)$ we can prove stability and related error estimates for both the primal and adjoint approximations, w_H and p_h, respectively. When considering the approximate adjoint p_h on a finer mesh than the primal w_H, we can use $\|p_h\|_{L^2(\Gamma)}$ as a posteriori error indicator for the error $\|w - w_H\|_{H^{-1}(\Gamma)}$ to drive an adaptive mesh refinement. Note that this defines an adaptive boundary element method also for regular boundary data $g \in H^{1/2}(\Gamma)$. Numerical examples confirm the theoretical results.

1 Introduction

The weak formulation of second order elliptic partial differential equations such as the Poisson equation or the system of linear elastostatics is usually considered in suitable subspaces of $H^1(\Omega)$, implying $g \in H^{1/2}(\Gamma)$ when considering Dirichlet boundary conditions $u = g$ on $\Gamma = \partial\Omega$. In this case we are also able to formulate equivalent boundary integral equations to describe the solution of the Dirichlet boundary value problem by using a representation formula, or an indirect single layer potential ansatz. In both cases we have to solve a first kind boundary integral equation $Vw = f$ with the single layer boundary integral operator $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, and a right hand side $f \in H^{1/2}(\Gamma)$. In this paper we are interested in the solution of the Dirichlet boundary value problem with less regular boundary data $g \in L^2(\Gamma)$, see, e.g., [1], as it appears in Dirichlet boundary control problems with a control in $L^2(\Gamma)$, see, e.g., [6, 14]. While in this case we may use an indirect double layer potential ansatz in order to solve a second kind boundary integral equation in $L^2(\Gamma)$, this would require that the double layer boundary integral operator is compact, which excludes polygonal or polyhedral bounded domains as well as Lipschitz domains in the case of the Laplacian, while the double layer boundary integral operator of linear elastostatics is never compact. In this paper we therefore consider the weakly singular boundary integral equation $Vw = f$ in $L^2(\Gamma)$ to find an unknown density function $w \in H^{-1}(\Gamma)$ by means of a least squares approach.

When considering less regular boundary data g, also the solution w of the boundary integral equation has a reduced regularity, implying a reduced order of convergence when using boundary element methods with respect to a globally quasi-uniform boundary mesh for discretization. Instead, and based on appropriate a posteriori error estimators, one can drive an adaptive mesh refinement algorithm to reach again an optimal order of onvergence, see, e.g., [8] for a review on adaptive boundary element methods. Since the underlying boundary integral equations are usually formulated in the Sobolev trace spaces $H^{\pm 1/2}(\Gamma)$ the localization of the involved non-local norms is in most cases done by using weighted sums. A rather simple a posteriori error estimator is the so called $h - h/2$ approach [9], where the difference of the computed boundary element solution to a solution on a refined mesh serves as error indicator.

Since the single layer boundary integral operator V is an isomorphism in a scale of Sobolev spaces [4, 24], we will solve the boundary integral equation $Vw = f$ in $L^2(\Gamma)$ by means of a least squares approach to find the minimizer $w \in H^{-1}(\Gamma)$ of the functional $\mathcal{J}(w) = \frac{1}{2}\|Vw - f\|^2_{L^2(\Gamma)}$ as solution of the gradient equation $V^*Vw = V^*f$ in $H^1(\Gamma)$. This corresponds to the solution of a mixed variational formulation [2]. For this particular application we observe that the adjoint variable $p := f - Vw \in L^2(\Gamma)$ is zero, i.e., its finite element approximation p_h may serve as an error indicator for the approximation error $w - w_h$ of the primal variable w, when the discretization is done with repect to different boundary element meshes. In contrast to standard mixed finite element approximations [2] we do not need any discrete inf-sup condition to establish unique solvability of the boundary element discretization. This is due to the use of nested boundary element spaces for the approximation of both the primal and adjoint variable, and since an appropriate block of the discrete single layer boundary integral operator matrix is invertible.

While at this time we only consider the least squares formulation of boundary integral equations which are related to the Laplace equation, this approach is intended to formulate and analyze stable boundary element approximations of boundary integral operators for the wave equation, following the generalized approach as considered in [22, 23], and already used in [13] for a space-time finite element discretization of a distributed control problem subject to the wave equation.

The remainder of this paper is structured as follows: In Section 2 we describe the use of boundary integral equations to solve the Dirichlet problem for the Laplacian when using the direct approach, and we discuss a related least squares formulation wich results in a mixed variational formulation. The boundary element discretization of the mixed formulation and its stability and error analysis is given in Section 3. The use of the discrete adjoint as error indicator for the primal variable is established in Section 4. Two numerical examples are given in Section 5, which confirm all the theoretical results. Some conclusions and comments on related and ongoing work are finally given in Section 6.

2 A least squares boundary integral formulation

As a model problem we consider the interior Dirichlet boundary value problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma \tag{2.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with Lipschitz boundary $\Gamma = \partial\Omega$, and where $g \in L^2(\Gamma)$ is given. The solution of (2.1) can be described by using the representation formula

$$u(x) = \int_\Gamma U^*(x, y) \frac{\partial}{\partial n_y}(y)\, ds_y - \int_\Gamma g(y) \frac{\partial}{\partial n_y} U^*(x, y)\, ds_y \quad \text{for } x \in \Omega, \tag{2.2}$$

where $U^*(x, y)$ is the well known fundamental solution of the Laplacian given as

$$U^*(x, y) = \begin{cases} -\dfrac{1}{2\pi} \log|x - y| & \text{for } n = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x - y|} & \text{for } n = 3. \end{cases}$$

In order to find the unknown normal derivative $w := n_y \cdot \nabla u$ on Γ we have to solve the related boundary integral equation

$$(Vw)(x) := \int_\Gamma U^*(x, y)\, w(y)\, ds_y = \frac{1}{2} g(x) + \int_\Gamma g(y) \frac{\partial}{\partial n_y} U^*(x, y)\, ds_y \;=:\; f(x) \tag{2.3}$$

for $x \in \Gamma$ almost everywhere. It is well known, e.g., [11], that the single layer boundary integral operator $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded and elliptic, where for $n = 2$ we assume $\operatorname{diam} \Omega < 1$ to ensure the latter. In fact, $\|w\|_V := \sqrt{\langle Vw, w\rangle_\Gamma}$ defines a norm in $H^{-1/2}(\Gamma)$. The properties of the single layer boundary integral operator V not only give unique solvability of the boundary integral equation (2.3), but also guarantee stability and convergence of conforming boundary element methods to solve (2.3) numerically, e.g., [15, 19]. Note that such an approach assumes $g \in H^{1/2}(\Gamma)$.

