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Boundary element methods in linear elasticity: Can we avoid the symmetric formulation?

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Abstract

In this paper we discuss the use of single and double integral equations for the numerical solution of linear elasticity problems with boundary conditions of mixed type, and the one–equation coupling of finite and boundary element methods. In particular we present a sufficient and necessary condition which ensures stability of the coupled approach for any choice of finite and boundary elements. These results justify the coupling of collocation and Galerkin one–equation boundary element methods with finite elements as used in many engineering and industrial applications.

1 Introduction

The symmetric formulation of boundary integral equations [24] and related Galerkin boundary element methods are well established both from a mathematical and a practical point of view, In particular when consider boundary value problems with boundary conditions of mixed type [25], domain decomposition methods [8], and the coupling of finite and boundary element methods [3]. But the symmetric formulation requires the use of the hypersingular boundary integral operator, in particular the integration of hypersingular surface integrals. Although one may use integration by parts [6, 12] to rewrite Cauchy singular and hypersingular surface integrals as weakly singular ones, such an approach always requires the use of a Galerkin discretization. Even there are efficient implementations by means of fast boundary element methods available, e.g. fast multipole methods [14], adaptive cross approximation [20], etc., there is still not a big acceptance of symmetric Galerkin boundary element methods in engineering and industrial applications.
Instead, standard boundary integral equations, either based on a direct or indirect approach, but only using single and double layer integral operators are still very popular in engineering and industrial applications. In the case of mixed boundary conditions a common approach is to reorder the degrees of freedom after discretization. The resulting

linear system combines the discrete single and double layer integral operators, i.e. boundary integral equations of first and second kind, so it is not obvious to design iterative solution procedures. An alternative approach is to consider a mixed formulation where the Neumann boundary condition is formulated as a constraint in addition to the boundary integral equation which is related to a Dirichlet boundary. Finally such an approach results in a Steklov–Poincaré operator equation of the first kind, where the numerical analysis is well established, but this approach requires a proper choice of boundary elements for the discretization, see, for example, [26].
The coupling of finite and boundary element methods is of particular interest when modeling nonlinear materials in a bounded region with a linear material in an exterior unbounded domain, see, e.g., [2, 4, 31]. The one–equation or Johnson–Nédélec coupling goes back to the late seventies [1, 11] but theoretical results on the stability of the coupled formulation, even for the simpler case of a Laplace equation, required the consideration of a smooth interface. Moreover, since the double layer integral operator of linear elasticity is not compact, the theory which was developed for the Laplace equation, could not be extended. Although numerical examples indicate stability, there was no rigorous proof available to establish stability of the one–equation coupling of finite and boundary element methods.
In a recent paper [23] it was shown that the Johnson–Nédéc coupling is stable for any choice of boundary and finite elements. Alternative proofs, including a sufficient condition on the ratio of the involved material parameters of the scalar Laplace operators in the interior and exterior domains, were given in [29]. In [16] this result was improved, i.e. it was shown that the condition on the material parameters is also necessary. The aim of this paper is to extend this approach to the case of linear elasticity, see also [5] for a related result. In fact, we prove stability if the ratio of the involved Lamé parameters in the interior and exterior domain is bounded below by the contraction constant of the double layer integral operator. While in this paper we consider the case of a free space transmission problem, this approach can be extended to boundary value problems in bounded domains, see [17] for the scalar case.
This paper is organised as follows: In Sect. 2 we recall the formulation of boundary integral equations for the solution of boundary value problems with boundary conditions of mixed type. Mapping properties of the single and double layer integral operators are summarized in Sect. 3, where we derive the definition of appropriate norms which are induced by the single layer integral operator. Moreover we focus on the contraction property of the double layer integral operator, and we discuss possible approaches to compute approximate values of the contraction constant by solving related eigenvalue problems. The main result of this paper is given in Sect. 4, where we derive a sufficent and necessary condition to ensure stability of the coupled approach for any choice of finite and boundary elements. We end this paper by some final remarks and conclusions.

2 Boundary integral equations

We consider the equilibrium equations of linear elastostatics, by using the Einstein sum convention,

$$\sigma_{ij,j}(u) + f_i = 0 \quad \text{in } \Omega \subset \mathbb{R}^3 \tag{2.1}$$

with the stress tensor σ_{ij} given by Hooke's law, i.e.

$$\sigma_{ij}(u) = \frac{E\nu}{(1+\nu)(1-2\nu)} e_{kk}(u) + \frac{E}{1+\nu} e_{ij}(u),$$

with $E > 0$ and $\nu \in (0, \frac{1}{2})$, and with the linearized strain tensor

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

Later we will use the Lamé constants

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

By

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} \right\}$$

we denote the space of rigid body motions satisfying

$$e_{ij}(v^k) = 0 \quad \text{for all } v^k \in \mathcal{R}.$$

In addition to the partial differential equations (2.1) we consider boundary conditions of mixed type, i.e. prescribed displacements (Dirichlet) or boundary stresses (Neumann),

$$\begin{aligned} u_i &= g_i^D \quad \text{on } \Gamma_i^D, \\ t_i := \sigma_{ij}(u) n_j &= g_i^N \quad \text{on } \Gamma_i^N, \end{aligned}$$

where n is the exterior normal vector which is defined almost everywhere on $\Gamma = \partial\Omega$. Note that we assume

$$\Gamma = \overline{\Gamma}_i^D \cup \overline{\Gamma}_i^N, \quad \Gamma_i^D \cap \Gamma_i^N = \emptyset, \quad i = 1, 2, 3,$$

in particular we allow to consider different boundary conditions in different components. Moreover, we may consider boundary conditions in tangential and normal direction as well, and Signorini boundary conditions to describe contact problems with or without friction. Any solution of the linear elasticity system (2.1) is given by the representation formula (Somigliana identity)