It is well known [4, 24] that in the case of a Lipschitz boundary Γ the single layer boundary integral operator $V : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is continuous for all $|s| \leq \frac{1}{2}$, see also [20]. If Γ is piecewise smooth, this remains true for larger values of $|s| < \sigma_0$ for some $\sigma_0 > \frac{1}{2}$: In the case of a polygonal bounded domain $\Omega \subset \mathbb{R}^2$ with J corner points and associated interior angles α_j we have [5]

$$\sigma_0 := \min_{j=1,\dots,J} \left\{ \min\left[\frac{\pi}{\alpha_j}, \frac{\pi}{2\pi - \alpha_j} \right] \right\}. \tag{2.4}$$

In particular we have that $V : L^2(\Gamma) \to H^1(\Gamma)$ is continuous and invertible, see Theorem 1 and the following Remark in [4], and [24, Theorem 3.3]. For the solution $w \in L^2(\Gamma)$ of the boundary integral equation $Vw = f$ in $H^1(\Gamma)$ we therefore conclude

$$\|w\|_{L^2(\Gamma)} = \|V^{-1}f\|_{L^2(\Gamma)} \leq c_2^{V^{-1}} \|f\|_{H^1(\Gamma)} = c_2^{V^{-1}} \|Vw\|_{H^1(\Gamma)}.$$

When using duality arguments we further obtain, for $w \in H^{-1}(\Gamma)$,

$$\begin{aligned}\|w\|_{H^{-1}(\Gamma)} &= \sup_{0\neq v\in H^1(\Gamma)} \frac{\langle w, v\rangle_\Gamma}{\|v\|_{H^1(\Gamma)}} = \sup_{0\neq v=Vq\in H^1(\Gamma), q\in L^2(\Gamma)} \frac{\langle w, Vq\rangle_\Gamma}{\|Vq\|_{H^1(\Gamma)}} \\ &\leq c_2^{V^{-1}} \sup_{0\neq q\in L^2(\Gamma)} \frac{\langle w, Vq\rangle_\Gamma}{\|q\|_{L^2(\Gamma)}},\end{aligned}$$

i.e., we have the stability condition

$$c_S \|w\|_{H^{-1}(\Gamma)} \leq \sup_{0\neq q\in L^2(\Gamma)} \frac{\langle Vw, q\rangle_{L^2(\Gamma)}}{\|q\|_{L^2(\Gamma)}} \quad \text{for all } w \in H^{-1}(\Gamma),\ c_S = \frac{1}{c_2^{V^{-1}}}. \tag{2.5}$$

For given $f \in L^2(\Gamma)$ we may therefore consider the boundary integral equation to find $w \in H^{-1}(\Gamma)$ such that $Vw = f$ in $L^2(\Gamma)$. Note that $f := (\frac{1}{2}I + K)g \in L^2(\Gamma)$ is well defined due to the mapping properties of the double layer boundary integral operator. Although we have $V : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ for all $|s| \leq \frac{1}{2}$, we now consider $V : H^{-1}(\Gamma) \to L^2(\Gamma)$, and we write $V^* : L^2(\Gamma) \to H^1(\Gamma)$ for its adjoint operator, even if it coincides with V. Instead of the boundary integral equation $Vw = f$ we may now consider the problem to minimize

$$\begin{aligned}\mathcal{J}(w) &= \frac{1}{2}\|Vw - f\|^2_{L^2(\Gamma)} \\ &= \frac{1}{2}\langle Vw - f, Vw - f\rangle_{L^2(\Gamma)} \\ &= \frac{1}{2}\langle V^*Vw, w\rangle_{L^2(\Gamma)} - \langle V^*f, w\rangle_{L^2(\Gamma)} + \frac{1}{2}\|f\|^2_{L^2(\Gamma)},\end{aligned}$$

whose minimizer is given as the unique solution of the gradient equation

$$V^*(Vw - f) = 0. \tag{2.6}$$

We introduce the adjoint

$$p := -(Vw - f) \in L^2(\Gamma)$$

to conclude the saddle point system of boundary integral equations

$$p + Vw = f, \quad V^*p = 0.$$

Hence we have to find $p \in L^2(\Gamma)$ and $w \in H^{-1}(\Gamma)$ such that

$$\langle p, q\rangle_{L^2(\Gamma)} + \langle Vw, q\rangle_{L^2(\Gamma)} = \langle g, q\rangle_{L^2(\Gamma)}, \quad \langle p, Vv\rangle_{L^2(\Gamma)} = 0 \tag{2.7}$$

is satisfied for all $q \in L^2(\Gamma)$ and for all $v \in H^{-1}(\Gamma)$. By construction, since V and therefore V^* are invertible, we have $p \equiv 0$. Unique solvability of the mixed variational formulation (2.7) follows from standard arguments [2], using the stability condition (2.5). Instead of (2.7) we may also consider the gradient equation (2.6), which is the Schur complement system of (2.7).

3 A least squares boundary element method

Let

$$S_H^0(\Gamma) = \operatorname{span}\{\psi_\ell\}_{\ell=1}^{N_H} \subset S_h^0(\Gamma) = \operatorname{span}\{\phi_j\}_{j=1}^{N_h} \subset L^2(\Gamma) \subset H^{-1}(\Gamma)$$

be two conforming nested boundary element spaces spanned by piecewise constant basis functions ψ_ℓ and ϕ_j which are defined with respect to some nested decompositions of Γ into simplicial shape regular boundary elements τ_ℓ^H and τ_j^h with volumes Δ_ℓ^H and Δ_j^h, and related mesh sizes H_ℓ and h_j, repectively, i.e.,

$$\Delta_\ell^H = \int_{\tau_\ell^H} ds_x, \quad \Delta_j^h = \int_{\tau_j^h} ds_x, \quad H_\ell = \left(\Delta_\ell^H\right)^{1/(n-1)}, \quad h_j = \left(\Delta_j^h\right)^{1/(n-1)}.$$

In addition, $H = \max H_\ell$ denotes the global mesh size.