$$\begin{aligned} u_i(x) &= \int_\Gamma U_{ij}^*(x,y) t_j(y) ds_y - \int_\Gamma T_{ij}^*(x,y) u_j(y) ds_y \\ &\quad + \int_\Omega U_{ij}^*(x,y) f_j(y) dy \quad \text{for } x \in \Omega \end{aligned} \tag{2.2}$$

where the fundamental solution of linear elasticity is given by the Kelvin tensor

$$
\begin{aligned}
U^*_{ij}(x,y) &= \frac{1}{8\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}\cdot \\
&\cdot\left[(3-4\nu)\frac{\delta_{ij}}{|x-y|}+\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right],
\end{aligned}
$$

and $T^*_{ij}(x,y)$ is the related stress fundamental solution.
To apply the representation formula (2.2) we need to determine the complete Cauchy data $[u,t]_{|\Gamma}$ by solving appropriate boundary integral equations. For this we consider the limiting process $x \to \Gamma$ in (2.2) to obtain the standard boundary integral equation of the direct approach,

$$
\begin{aligned}
\int_\Gamma U^*_{ij}(x,y)t_j(y)ds_y &= c_{ij}u_j(x)+\int_\Gamma T^*_{ij}(x,y)u_j(y)ds_y \\
&\quad -\int_\Omega U^*_{ij}(x,y)f_j(y)dy \quad \text{for } x\in\Gamma,
\end{aligned} \tag{2.3}
$$

where the integral free coefficients c_{ij} reflect the jump conditions of the double layer potential in corners and along edges. Without loss of generality we only consider the case $c_{ij} = \frac{1}{2}\delta_{ij}$. Hence we can write the boundary integral equation (2.3) as

$$
Vt = (\frac{1}{2}I + K)u - Nf \quad \text{on } \Gamma \tag{2.4}
$$

where we used the standard notations for the single layer integral operator V, for the double layer integral operator K, and for the Newton potential N. Depending on the given boundary conditions we will use the boundary integral equation (2.4) to find the yet unknown Cauchy data. Instead of the general case of boundary conditions of mixed type, we first consider the particular cases of pure Dirichlet or Neumann boundary conditions. In the case of Dirichlet boundary conditions $u = g^D$ on Γ, (2.4) results in the first kind boundary integral equation to find the Neumann datum t satisfying

$$
Vt = \frac{1}{2}g^D + Kg^D - Nf \quad \text{on } \Gamma. \tag{2.5}
$$

In the case of a Neumann boundary value problem we need to assume the equilibrium conditions

$$
\int_\Omega f\cdot v^k\,dx + \int_\Gamma g^N\cdot v^k_{|\Gamma}\,ds_x = 0 \quad \text{for all } v^k \in \mathcal{R},
$$

where $\mathcal{R}$ is the space of all rigid body motions. To find the yet unknown Dirichlet datum $u_{|\Gamma}$ we may consider the second kind boundary integral equation

$$
(\frac{1}{2}I + K)u_{|\Gamma} = Vg^N + Nf \quad \text{on } \Gamma. \tag{2.6}
$$

Instead of (2.6) we may also consider a mixed formulation to find $(u_{|\Gamma}, t)$ satisfying

$$Vt - (\frac{1}{2}I + K)u_{|\Gamma} = -Nf, \quad t = g^N \quad \text{on } \Gamma. \tag{2.7}$$

Since the first equation in (2.7) can be identified with the boundary integral equation (2.5) of the Dirichlet boundary value problem, we can solve this equation to obtain for the Neumann datum

$$t = V^{-1}(\frac{1}{2}I + K)u_{|\Gamma} - V^{-1}Nf \quad \text{on } \Gamma.$$

The operator

$$S^{\text{int}} := V^{-1}(\frac{1}{2}I + K) \tag{2.8}$$

is called the Steklov–Poincaré operator which realizes the Dirichlet to Neumann map which is associated to a function u satisfying the homogeneous partial differential equations (2.1), i.e. $f = 0$. Hence we can rewrite (2.7) as a first kind boundary integral equation to find $u_{|\Gamma}$ such that

$$S^{\text{int}}u_{|\Gamma} = g^N + V^{-1}Nf \quad \text{on } \Gamma. \tag{2.9}$$

In the case of mixed boundary conditions instead of (2.7) we consider the mixed formulation to find $(u_{|\Gamma}, t)$ such that

$$\begin{array}{rcll} Vt - (\frac{1}{2}I + K)u_{|\Gamma} & = & -Nf & \text{on } \Gamma, \\ t_i & = & g_i^N & \text{on } \Gamma_i^N, \\ u_i & = & g_i^D & \text{on } \Gamma_i^D. \end{array} \tag{2.10}$$

In fact, when eliminating the boundary stresses t we end up with a Steklov–Poincaré operator equation to find $u_{|\Gamma}$ such that

$$u_i = g_i^D \text{ on } \Gamma_i^D, \; (Su_{|\Gamma})_i = g_i^N + (V^{-1}Nf)_i \text{ on } \Gamma_i^N. \tag{2.11}$$

Note that the Steklov–Poincaré boundary integral equation (2.11) can be generalized to the case when describing boundary conditions in normal or tangential directions.

3 Boundary integral operators

The analysis of the boundary integral formulations (2.5)–(2.11) is based on the mapping properties of the involved boundary integral operators, see, e.g. [10, 13, 22, 28]. However, in what follows we will summarize some properties of boundary integral operators as required later on.

3.1 Single layer integral operator

For an arbitrary chosen density function w we define the single layer potential

$$u_i(x) = \int_\Gamma U^*_{ij}(x,y) w_j(y) ds_y \quad \text{for } x \in \mathbb{R}^3 \backslash \Gamma$$

which is a solution of the homogeneous linear elasticity system (2.1) for both the interior domain Ω, and for the exterior domain $\Omega^c := \mathbb{R}^3 \backslash \overline{\Omega}$. Betti's first formula for the interior problem gives

$$\int_\Omega \sigma_{ij}(u) e_{ij}(v) dx = \int_\Gamma t_i v_i ds_x. \tag{3.1}$$

In particular for $v = u$ and by using the jump relations of the single layer potential we further obtain

$$\int_\Omega \sigma_{ij}(u) e_{ij}(u) dx = \int_\Gamma [(\frac{1}{2} I - K')w]^\top V w \, ds_x, \tag{3.2}$$

where K' is the adjoint double layer integral operator, which results from the application of the boundary stress operator to the single layer potential. Accordingly we find for the exterior Dirichlet boundary value problem, and by using the decay behavior of the single layer potential in $\mathbb{R}^3$,

$$\int_{\Omega^c} \sigma_{ij}(u) e_{ij}(u) dx = \int_\Gamma [(\frac{1}{2} I + K')w]^\top V w \, ds_x. \tag{3.3}$$

By taking the sum of (3.2) and (3.3) this gives

$$\int_{\mathbb{R}^3 \backslash \Gamma} \sigma_{ij}(u) e_{ij}(u) dx = \langle V w, w \rangle_\Gamma. \tag{3.4}$$

Hence, by using Korn's inequality, this shows that the single layer integral operator V is elliptic, and therefore invertible, see also [4, 9, 28]. Recall that the ellipticity constant of the single layer integral operator degenerates for almost incompressible materials [27]. For simplicity we therefore assume $\nu < \frac{1}{2}$. In particular, we conclude the unique solvability of the boundary integral equation (2.5) which is related to the Dirichlet boundary value problem. Moreover, the Steklov–Poincaré operator S^{int} as given in (2.8) and which is related to the interior Dirichlet problem is well defined.
By taking the Dirichlet trace of the single layer potential we obtain $u_{|\Gamma} = Vw$, and since the single layer integral operator V is invertible, $w = V^{-1} u_{|\Gamma}$ follows. From (3.4) we therefore obtain

$$\int_{\mathbb{R}^3 \backslash \Gamma} \sigma_{ij}(u) e_{ij}(u) dx = \langle V^{-1} u, u \rangle_\Gamma,$$

i.e.

$$\|u\|^2_{V^{-1}} := \langle V^{-1} u, u \rangle_\Gamma \tag{3.5}$$

defines an equivalent norm in the space of given Dirichlet data, i.e. in $H^{1/2}(\Gamma)$.

3.2 Double layer integral operator

As for the single layer potential, for an arbitrary chosen density function v we now consider the double layer potential

$$u_i(x) = -\int_\Gamma T^*_{ij}(x,y)v_j(y)ds_y \quad \text{for } x \in \mathbb{R}^3\backslash\Gamma$$

which is again a solution of the homogeneous linear elasticity system (2.1). In the case of a Dirichlet problem, this indirect ansatz results in a second kind boundary integral equation to be solved, i.e.

$$(\frac{1}{2}I - K)v = g^D \quad \text{on } \Gamma. \tag{3.6}$$

The solution of (3.6) is given by the Neumann series

$$v = \sum_{\ell=0}^\infty (\frac{1}{2}I + K)^\ell g^D \quad \text{on } \Gamma,$$

which is convergent when $\frac{1}{2}I + K$ is a contraction. Recall that we have

$$(\frac{1}{2}I + K)v^k = 0 \quad \text{for all } v^k \in \mathcal{R}.$$

When using the norm as defined in (3.5) it is possible to prove the contraction estimate [30]

$$\|(\frac{1}{2}I + K)v\|_{V^{-1}} \le c_K \,\|v\|_{V^{-1}} \quad \text{for } v \in H^{1/2}(\Gamma) \tag{3.7}$$

with

$$c_K = \frac{1}{2} + \sqrt{\frac{1}{4} - c_0} < 1, \tag{3.8}$$

and

$$c_0 = \min_{0\neq v\in H^{1/2}_{\mathcal{R}}(\Gamma)} \frac{\langle Dv, v\rangle_\Gamma}{\langle V^{-1}v, v\rangle_\Gamma} > 0. \tag{3.9}$$

Note that D is the so–called hypersingular boundary integral operator of linear elasticity. In addition, $H^{1/2}_{\mathcal{R}}(\Gamma)$ is the space of functions which are orthogonal to the rigid body motions, in particular for $v \in H^{1/2}_{\mathcal{R}}$ we have

$$\langle v, V^{-1}v^k_{|\Gamma}\rangle_\Gamma = 0 \quad \text{for all } v^k \in \mathcal{R}.$$

Note that the orthogonality is considered with respect to an inner product which is induced by the inverse single layer integral operator V^{-1}.

For several applications, e.g., for the one–equation coupling of finite and boundary element methods, an explicit knowledge of the constant c_K as given in (3.8) may be useful. But only for rather few cases it may be possible to find c_K analytically. Instead, a numerical approximation of related eigenvalue problems is required in general.