The Galerkin formulation of (2.7) is to find $p_h \in S_h^0(\Gamma)$ and $w_H \in S_H^0(\Gamma)$ such that

$$\langle p_h, q_h \rangle_{L^2(\Gamma)} + \langle V w_H, q_h \rangle_{L^2(\Gamma)} = \langle f, q_h \rangle_{L^2(\Gamma)}, \quad \langle p_h, V v_H \rangle_{L^2(\Gamma)} = 0 \tag{3.1}$$

is satisfied for all $q_h \in S_h^0(\Gamma)$ and for all $v_H \in S_H^0(\Gamma)$. This is equivalent to a linear system of algebraic equations,

$$\begin{pmatrix} D_h & V_h \\ V_h^\top & \end{pmatrix} \begin{pmatrix} \underline{p} \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{0} \end{pmatrix}, \tag{3.2}$$

where for $i, j = 1, \ldots, N_h$ and for $k = 1, \ldots, N_H$ we have

$$D_h[j,i] = \int_\Gamma \phi_i(x)\phi_j(x)\, ds_x, \quad V_h[j,k] = \int_\Gamma (V\psi_k)(x)\phi_j(x)\, ds_x, \quad f_j = \int_\Gamma f(x)\phi_j(x)\, ds_x.$$

Since the diagonal matrix D_h is invertible, we can eliminate $\underline{p} = D_h^{-1}[\underline{f} - V_h \underline{w}]$ to end up with the Schur complement system

$$S_h \underline{w} := V_h^\top D_h^{-1} V_h \underline{w} = V_h^\top D_h^{-1} \underline{w}, \tag{3.3}$$

which is nothing than a Galerkin approximation of the gradient equation (2.6).

In the particular situation $S_H^0(\Gamma) = S_h^0(\Gamma)$ we obtain the standard Galerkin stiffness matrix $V_h = V_H$ of the single layer boundary integral operator which is symmetric and positive definite. From the second equation in (3.2), $V_H \underline{p} = \underline{0}$, we then conclude $\underline{p} = \underline{0}$, and it remains to solve $V_H \underline{w} = \underline{f}$ which is the standard boundary element formulation for the Dirichlet problem when using the direct approach.

In the general situation $S_H^0(\Gamma) \subset S_h^0(\Gamma)$ we may consider the orthogonal decomposition

$$S_h^0(\Gamma) = S_H^0(\Gamma) \oplus S_\perp^0(\Gamma) = \operatorname{span}\{\psi_\ell\}_{\ell=1}^{N_H} \oplus \operatorname{span}\{\phi_j^\perp\}_{j=N_H+1}^{N_h}, \tag{3.4}$$

i.e.,

$$\int_\Gamma q_H(x)\, q_h^\perp\, ds_x = 0 \quad \text{for all } q_H \in S_H^0(\Gamma),\ q_h^\perp \in S_h^\perp(\Gamma).$$

When considering a related decomposition of the coefficient vector $\underline{p} = (\underline{p}_H^\top, \underline{p}_\perp^\top)$ we can write the linear system (3.2) as

$$\begin{pmatrix} D_H & & V_H \\ & D_\perp & V_\perp \\ V_H & V_\perp^\top & \end{pmatrix} \begin{pmatrix} \underline{p}_H \\ \underline{p}_\perp \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f}_H \\ \underline{f}_\perp \\ \underline{0} \end{pmatrix}, \tag{3.5}$$

where

$$V_H[\ell, k] = \langle V\psi_k, \psi_\ell \rangle_\Gamma, \quad V_\perp[j,k] = \langle V\psi_k, \psi_j^\perp \rangle_\Gamma$$

and

$$D_H[\ell, k] = \langle \psi_k, \psi_\ell \rangle_{L^2(\Gamma)}, \quad D_\perp[j,i] = \langle \phi_i^\perp, \phi_j^\perp \rangle_{L^2(\Gamma)}$$

for $k, \ell = 1, \ldots, N_H$ and $i, j = N_H + 1, \ldots, N_h$, and the vector $\underline{f}$ of the right hand side is defined accordingly. Again we can eliminate $\underline{p}_H = D_H^{-1}[\underline{f}_H - V_H \underline{w}]$ and $\underline{p}_\perp = D_\perp^{-1}[\underline{f}_\perp - V_\perp \underline{w}]$ to conclude the Schur complement system

$$\Big(V_H D_H^{-1} V_H + V_\perp^\top D_\perp^{-1} V_\perp \Big) \underline{w} = V_H D_H^{-1} \underline{f}_H + V_\perp^\top D_\perp^{-1} \underline{f}_\perp \,. \tag{3.6}$$

The stiffness matrix of the Schur complement system (3.6) consists of a symmetric and positive definite part $V_H D_H^{-1} V_H$, and a symmetric remainder $V_\perp^\top D_\perp^{-1} V_\perp$ which is at least positive semi-definite. Hence we conclude unique solvability of (3.6), and therefore of (3.5) as well as of (3.2).

It remains to provide an a priori error estimate for the unique solution of (3.2).

Lemma 3.1 *Let $(w_H, p_h) \in S_H^0(\Gamma) \times S_h^0(\Gamma)$ be the unique solution of the mixed variational formulation* (3.1)*. Assume $w \in H^s(\Gamma)$ for some $s \in [-1, 1]$. Then there holds the error estimate, recall that $p \equiv 0$,*

$$\|p_h\|_{L^2(\Gamma)} \leq c\, H^{1+s}\, |w|_{H^s(\Gamma)} \,. \tag{3.7}$$

Proof. When using $f = Vw$ we can write the mixed variational formulation (3.1) for the unique solution $(w_H, p_h) \in S_H^0(\Gamma) \times S_h^0(\Gamma)$ by means of the Galerkin orthogonalities

$$\langle p_h, q_h \rangle_{L^2(\Gamma)} = \langle V(w - w_H), q_h \rangle_{L^2(\Gamma)} \quad \text{for all } q_h \in S_h^0(\Gamma), \tag{3.8}$$

and

$$\langle p_h, V v_H \rangle_{L^2(\Gamma)} = 0 \quad \text{for all } v_H \in S_H^0(\Gamma). \tag{3.9}$$

In particular for $q_h = p_h$ we then obtain, for any $v_H \in S_H^0(\Gamma)$,

$$\begin{aligned} \|p_h\|^2_{L^2(\Gamma)} &= \langle p_h, p_h \rangle_{L^2(\Gamma)} \\ &= \langle V(w - w_H), p_h \rangle_{L^2(\Gamma)} \\ &= \langle V(w - v_H), p_h \rangle_{L^2(\Gamma)} + \langle V(v_H - w_H), p_h \rangle_{L^2(\Gamma)} \\ &= \langle V(w - v_H), p_h \rangle_{L^2(\Gamma)} \\ &= \langle w - v_H, V p_h \rangle_{L^2(\Gamma)} \\ &\leq \|w - v_H\|_{H^{-1}(\Gamma)} \|V p_h\|_{H^1(\Gamma)} \\ &\leq c\, \|w - v_H\|_{H^{-1}(\Gamma)} \|p_h\|_{L^2(\Gamma)}, \end{aligned}$$

i.e.,

$$\|p_h\|_{L^2(\Gamma)} \leq c\,\|w - v_H\|_{H^{-1}(\Gamma)} \quad \text{for all } v_H \in S_H^0(\Gamma). \tag{3.10}$$