By using (3.9) we can find $c_0 = \lambda_{\min}^2$ from the minimal eigenvalue of the generalized eigenvalue problem

$$Dv = \lambda^2 V^{-1} v \quad \text{in } H^{1/2}_{\mathcal{R}}(\Gamma).$$

For $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$ we introduce the transformation

$$w = \lambda V^{-1} v \in H^{-1/2}_{\mathcal{R}}(\Gamma),$$

in particular we have

$$\langle w, v^k \rangle_\Gamma = \lambda \langle v, V^{-1} v^k \rangle_\Gamma = 0 \quad \text{for all } v^k \in \mathcal{R}.$$

Hence we finally have to consider the operator eigenvalue problem

$$\begin{pmatrix} D & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \lambda \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \tag{3.10}$$

for $(v, w) \in H^{1/2}_{\mathcal{R}}(\Gamma) \times H^{-1/2}_{\mathcal{R}}(\Gamma)$. Although the eigenvalue problem (3.10) can be used to determine the minimal eigenvalue $c_0 = \lambda_{\min}^2$, and therefore c_K, it requires the use of the hypersingular boundary integral operator D, and a Galerkin discretization in the appropriate factor spaces $H^{1/2}_{\mathcal{R}}(\Gamma) \times H^{-1/2}_{\mathcal{R}}(\Gamma)$ is mandatory. Moreover, numerical algorithms to compute minimal eigenvalues may not be optimal with respect to efficiency and stability, e.g., the inverse power iteration requires the use of inverse matrices.
Hence we are interested in an approach to determine c_K which does not use the hypersingular boundary integral operator D, and where only the computation of a maximal eigenvalue is required. The contraction constant c_K of the contraction estimate (3.7) can be characterized by using a Rayleigh quotient, i.e.

$$\begin{aligned} c_K^2 &= \max_{v \in H^{1/2}(\Gamma)} \frac{\|(\frac{1}{2}I + K)v\|^2_{V^{-1}}}{\|v\|^2_{V^{-1}}} \\ &= \max_{v \in H^{1/2}(\Gamma)} \frac{\langle (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v, v \rangle_\Gamma}{\langle V^{-1} v, v \rangle_\Gamma}, \end{aligned}$$

and therefore $c_K = \lambda_{\max}$ is the maximal eigenvalue of the generalized operator eigenvalue problem

$$(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v = \lambda^2 V^{-1} v \quad \text{in } H^{1/2}(\Gamma). \tag{3.11}$$

Since all eigenvalues are non–negative, we may introduce the transformations

$$w = \lambda V^{-1} v, \quad z = V^{-1}(\frac{1}{2}I + K)v$$

which result in the generalized eigenvalue problem

$$\begin{pmatrix} V & -(\frac{1}{2}I + K) & 0 \\ (\frac{1}{2}I + K') & 0 & 0 \\ 0 & 0 & V \end{pmatrix} \begin{pmatrix} z \\ v \\ w \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} z \\ v \\ w \end{pmatrix}$$

which does not require neither the use of the hypersingular boundary integral operator D, nor the use of any factor space. Hence we can use standard boundary element approximations, and a simple power method to find an approximation of the maximal eigenvalue $\lambda_{\max}$, and therefore of c_K.
From the contraction estimate (3.7) we finally conclude the ellipticiy estimate

$$\langle S^{\rm int} v, v\rangle_\Gamma > (1 - c_K)\|v\|^2_{V^{-1}} \quad \text{for all } v \in H^{1/2}_{\mathcal{R}}(\Gamma)$$

which ensures unique solvability of the boundary integral equation (2.9) in the appropriate factor space $H^{1/2}_{\mathcal{R}}$, i.e. the solution of the Neumann boundary value problem is only unique up to the rigid body motions.
Moreover, there also holds the ellipticity estimate, see [16], for $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$

$$\frac{1}{c_K}\,\|(\frac{1}{2}I + K)v\|^2_{V^{-1}} \le \langle S^{\rm int} v, v\rangle_\Gamma. \tag{3.12}$$

4 Boundary element methods

In this section we discuss the numerical solution of boundary integral equations which are related to different boundary value problems as discussed before, by using Galerkin and collocation schemes. For a more general discussion on the numerical analysis of boundary element methods and on the design of fast methods, see, for example, [20, 22, 28].

4.1 Boundary element spaces

For $N \in \mathbb{N}$ we consider a sequence of admissible boundary element meshes $\Gamma_h = \cup_{\ell=1}^N \tau_\ell$. In the most simple case, we assume that Γ is polyhedral and that each boundary element mesh Γ_h consists of N plane shape regular triangular boundary elements τ_ℓ with mid points x_ℓ^*, with the area $\Delta_\ell = \int_{\tau_\ell} ds_x$, and with the local mesh size $h_\ell = \sqrt{\Delta_\ell}$. With respect to Γ_h we introduce, just for simplicity, lowest order boundary element spaces

$$S_h^0(\Gamma) = \text{span}\{\psi_\ell\}_{\ell=1}^N, \quad S_h^1(\Gamma) = \text{span}\{\varphi_i\}_{i=1}^M$$

of piecewise constant basis functions ψ_ℓ, and piecewise linear continuous nodal basis functions φ_i. M is the number of boundary element nodes, and we assume, that the boundary element meshes resolve the change in the boundary conditions when considering a mixed boundary value problem.

4.2 Dirichlet boundary value problem

In the case of a Dirichlet boundary value problem we consider the boundary integral equation (2.5). Let $g_h^D \in [S_h^1(\Gamma)]^3$ be some approximation of the given Dirichlet datum, which is obtained either by a simple interpolation, or by using a L_2 projection. The

Galerkin variational formulation of the boundary integral equation (2.5) is to find $t_h \in [S_h^0(\Gamma)]^3$ such that

$$\langle V t_h, \tau_h \rangle_\Gamma = \langle (\frac{1}{2} I + K) g_h^D, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in [S_h^0(\Gamma)]^3,$$

which is equivalent to a linear system of algebraic equations,

$$V_h \underline{t} = (\frac{1}{2} M_h + K_h) \underline{g} - \underline{f}^N. \tag{4.1}$$

Note that V_h and K_h are the Galerkin matrices of the single and double layer integral operators V and K, respectively, and M_h is the mass matrix

$$M_h = \begin{pmatrix} \mathcal{M} & & \\ & \mathcal{M} & \\ & & \mathcal{M} \end{pmatrix}, \quad \mathcal{M}[j,k] = \int_\Gamma \psi_k(x) \varphi_j(x) ds_x,$$

and $k = 1, \ldots, N$, $j = 1, \ldots, M$. For the evaluation of all Galerkin integrals, see, for example, [14, 19, 20, 21]. Finally, $\underline{f}^N$ is the contribution due to possible volume forces, for an efficient evaluation, see, e.g., [18]. When we approximate one integral in the computation of the Galerkin matrices by a simple mid point rule, i.e. for some boundary integral operator A which is discretized by using piecewise constant test functions we compute

$$A[\ell, k] = \int_\Gamma \psi_\ell(x) (A\psi_k)(x) ds_x \approx \Delta_\ell (A\psi_k)(x_\ell^*),$$

which corresponds to a weighted collocation scheme. Although there is still no proof for stability available, the resulting matrices are simple approximations of the related Galerkin matrices, and numerical examples indicate the applicability of such an approach. From a mathematical point of view, however, Galerkin methods are well established also from a theoretical point. In what follows we will not distinguish between Galerkin and collocation discretizations but we need to assume stability for the latter one.

4.3 Neumann boundary value problem

In the case of a Neumann boundary value problem we consider the mixed formulation (2.7) where the discretization of the first equation corresponds to the linear system (4.1) but now with unknown Dirichlet data, and where the weak formulation of the Neumann boundary condition reads

$$\int_\gamma t_{k,h}(x) \varphi_j(x) ds_x = \int_\Gamma g_i^N(x) \varphi_k(x) ds_x, \; k = 1, \ldots, M.$$

Hence we end up with the linear system

$$\begin{pmatrix} V_h & -\frac{1}{2} M_h - K_h \\ M_h^\top & \end{pmatrix} \begin{pmatrix} \underline{t} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} -\underline{f}^N \\ \underline{g}^N \end{pmatrix}. \tag{4.2}$$

Since the discrete single layer integral operator V_h is invertible, instead of (4.2) we may consider the Schur complement system

$$M_h^\top V_h^{-1}(\frac{1}{2}M_h + K_h)\underline{u} = \underline{g}^N + M_h^\top V_h^{-1}\underline{f}^N \tag{4.3}$$

which is nothing than a boundary element approximation of the Steklov–Poincaré operator equation (2.9). Note that

$$S_h^{\mathrm{BEM}} := M_h^\top V_h^{-1}(\frac{1}{2}M_h + K_h) \tag{4.4}$$

is an in general non–symmetric discrete approximation of the self–adjoint operator S, and this approximation is in general not stable. In particular when using piecewise constant and piecewise linear continuous basis functions to approximate the boundary stresses and displacements, respectively, oscillations may appear. In fact, a stable approximation of the Steklov–Poincaré operator S as defined in (2.8) requires an appropriate choice of the boundary element spaces to be used. The most common approach is to define boundary element spaces $S_h^0(\Gamma)$ and $S_H^1(\Gamma)$ with respect to boundary element meshes of different mesh size, in particular we need to assume $h < c_M H$ where c_M is in general unknown, see, e.g., [26, 31]. From a practical point of view, $c_M = \frac{1}{2}$ seems to be sufficient for most applications. In fact, the boundary element mesh as used to approximate the boundary displacements by using piecewise linear basis functions is refined once again to define piecewise constant basis functions to approximate the boundary stresses. An alternative approach is to use piecewise linear but discontinuous basis functions for the approximation of the boundary stresses [26] on the same mesh as for the boundary displacements. In both cases, however, the number of degrees of freedom increases compared to a naive approximation by using piecewise constant and piecewise linear basis functions on the same mesh.
Since the Neumann boundary value problem is only unique up to the rigid body motions one may introduce an additional stabilization of the discrete Steklov–Poincaré operator S_h^{BEM}, see, e.g., [15], in the case of a hypersingular boundary integral equation in the Laplace case.

4.4 Mixed boundary value problem

In the case of a mixed boundary value problem we consider der Steklov–Poincaré operator equation (2.11) and we follow the discretization approach as for a Neumann boundary value problem. The structure of the linear systems (4.2) and (4.3) remains the same, with different right hand sides, and a smaller dimension of the unknown vector $\underline{u}$. Note that for a mixed boundary value problem no stabilization of the discrete Steklov–Poincaré operator is required as for the Neumann problem. However, to ensure stability of the discrete Steklov–Poincaré operator S_h^{BEM} again an appropriate choice of boundary element spaces is mandatory, as discussed before.

5 Non–symmetric BEM/FEM coupling

As a model problem we consider a free space transmission problem to find displacement fields u^{int} and u^{ext} satisfying the equilibrium equations

$$\sigma^{\text{int}}_{ij,j}(u^{\text{int}}) + f_i = 0 \quad \text{in } \Omega \subset \mathbb{R}^3 \tag{5.1}$$

and

$$\sigma^{\text{ext}}_{ij,j}(u^{\text{ext}}) = 0 \quad \text{in } \Omega^c := \mathbb{R}^3 \backslash \overline{\Omega} \tag{5.2}$$

together with the transmission conditions on Γ

$$u^{\text{int}} = u^{\text{ext}}, \quad t := \sigma_{ij}(u^{\text{int}})n_j = \sigma_{ij}(u^{\text{ext}})n_j, \tag{5.3}$$

where n is the exterior normal vector whith respect to Ω which is defined almost everywhere on $\Gamma = \partial\Omega$. In addition we assume the radiation condition

$$|u(x)| = \mathcal{O}\Big(\frac{1}{|x|}\Big) \quad \text{as } |x| \to \infty. \tag{5.4}$$