So it remains to prove an approximation property of $S_H^0(\Gamma)$ in $H^{-1}(\Gamma)$:

Let $A : H^{-1}(\Gamma) \to H^1(\Gamma)$ be some bounded and invertible operator realizing the norm in $H^{-1}(\Gamma)$, i.e.,

$$\|\psi\|^2_{H^{-1}(\Gamma)} = \langle A\psi, \psi\rangle_\Gamma \quad \text{for all } \psi \in H^{-1}(\Gamma).$$

A possible choice of A is the inverse Laplace–Beltrami operator. For any $\varphi \in H^{-1}(\Gamma)$ we define $\varphi_H = P_H\varphi \in S_H^0(\Gamma) \subset L^2(\Gamma) \subset H^{-1}(\Gamma)$ as the unique solution of the Galerkin variational formulation

$$\langle A\varphi_H, \psi_H\rangle_\Gamma = \langle A\varphi, \psi_H\rangle_\Gamma \quad \text{for all } \psi_H \in S_H^0(\Gamma),$$

and we conclude Cea's lemma,

$$\|\varphi - \varphi_H\|_{H^{-1}(\Gamma)} \leq \|\varphi - \psi_H\|_{H^{-1}(\Gamma)} \quad \text{for all } \psi_H \in S_H^0(\Gamma). \tag{3.11}$$

In particular for $\psi_H \equiv 0$ this gives

$$\|\varphi - \varphi_H\|_{H^{-1}(\Gamma)} \leq \|\varphi\|_{H^{-1}(\Gamma)}.$$

For $\varphi \in H^1(\Gamma)$ we define the piecewise constant L^2 projection $Q_H\varphi \in S_H^0(\Gamma)$ as the unique solution of the Galerkin variational formulation

$$\langle Q_H\varphi, \psi_H\rangle_{L^2(\Gamma)} = \langle \varphi, \psi_H\rangle_{L^2(\Gamma)} \quad \text{for all } \psi_H \in S_H^0(\Gamma).$$

When using standard arguments we then conclude the error estimates

$$\|\varphi - Q_H\varphi\|_{L^2(\Gamma)} \leq c\,H\,|\varphi|_{H^1(\Gamma)},$$

and, using duality arguments,

$$\|\varphi - Q_H\varphi\|_{H^{-1}(\Gamma)} \leq c\,H^2\,|\varphi|_{H^1(\Gamma)}.$$

Now, using (3.11) for $\psi_H = Q_H\varphi$ this gives

$$\|\varphi - \varphi_H\|_{H^{-1}(\Gamma)} \leq \|\varphi - Q_H\varphi\|_{H^{-1}(\Gamma)} \leq c\,H^2\,|\varphi|_{H^1(\Gamma)}\,,$$

and using a space interpolation argument we further obtain

$$\|\varphi - \varphi_H\|_{H^{-1}(\Gamma)} \leq c\,H^{s+1}(\Gamma)\,\|\varphi\|_{H^s(\Gamma)}$$

when assuming $\varphi \in H^s(\Gamma)$ for some $s \in [-1, 1]$. This gives the required approximation property, i.e., using in (3.10) $v_H = P_H w$ this gives the assertion. ■

Lemma 3.2 *Let* $(w_H, p_h) \in S_H^0(\Gamma) \times S_h^0(\Gamma)$ *be the unique solution of* (3.1) *where the underlying boundary element mesh is globally quasi-uniform. Assume* $w \in H^s(\Gamma)$ *for some* $s \in [-1, 1]$. *Let* $\tau \in [-2, -1]$ *such that* $\tau > -\frac{1}{2} - \sigma_0$ *is satisfied. Then there holds the error estimate*

$$\|w - w_H\|_{H^\tau(\Gamma)} \leq c\, H^{s-\tau}\, |w|_{H^s(\Gamma)}\,. \tag{3.12}$$

Proof. For some $\tau \in [-2, -1]$ satisfying $\tau > -\frac{1}{2} - \sigma_0$ let $v = Vq \in H^{-\tau}(\Gamma)$ for $q \in H^{-\tau-1}(\Gamma)$. We then have, using (3.8) and $Vw = f$,

$$\begin{aligned}
\|w - w_H\|_{H^\tau(\Gamma)} &= \sup_{0 \neq v \in H^{-\tau}(\Gamma)} \frac{\langle w - w_H, v\rangle_\Gamma}{\|v\|_{H^{-\tau}(\Gamma)}} = \sup_{0 \neq q \in H^{-\tau-1}(\Gamma)} \frac{\langle w - w_H, Vq\rangle_\Gamma}{\|Vq\|_{H^{-\tau}(\Gamma)}} \\
&\leq c \sup_{0 \neq q \in H^{-\tau-1}(\Gamma)} \frac{\langle f - Vw_H, q\rangle_\Gamma}{\|q\|_{H^{-\tau-1}(\Gamma)}} \\
&= c \sup_{0 \neq q \in H^{-\tau-1}(\Gamma)} \frac{\langle f - Vw_H, q - q_H\rangle_\Gamma + \langle f - Vw_H, q_H\rangle_\Gamma}{\|q\|_{H^{-\tau-1}(\Gamma)}} \\
&= c \sup_{0 \neq q \in H^{-\tau-1}(\Gamma)} \frac{\langle f, q - q_H\rangle_\Gamma + \langle p_h, q_H\rangle_\Gamma}{\|q\|_{H^{-\tau-1}(\Gamma)}},
\end{aligned}$$

when $q_H \in S_H^0(\Gamma)$ is the unique solution of the Galerkin variational formulation

$$\langle Vq_H, v_H\rangle_{L^2(\Gamma)} = \langle Vq, v_H\rangle_{L^2(\Gamma)} \quad \text{for all } v_H \in S_H^0(\Gamma). \tag{3.13}$$