For both the interior and exterior linear elasticity system we will consider Hooke's law for the interior stress tensor σ^{int}_{ij} and for the exterior σ^{ext}_{ij}, but with different material parameters $(E^{\text{int}}, \nu^{\text{int}})$ in Ω, and $(E^{\text{ext}}, \nu^{\text{ext}})$ in Ω^c, respectively.
The variational formulation of the interior problem (5.1), when inserting the Neumann transmission condition, is to find $u \in [H^1(\Omega)]^3$ such that

$$\int_\Omega \sigma^{\text{int}}_{ij}(u^{\text{int}})e_{ij}(v)dx - \int_\Gamma t_i v_i ds_x = \int_\Omega f_i v_i dx \tag{5.5}$$

is satisfied for all $v \in [H^1(\Omega)]^3$. On the other hand, the boundary integral equation which is related to the exterior Dirichlet boundary value problem is, by using the Dirichlet transmission condition,

$$Vt = (-\frac{1}{2}I + K)u^{\text{int}} \quad \text{on } \Gamma. \tag{5.6}$$

Since the single layer integral operator V is invertible, we can solve the boundary integral equation (5.6) to obtain

$$t^{\text{ext}} = -V^{-1}(\frac{1}{2}I - K)u^{\text{int}} =: -S^{\text{ext}}u^{\text{int}} \quad \text{on } \Gamma, \tag{5.7}$$

where

$$S^{\text{ext}} = V^{-1}(\frac{1}{2}I - K)$$

is the Steklov–Poincaré operator which is related to the exterior Dirichlet boundary value problem.

When inserting the exterior Dirichlet to Neumann map (5.7) into the variational formulation (5.5) we obtain, by setting $u = u^{\rm int}$,

$$\int_\Omega \sigma_{ij}^{\rm int}(u) e_{ij}(v) dx + \int_\Gamma (S^{\rm ext} u_{|\Gamma})_i v_i ds_x = \int_\Omega f_i v_i dx \qquad (5.8)$$

where the related bilinear form

$$a(u,v) = \int_\Omega \sigma_{ij}^{\rm int}(u) e_{ij}(v) dx + \int_\Gamma (S^{\rm ext} u_{|\Gamma})_i v_i ds_x$$

is elliptic. This ensures unique solvability of the variational formulation (5.8), as well as stability and quasi–optimality of associated Galerkin discretizations. However, the exterior Steklov–Poincaré operator $S^{\rm ext}$ involves the inversion of the single layer integral operator V, and therefore a stable boundary element approximation has to be used. At a first glance, and as for the interior Neumann boundary value problem, this approach restricts the choice of finite and boundary element spaces to be used, see, e.g., [26, 31].
Instead of the reduced variational formulation (5.8) we now consider a coupled variational formulation which combines the interior variational problem (5.5) and the exterior boundary integral equation (5.6). For this we define the bilinear form

$$\begin{aligned} a(u,t;v,\tau) &= \int_\Omega \sigma_{ij}^{\rm int}(u) e_{ij}(v) dx - \langle t, v\rangle_\Gamma \\ &\qquad + \langle Vt, \tau\rangle_\Gamma + \langle (\frac{1}{2} I - K) u, \tau \rangle_\Gamma \end{aligned}$$

and we consider the variational problem to find $(u,t) \in [H^1(\Omega)]^3 \times [H^{-1/2}(\Gamma)]^3$ such that

$$a(u,t;v,\tau) = \int_\Omega f_i v_i dx \qquad (5.9)$$

is satisfied for all $(v,\tau) \in [H^1(\Omega)]^3 \times [H^{-1/2}(\Gamma)]^3$. Unique solvability of the variational formulation (5.9) as well as stability and quasi–optimality of related Galerkin solutions follow when ellipticity of the bilinear form $a(\cdot,\cdot)$ can be ensured.
Since the finite element part of the bilinear form $a(\cdot,\cdot)$ only defines a semi–norm, we first introduce an alternative representation.
Chosing $v = v^k \in \mathcal{R}$ and $\tau = 0$ as test functions in the variational formulation(5.9) we first obtain

$$-\int_\Gamma t_i v_i^k ds_x = \int_\Omega f_i v_i^k \, dx, \quad k = 1, \ldots, 6.$$

On the other hand, by chosing $\tau = t^k = V^{-1} v^k$ and $v = 0$ gives

$$\begin{aligned} \langle t, v^k\rangle_\Gamma &= \langle t, Vt^k\rangle_\Gamma = \langle Vt, t^k\rangle_\Gamma \\ &= \langle (-\frac{1}{2} I + K) u, t^k\rangle_\Gamma \\ &= \langle (\frac{1}{2} I + K) u, t^k\rangle_\Gamma - \langle u, t^k\rangle_\Gamma = -\langle u, t^k\rangle_\Gamma \end{aligned}$$

due to

$$\begin{aligned}\langle(\frac{1}{2}I+K)u,t^k\rangle_\Gamma &= \langle(\frac{1}{2}I+K)u,V^{-1}v^k\rangle_\Gamma \\ &= \langle u,V^{-1}(\frac{1}{2}I+K)v^k\rangle_\Gamma = 0.\end{aligned}$$

Note that we have used the symmetry $KV = VK'$ and $(\frac{1}{2}I + K)v^k = 0$ for all $v^k \in \mathcal{R}$. Hence we conclude

$$\langle u, t^k\rangle_\Gamma = \langle f, v^k\rangle_\Omega \quad \text{for all } v^k \in \mathcal{R}.$$