Due to $q \in H^{-\tau-1}(\Gamma) \subset L^2(\Gamma)$ this is the standard boundary element formulation for the single layer boundary integral operator V. Since we assume $f \in H^{1+s}(\Gamma)$ for some $s \in [-1, 1]$ we can further estimate

$$\begin{aligned}
\langle f, q - q_H\rangle_\Gamma &\leq \|f\|_{H^{1+s}(\Gamma)}\|q - q_H\|_{H^{-1-s}(\Gamma)} \\
&\leq c\, H^{(-\tau-1)-(-1-s)}\, \|f\|_{H^{1+s}(\Gamma)}\|q\|_{H^{-\tau-1}(\Gamma)} \\
&= c\, H^{s-\tau}\, \|f\|_{H^{1+s}(\Gamma)}\|q\|_{H^{-\tau-1}(\Gamma)}.
\end{aligned}$$

It remains to consider, using (3.7) and standard error estimates for the Galerkin solution q_H of (3.13),

$$\begin{aligned}
\langle p_h, q_H\rangle_{L^2(\Gamma)} &= \langle p_h, q_H - q\rangle_{L^2(\Gamma)} + \langle p_h, q\rangle_{L^2(\Gamma)} \\
&\leq \|p_h\|_{L^2(\Gamma)}\|q - q_H\|_{L^2(\Gamma)} + \langle p_h, q\rangle_{L^2(\Gamma)} \\
&\leq c\, H^{1+s}\, |w|_{H^s(\Gamma)}\, H^{-\tau-1}\, \|q\|_{H^{-\tau-1}(\Gamma)} + \langle p_h, q\rangle_{L^2(\Gamma)} \\
&\leq c\, H^{s-\tau}\, |w|_{H^s(\Gamma)}\|q\|_{H^{-\tau-1}(\Gamma)} + \langle p_h, q\rangle_{L^2(\Gamma)}.
\end{aligned}$$

Due to the properties of $V : H^{-\tau-2}(\Gamma) \to H^{-\tau-1}(\Gamma)$ for $\tau \in (-\frac{1}{2} - \sigma_0, -1]$ we can write $q = Vz \in H^{-\tau-1}(\Gamma)$ for $z \in H^{-\tau-2}(\Gamma)$ to conclude, using the Galerkin orthogonity (3.9) for the L^2 projection $v_H = Q_H z \in S_H^0(\Gamma)$, the error estimate (3.7), and the Aubin–Nitsche trick for the L^2 projection $Q_H z$,

$$\begin{aligned}
\langle p_h, q\rangle_{L^2(\Gamma)} &= \langle p_h, Vz\rangle_{L^2(\Gamma)} = \langle p_h, V(z - Q_H z)\rangle_\Gamma \;\leq\; \|p_h\|_{L^2(\Gamma)} \|V(z - Q_H z)\|_{L^2(\Gamma)} \\
&\leq\; c\,\|p_h\|_{L^2(\Gamma)} \|z - Q_H z\|_{H^{-1}(\Gamma)} \leq c\, H^{1+s}\, |w|_{H^s(\Gamma)} H^{-\tau-2+1}\, \|z\|_{H^{-\tau-2}(\Gamma)} \\
&= c\, H^{s-\tau}\, |w|_{H^s(\Gamma)} \|q\|_{H^{-\tau-1}(\Gamma)}.
\end{aligned}$$

Finally, when collecting all contributions, this gives the assertion. ■

4 A posteriori error estimator

From the Galerkin orthogonality (3.8) we observe that $p_h = Q_h \overline{p} \in S_h^0(\Gamma)$ is the L^2 projection of $\overline{p} = V(w - w_H) \in L^2(\Gamma)$, satisfying

$$\langle p_h, q_h\rangle_{L^2(\Gamma)} = \langle \overline{p}, q_h\rangle_{L^2(\Gamma)} \quad \text{for all } q_h \in S_h^0(\Gamma). \tag{4.1}$$

From this we immediately conclude

$$\|p_h\|_{L^2(\Gamma)} \leq \|\overline{p}\|_{L^2(\Gamma)} = \|V(w - w_H)\|_{L^2(\Gamma)} \;\leq\; c\,\|w - w_H\|_{H^{-1}(\Gamma)}. \tag{4.2}$$

To prove the opposite direction we consider, similar as in (3.1), the mixed variational formulation to find $\widehat{p}_h \in S_h^0(\Gamma)$ and $\widehat{w}_h \in S_h^0(\Gamma)$ such that

$$\langle \widehat{p}_h, q_h\rangle_{L^2(\Gamma)} + \langle V\widehat{w}_h, q_h\rangle_{L^2(\Gamma)} = \langle f, q_h\rangle_{L^2(\Gamma)} \quad \langle \widehat{p}_h, Vv_h\rangle_{L^2(\Gamma)} = 0$$

is satisfied for all $q_h \in S_h^0(\Gamma)$ and for all $v_h \in S_h^0(\Gamma)$. Since the boundary element spaces for both the primal and adjoint unknows $\widehat{w}_h$ and $\widehat{w}_h$ coincide, and since the discrete single layer boundary integral operator V_h is square and invertible in this case, $\widehat{p}_h \equiv 0$ follows. In fact, $\widehat{w}_h \in S_h^0(\Gamma)$ solves

$$\langle V\widehat{w}_h, q_h\rangle_{L^2(\Gamma)} = \langle f, q_h\rangle_{L^2(\Gamma)} \quad \text{for all } q_h \in S_h^0(\Gamma).$$

Together with the first equation in (3.1) this gives

$$\langle V(\widehat{w}_h - w_H), q_h\rangle_{L^2(\Gamma)} = \langle p_h, q_h\rangle_{L^2(\Gamma)} \quad \text{for all } q_h \in S_h^0(\Gamma).$$

Now, using the triangle inequality

$$\|w - w_H\|_{H^{-1}(\Gamma)} \leq \|w - \widehat{w}_h\|_{H^{-1}(\Gamma)} + \|\widehat{w}_h - w_H\|_{H^{-1}(\Gamma)}$$

and the saturation assumption, note that we can choose h sufficiently small,

$$\|w - \widehat{w}_h\|_{H^{-1}(\Gamma)} \;\leq q\,\|w - w_H\|_{H^{-1}(\Gamma)} \tag{4.3}$$

for some $q < 1$, this gives

$$\|w - w_H\|_{H^{-1}(\Gamma)} \leq \frac{1}{1-q}\,\|\widehat{w}_h - w_H\|_{H^{-1}(\Gamma)}\,.$$

For $\phi \in L^2(\Gamma)$ let $\phi_h \in S_h^0(\Gamma)$ be the unique solution of the Galerkin variational formulation

$$\langle V\phi_h, \psi_h\rangle_\Gamma = \langle V\phi, \psi_h\rangle_\Gamma \quad \text{for all } \psi_h \in S_h^0(\Gamma).$$

When using boundedness and ellipticity of $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ we obviously conclude the stability estimate

$$\|\phi_h\|_{H^{-1/2}(\Gamma)} \leq \frac{c_2^V}{c_1^V} \|\phi\|_{H^{-1/2}(\Gamma)} .$$

Here we also assume the stability estimate

$$\|\phi_h\|_{L^2(\Gamma)} \leq c_S \|\phi\|_{L^2(\Gamma)} \quad \text{for all } \phi \in L^2(\Gamma). \tag{4.4}$$

This stability estimate, as well as related boundary element error estimates in $L^2(\Gamma)$ follow from the energy error estimate in $H^{-1/2}(\Gamma)$, when using an inverse inequality, and therefore assuming a global quasi-uniform boundary mesh. In the case of an adaptive mesh, which is only locally quasi-uniform, we may proceed similar as in proving the stability of the L^2 projection in fractional Sobolev spaces [18]. But since this is far behind the scope of this paper, at this time we just assume (4.4) which is also confirmed by numerical results.