For any $u \in [H^1(\Omega)]^3$ we can therefore write

$$u = \widetilde{u} + \sum_{k=1}^{6} \alpha_k v^k, \quad \langle \widetilde{u}, t^k\rangle_\Gamma = 0 \quad \text{for } k = 1, \ldots, 6$$

where the coefficients α_k are determined by the solution of the linear system

$$\sum_{k=1}^{6} \alpha_k \langle v^k, V^{-1}v^\ell\rangle_\Gamma = \langle f, v^\ell\rangle_\Omega \quad \text{for } \ell = 1, \ldots, 6.$$

Hence, instead of (5.5) and (5.6) we consider a modified variational formulation to find $(\widetilde{u}, t) \in [H^1(\Omega)]^3 \times [H^{-1/2}(\Gamma)]^3$ such that

$$\int_\Omega \sigma_{ij}^{\text{ext}}(\widetilde{u}) e_{ij}(v) dx + \sum_{k=1}^{6} \langle \widetilde{u}, t^k\rangle_\Gamma \langle v, t^k\rangle_\Gamma - \langle t, v\rangle_\gamma = \langle f, v\rangle_\Gamma$$

and

$$\langle Vt, \tau\rangle_\Gamma + \langle(\frac{1}{2}I - K)\widetilde{u}, \tau\rangle_\Gamma = -\sum_{k=1}^{6} \alpha_k \langle v^k, \tau\rangle_\Gamma$$

are satisfied for all $(v, \tau) \in [H^1(\Omega)]^3 \times [H^{-1/2}(\Gamma)]^3$. The related bilinear form is given by

$$\begin{aligned}\widetilde{a}(u, t; v, \tau) &= \int_\Omega \sigma_{ij}^{\text{int}}(u) e_{ij}(v) dx + \sum_{k=1}^{6} \langle u, t^k\rangle_\Gamma \langle v, t^k\rangle_\Gamma \\ &\quad -\langle t, v\rangle_\Gamma + \langle Vt, \tau\rangle_\Gamma + \langle(\frac{1}{2}I - K)u, \tau\rangle_\Gamma.\end{aligned}$$

To ensure ellipticity of the bilinear form $\widetilde{a}(\cdot;\cdot)$ we consider for $(v, \tau) \in [H^1(\Omega)]^3 \times [H^{-1/2}(\Gamma)]^3$

$$\begin{aligned}\widetilde{a}(v, \tau; v, \tau) &= \int_\Omega \sigma_{ij}^{\text{int}}(v) e_{ij}(v) dx + \sum_{k=1}^{6} [\langle v, t^k\rangle_\Gamma]^2 \\ &\quad +\langle V\tau, \tau\rangle_\Gamma - \langle(\frac{1}{2}I + K)v, \tau\rangle_\Gamma.\end{aligned}$$

We can write the finite element part of the bilinear form $\widetilde{a}(\cdot;\cdot)$ as

$$\begin{aligned}
&\int_\Omega \sigma_{ij}^{\rm int}(v)e_{ij}(v)dx \\
&= 2\mu^{\rm int}\int_\Omega e_{ij}(v)e_{ij}(v)dx + \lambda^{\rm int}\int_\Omega [\operatorname{div} v]^2 dx \\
&\geq \eta\left[2\mu^{\rm ext}\int_\Omega e_{ij}(v)e_{ij}(v)dx + \lambda^{\rm ext}\int_\Omega [\operatorname{div} v]^2 dx\right] \\
&= \eta\int_\Omega \sigma_{ij}^{\rm ext}(v)e_{ij}(v)dx
\end{aligned}$$

with

$$\eta := \min\left\{\frac{\mu^{\rm int}}{\mu^{\rm ext}}, \frac{\lambda^{\rm int}}{\lambda^{\rm ext}}\right\}.$$

For $v \in [H^1(\Omega)]^3$ let $v_\Gamma \in [H^1(\Omega)]^3$ the unique weak solution of the Dirichlet boundary value problem

$$\sigma_{ij,j}^{\rm ext}(v_\Gamma) = 0 \quad \text{in } \Omega, \quad v_\Gamma = v_{|\Gamma} \quad \text{on } \Gamma,$$

and we set $\widehat{v} := v - v_\Gamma \in [H_0^1(\Omega)]^3$. Then there holds the orthogonality

$$\int_\Omega \sigma_{ij}^{\rm ext}(v_\Gamma)e_{ij}(\widehat{v})dx = 0.$$

Hence we conclude

$$\begin{aligned}
\widetilde{a}(v,\tau;v,\tau) \quad &\geq \quad \eta\int_\Omega \sigma_{ij}^{\rm ext}(\widehat{v})e_{ij}(\widehat{v})dx + \sum_{k=1}^6 [\langle v, t^k\rangle_\Gamma]^2 \\
&+\eta\int_\Omega \sigma_{ij}^{\rm ext}(v_\Gamma)e_{ij}(v_\Gamma)dx + \langle V\tau,\tau\rangle_\Gamma - \langle(\frac{1}{2}I+K)v,\tau\rangle_\Gamma
\end{aligned}$$

and it remains to consider the second line. Since v_Γ is a solution of the homogeneous equilibrium equations, by using Betti's first formula we have

$$\int_\Omega \sigma_{ij}^{\rm ext}(v_\Gamma)e_{ij}(v_\Gamma)dx = \int_\Gamma S^{\rm int/ext}v_\Gamma \cdot v_\Gamma ds_x$$

where $S^{\rm int/ext}$ is the Steklov–Poincaré operator which is related to an interior Dirichlet boundary value problem, but with the material parameters as defined in the exterior domain. On the other hand, doe to (3.12)

$$\begin{aligned}
\langle(\frac{1}{2}I+K)v,\tau\rangle_\Gamma \quad &\leq \quad \|(\frac{1}{2}I+K)v\|_{V^{-1}}\|\tau\|_V \\
&\leq \quad \sqrt{c_K\langle S^{\rm int/ext}v, v\rangle_\Gamma}\|\tau\|_V.
\end{aligned}$$