Due to

$$\|w_h\|_{H^{-1}(\Gamma)} = \sup_{0\neq v\in H^1(\Gamma)} \frac{\langle w_h, v\rangle_\Gamma}{\|v\|_{H^1(\Gamma)}} = \sup_{0\neq v=Vq\in H^1(\Gamma), q\in L^2(\Gamma)} \frac{\langle w_h, Vq\rangle_\Gamma}{\|Vq\|_{H^1(\Gamma)}}$$

$$\begin{aligned}
(1-q)\,\|w - w_H\|_{H^{-1}(\Gamma)} &\leq \|\widehat{w}_h - w_H\|_{H^{-1}(\Gamma)} \\
&= \sup_{0\neq v\in H^1(\Gamma)} \frac{\langle \widehat{w}_h - w_H, v\rangle_{L^2(\Gamma)}}{\|v\|_{H^1(\Gamma)}} \\
&= \sup_{0\neq v=Vq\in H^1(\Gamma), q\in L^2(\Gamma)} \frac{\langle \widehat{w}_h - w_H, Vq\rangle_{L^2(\Gamma)}}{\|v\|_{H^1(\Gamma)}} \\
&= \sup_{0\neq v=Vq\in H^1(\Gamma), q\in L^2(\Gamma)} \frac{\langle \widehat{w}_h - w_H, Vq_h\rangle_{L^2(\Gamma)}}{\|v\|_{H^1(\Gamma)}}
\end{aligned}$$

when $q_h \in S_h^0(\Gamma)$ is the unique solution of the Galerkin equations

With the stability estimate (4.4) we further conclude

$$\begin{aligned}
\langle \widehat{w}_h - w_H, Vq_h\rangle_{L^2(\Gamma)} &= \langle V(\widehat{w}_h - w_H), q_h\rangle_{L^2(\Gamma)} \\
&= \langle p_h, q_h\rangle_{L^2(\Gamma)} \\
&\leq \|p_h\|_{L^2(\Gamma)}\|q_h\|_{L^2(\Gamma)} \\
&\leq c_S\,\|p_h\|_{L^2(\Gamma)}\|q\|_{L^2(\Gamma)} \\
&\leq c\,\|p_h\|_{L^2(\Gamma)}\|Vq\|_{H^1(\Gamma)},
\end{aligned}$$

i.e.,

$$\|w - w_H\|_{H^{-1}(\Gamma)} \leq \frac{c}{1-q}\,\|p_h\|_{L^2(\Gamma)} \tag{4.5}$$

follows. Now, when combining the upper estimate (4.5) with the lower estimate (4.2) this shows that $\|p_h\|_{L^2(\Gamma)}$ defines an error indicator for $\|w - w_H\|_{H^{-1}(\Gamma)}$, i.e.,

$$\eta_\ell := \|p_h\|^2_{L^2(\tau_\ell^H)}, \quad \sum_{\ell=1}^{N_H} \eta_\ell = \|p_h\|^2_{L^2(\Gamma)} \simeq \|w - w_H\|^2_{H^{-1}(\Gamma)},$$

and we refine all boundary elements τ_ℓ^H where

$$\eta_\ell \geq \theta \max_{j=1,\ldots,N_H} \eta_j \tag{4.6}$$

is satisfied for some $\theta \in (0,1)$.

5 Numerical results

As numerical example we consider a Dirichlet boundary value problem for the Laplace equation in the L shaped domain $\Omega \subset \mathbb{R}^2$ as sketched in Figure 1d). The Dirichlet data g are given in such a way that the solution of (2.1) is

$$u(x) = u(r, \varphi) = r^{2/3} \sin \frac{2}{3}\varphi \tag{5.1}$$

when using polar coordinates. This is a standard example for adaptive boundary element methods, e.g., [3, 7, 8, 16]. For the solution u of the Dirichlet boundary value problem (2.1) we have $u \in H^{5/3-\varepsilon}(\Omega)$ for any $\varepsilon > 0$, and hence $w \in H^{1/6-\varepsilon}(\Gamma)$ follows.

To compute a piecewise constant boundary element approximation $w_H \in S_H^0(\Gamma)$ we used the Galerkin variational formulation (3.1) where the adjoint $p_h \in S_h^0(\Gamma)$ is piecewise constant with respect to a refined boundary element mesh with local mesh size $h = \frac{1}{2}H$ which is well defined for both the uniform and the adaptive refinement strategy. All integrations were done using analytical integration formulae. The Dirichlet data g are approximated by a piecewise linear and continuous L^2 projection $g_h = Q_h g$. Note that this additional error can be analyzed using the Strang lemma, but this additional error will not disturb the error estimates as discussed in this paper, see, e.g., [19, Theorem 12.7]. For the solution of the resulting symmetric and positive definite Schur complement system (3.3) a conjugate gradient scheme with simple diagonal preconditioning was used. In all examples, we evaluate the representation formula (2.2) in $(0.04, 0.03) \in \Omega$ when using w_H instead of the normal derivative.

Let us first consider the error indicator $\|p_h\|_{L^2(\Gamma)}$ for $\|w - w_H\|_{H^{-1}(\Gamma)}$. In the case of an uniform refinement we expect the order of convergence to be $\frac{5}{3}$ which is confirmed by the numerical results as given in Table 1, and in Figure 1a). As in all plots, we compare the computed error line with a straight line which corresponds to the theoretical order of convergence. Since the exact solution (5.1) of (2.1) is known, we can also compute the error $\|w - w_H\|_{L^2(\Gamma)}$ where we expect the order of convergence to be $\frac{1}{6}$. Again, this is confirmed by the numerical results, see also Figure 1c). Finally, we consider the evaluation of the representation formula (2.2) when replacing the normal derivative by its boundary element approximation w_H. Due to the Aubin–Nitsche trick [12] we may expect the order of convergence to be $\frac{13}{6}$. But this estimate assumes that the single layer boundary integral operator $V : H^1(\Gamma) \to H^2(\Gamma)$ is continuous and bijective, see, e.g., [19, Theorem 12.3]. This condition is not satisfied, even for convex and polygonal bounded domains [5]. Hence, and using $V : L^2(\Gamma) \to H^1(\Gamma)$, we can ensure only an order of convergence to be $\frac{7}{6}$. Asymptotically, this reduced order of convergence is observed as seen in Figure 1b), but we also observe a convergence of $\frac{13}{6}$ initially. Note that in the case of equal order approximations of both the Dirichlet and Neumann data such a behavior was analyzed in [17]. However, the analysis of this behavior is not within the scope of this paper.

Instead, we now consider the adaptive boundary element approximation of (3.1) when using the a posteriori error indicator $\|p_h\|_{L^2(\Gamma)}$ and the refinement strategy (4.6) with $\theta = 0.5$. In this case, we observe a second order convergence of $\|p_h\|_{L^2(\Gamma)} \simeq \|w - w_H\|_{H^{-1}(\Gamma)}$, see Figure 1a), and a cubic order convergence for the approximate representation formula, see Figure 1b), as expected. But when computing the error $\|w - w_H\|_{L^2(\Gamma)}$ we observe $\frac{1}{2}$ as order of convergence. But this is due to the fact, that we design the adaptive boundary element mesh with respect to the error measured in $H^{-1}(\Gamma)$, and not in $L^2(\Gamma)$ which will result in different meshes. The final adaptive mesh is depicted in Figure 1d).

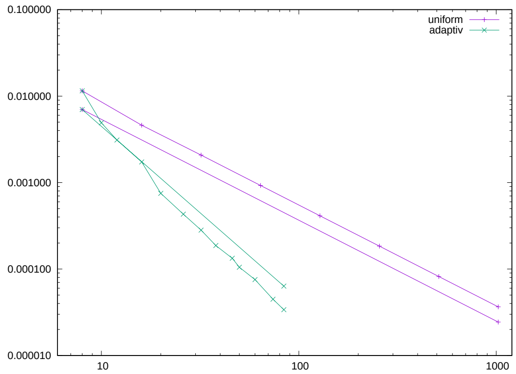
N_H	N_h	$\|p_h\|_{L^2(\Gamma)}$	eoc	$\|w - w_H\|_{L^2(\Gamma)}$	eoc	$\lvert u(\widetilde{x}) - \widetilde{u}(\widetilde{x})\rvert$	eoc
8	16	1.150 –2		5.454 –1		1.547 –2	
16	32	4.615 –3	1.317	4.816 –1	0.179	2.453 –3	2.657
32	64	2.079 –3	1.150	4.267 –1	0.175	5.131 –4	2.257
64	128	9.270 –4	1.165	3.800 –1	0.167	1.353 –4	1.923
128	256	4.131 –4	1.166	3.385 –1	0.167	1.096 –4	0.304
256	512	1.840 –4	1.167	3.016 –1	0.167	4.651 –5	1.237
512	1024	8.198 –5	1.166	2.687 –1	0.167	1.900 –5	1.292
1024	2048	3.652 –5	1.167	2.394 –1	0.167	7.635 –6	1.315
Theory			1.167		0.167		1.333

Table 1: Estimated order of convergence for $w \in H^{1/6-\varepsilon}(\Gamma)$, $\varepsilon > 0$.

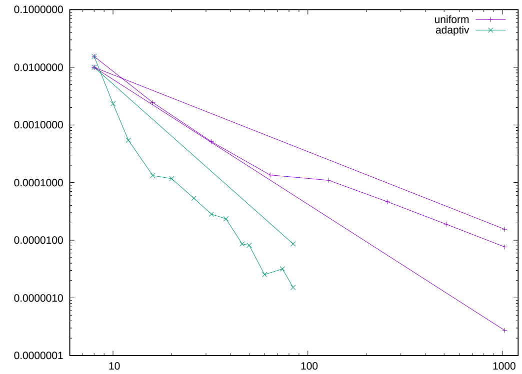
The numerical solution of the Dirichlet boundary value problem (2.1) for the solution (5.1) does not require the use of the mixed boundary element formulation (3.1). However, also in this case we can use p_h as error indicator, as seen above. In a second example we consider the harmonic function

$$u(x) = u(r, \varphi) = -r^{-1/10} \sin \frac{1}{10}\varphi, \tag{5.2}$$

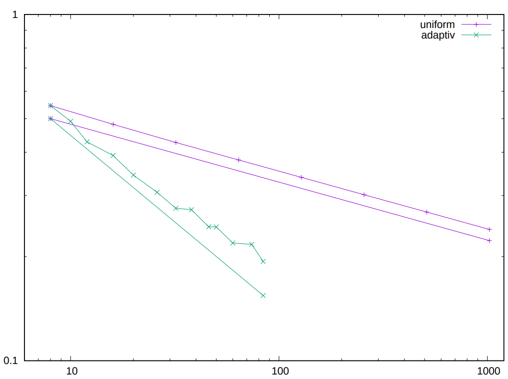
where we have $u \in H^{9/10-\varepsilon}(\Omega)$, $\varepsilon > 0$, only. In particular, the related Dirichlet datum $g \in H^{2/5-\varepsilon}(\Gamma)$ is discontinuous. Hence we should not use a piecewise linear and continuous approximation g_h, since this would reduce the expected order of convergence significantly.



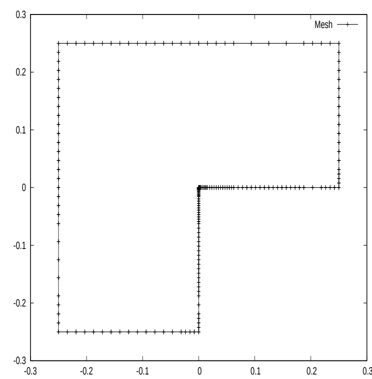
a) Error indicator $\|p_h\|_{L^2(\Gamma)}$



b) Pointwise error of the representation formula in $(0.04, 0.03)$



c) Error $\|w - w_H\|_{L^2(\Gamma)}$



d) Adaptive mesh at final level

Figure 1: Convergence results for $w \in H^{1/6-\varepsilon}(\Gamma)$, $\varepsilon > 0$.