Hence we conclude, for all $\gamma > 0$,

$$\begin{aligned}
&\eta \int_\Omega \sigma_{ij}^{\text{ext}}(v_\Gamma) e_{ij}(v_\Gamma) dx + \langle V\tau, \tau\rangle_\Gamma - \langle (\frac{1}{2}I + K)v, \tau\rangle_\Gamma \\
&\geq \eta \langle S^{\text{int/ext}} v, v\rangle_\Gamma + \|\tau\|_V^2 - \sqrt{c_K \langle S^{\text{int/ext}} v, v\rangle_\Gamma} \|\tau\|_V \\
&= \left(\eta - \frac{1}{2} c_K \frac{1}{\gamma^2}\right) \langle S^{\text{int/ext}} v, v\rangle_\Gamma + \left(1 - \frac{1}{2}\gamma^2\right) \|\tau\|_V^2 \\
&\qquad + \frac{1}{2}\left(\frac{1}{\gamma}\sqrt{c_K \langle S^{\text{int/ext}} v, v\rangle_\Gamma} - \gamma \|\tau\|_V\right)^2 \\
&\geq \left(\eta - \frac{1}{2}\gamma_*^2\right)\left(\langle S^{\text{int/ext}} v, v\rangle_\Gamma + \|\tau\|_V^2\right)
\end{aligned}$$

if

$$\eta - \frac{1}{2} c_K \frac{1}{\gamma_*^2} = 1 - \frac{1}{2}\gamma_*^2 > 0$$

is satisfied. From this condition we first find

$$\gamma_*^2 = 1 - \eta + \sqrt{[\eta - 1]^2 + c_K}$$

and therefore we obtain that

$$1 - \frac{1}{2}\gamma_*^2 = \frac{1}{2}\left[1 + \eta - \sqrt{[\eta - 1]^2 + c_K}\right] > 0$$

is satisfied for

$$\eta > \frac{1}{4} c_K.$$

In particular we finally conclude the ellipticity estimate

$$\begin{aligned}
\widetilde{a}(v, \tau; v, \tau) \;\; &\geq \;\; \eta \int_\Omega \sigma_{ij}^{\text{ext}}(\widehat{v}) e_{ij}(\widehat{v}) dx + \sum_{k=1}^{6} [\langle v, t^k\rangle_\Gamma]^2 \\
&\qquad + \left(\eta - \frac{1}{2}\gamma_*^2\right)\left(\langle S^{\text{int/ext}} v, v\rangle_\Gamma + \|\tau\|_V^2\right) \\
&\geq c_1^A \left\{ \int_\Omega \sigma_{ij}^{\text{ext}}(v) e_{ij}(v) dx + \sum_{k=1}^{6} [\langle v, V^{-1} v\rangle_\Gamma]^2 + \|\tau\|_V^2 \right\}
\end{aligned}$$

with the ellipticity constant

$$c_1^A = \frac{1}{2}\left[1 + \eta - \sqrt{[\eta - 1]^2 + c_K}\right] > 0$$

if we assume

$$\min\left\{\frac{\mu^{\text{int}}}{\mu^{\text{ext}}}, \frac{\lambda^{\text{int}}}{\lambda^{\text{ext}}}\right\} > \frac{1}{4} c_K. \tag{5.10}$$

Condition (5.10) is sufficient to ensure ellipticity of the modified bilinear form $\widetilde{a}(\cdot;\cdot)$. As a consequence we conclude stability for any conformal discretization, i.e. for any choice of standard finite and boundary elements. It turns out, see [16] in the case of the Laplace equation, that (5.10) is also necessary, i.e. for

$$\mu^{\text{ext}} = \overline{\eta}\mu^{\text{int}}, \quad \lambda^{ext} = \overline{\eta}\lambda^{\text{int}}$$

with

$$\overline{\eta} \leq \frac{1}{4}c_K$$

there exist $(\overline{v}, \overline{\tau}) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\widetilde{a}(\overline{v}, \overline{\tau}; \overline{v}, \overline{\tau}) = 0.$$

In this case, the bilinear form $\widetilde{a}(\cdot;\cdot)$ fails to be elliptic, and hence we can not ensure stability of the coupled finite and boundary elements by using the above arguments. However, in this case we may use the Steklov–Poincaré operator formulation (5.8) which is elliptic for any combination of material parameters. But a stable discretization then requires an appropriate choice of finite and boundary elements.

6 Conclusions

While the use of the Steklov–Poincaré operator equation for the solution of mixed boundary value problems requires an appropriate choice of boundary elements to approximate the boundary displacements and boundary stresses, the one–equation coupling of finite and boundary element methods turns out to be stable for any choice of finite and boundary elements, when a certain condition on the ratio of the material parameters in interior and exterior domain is satisfied. Although in this paper we have not presented an explicite proof, this condition turns out to be also necessery, see [16], where also numerical examples are given for illustration. While in this paper we have considered the model problem of a free space transmission problem, this approach can be extended to analyse the coupling of finite and boundary element methods to tackle boundary value problems in bounded domains [17]. In this case we have to analyse eigenvalue problems which relate the energy with respect to a bounded domain with the energy of an associated exterior domain. As for the computation of the contraction constant, the corresponding eigenvalue can be computed by using boundary element methods too. Other possible extensions include the consideration of more complicated materials, e.g. poroelasticity, elastoplasticity, etc., and the design and analysis of appropriate precondition iterative solution strategies for the resulting linear systems.

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