Of course, one may also use a piecewise constant approximation $g_h = Q_h g \in S_h^0(\Gamma)$, since $(\frac{1}{2}I + K)g_h$ is well defined also for $g_h \in L^2(\Gamma)$. But when applying the Aubin–Nitsche trick this would again reduce the expected order of convergence since we can consider $K : H^1(\Gamma) \to H^1(\Gamma)$ only. In this case, we therefore consider the indirect single layer potential ansatz

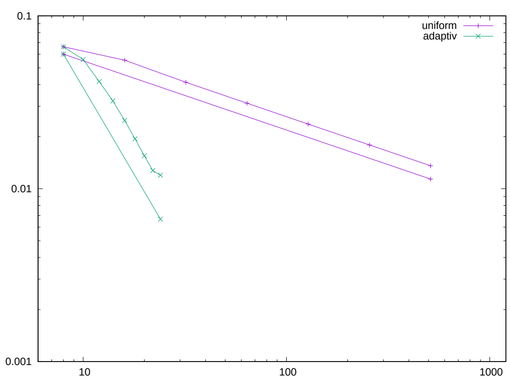
$$u(x) = \int_\Gamma U^*(x,y)w(y)\, ds_y \quad \text{for } x \in \Gamma,$$

which results in the boundary integral equation $Vw = g$ to be solved. For the numerical solution of this equation we can use the mixed variational formulation (3.1) for $f = g$ instead of $f = (\frac{1}{2}I + K)g$, and we can easily evaluate the right hand side also for given discontinuous Dirichlet data.

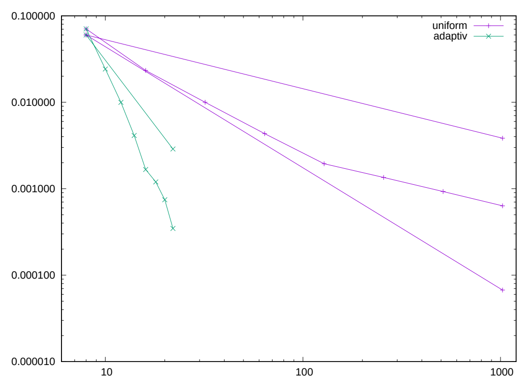
For the solution u as given in (5.2) we have $w \in H^{-3/5-\varepsilon}(\Gamma)$ for any $\varepsilon > 0$. In the case of an uniform refinement, we therefore expect the order order of convergence to be $\frac{2}{5}$ when considering $\|p_h\|_{L^2(\Gamma)} \simeq \|w - w_H\|_{H^{-1}(\Gamma)}$, and the approximate evaluation of the representation formula (2.2). Again we observe some higher order convergence initially, which corresponds to $\frac{7}{5}$, see Table 2, and Figure 1a,b). When using the adaptive boundary element approach, we observe again a second order of convergence for $\|p_h\|_{L^2(\Gamma)} \simeq \|w - w_H\|_{H^{-1}(\Gamma)}$, and a cubic one for the approximate representation formula, as expected.

N_H	N_h	$\|p_h\|_{L^2(\Gamma)}$	eoc	$\|u(\widetilde{x}) - \widetilde{u}(\widetilde{x})\|$	eoc
8	16	6.617 –2		7.048 –2	
16	32	5.539 –2	0.257	2.331 –2	1.596
32	64	4.125 –2	0.425	1.000 –2	1.221
64	128	3.120 –2	0.403	4.331 –3	1.207
128	256	2.364 –2	0.400	1.945 –3	1.155
256	512	1.791 –2	0.400	1.349 –3	0.528
512	1024	1.357 –2	0.400	9.272 –4	0.541
1024	2048	1.029 –2	0.399	6.338 –4	0.549
Theory			0.4		0.567

Table 2: Estimated order of convergence for $w \in H^{-3/5-\varepsilon}(\Gamma)$, $\varepsilon > 0$.



a) Error indicator $\|p_h\|_{L^2(\Gamma)}$



b) Pointwise error of the representation formula in $(0.04, 0.03)$

Figure 2: Convergence results for $w \in H^{-3/5-\varepsilon}(\Gamma)$, $\varepsilon > 0$.

6 Conclusions

In this paper we have formulated and analyzed an adaptive least squares boundary element method for the numerical solution of the first kind boundary integral equation $Vw = f$ by minimizing $\frac{1}{2}\|Vw - f\|^2_{L^2(\Gamma)}$. At one hand, this allows to consider less regular Dirichlet boundary data $g \notin H^{1/2}(\Gamma)$, on the other hand, this defines a simple a posteriori error estimator when using the adjoint p_h as error indicator. The numerical examples show the potential of the propesed approach, in particular for problems in three space dimensions, and when considering other partial differential equations. So far, our focus was not on the efficient solution of the Schur complement system (3.3) which is an approximation of the composed operator $V^*V : H^{-1}(\Gamma) \to H^1(\Gamma)$. Possible preconditioners for the conjugate gradient scheme include operator preconditioning [10, 21]. Hence we may use a stable discretization of the Laplace–Beltrami operator to define a suitable preconditioner, also in combination with fast boundary elements to solve problems in 3D.

For the solution of the boundary integral equation $Vw = f$, and in the case of sufficient regular given boundary data g, we may also consider the minimization of $\|Vw - f\|_{H^1(\Gamma)}$ to determine $w \in L^2(\Gamma)$. When introducing the adjoint $p := f - Vw \in H^1(\Gamma)$ and using a bounded and invertible operator $A : H^1(\Gamma) \to H^{-1}(\Gamma)$ we can use $\|p\|^2_{H^1(\Gamma)} = \langle Ap, p\rangle_\Gamma$ to conclude a mixed variational formulation to find $w \in L^2(\Gamma)$ and $p \in H^1(\Gamma)$. While we can still use piecewise constants to approximate the primal unknown w, we have to use continuous basis functions to approximate the adjoint p. However, its approximation p_h still serves as an error estimator for $\|w - w_H\|_{L^2(\Gamma)}$. A possible candidate for A is again the Laplace–Beltrami operator. It is more or less obvious, that we may apply these approaches also to other boundary integral equations including the hpersingular boundary integral operator, the double layer boundary integral operator, and its adjoint.

But of more interest is the application of this concept of least squares boundary element methods to the numerical solution of boundary integral equations which are related to time dependent partial differential equations, in particular for the wave equation. Based on [23] we have already presented a new approach to space-time boundary integral equations for the wave equation [22]. In future work we intend to use a least squares boundary element formulation for wave problems, similar as described here in the case of the Laplacian.

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