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# Technische Universität Graz

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# On space-time finite element domain decomposition methods for the heat equation

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## Abstract

For the space-time finite element discretization of the heat equation we propose to use a non-overlapping domain decomposition method. The space-time domain is decomposed into  $p$  subdomains which can be arbitrary in space and time. All interior degrees of freedom are eliminated by using direct solvers locally, and the resulting Schur complement system is solved by using a global GMRES method. First numerical results indicate the potential of space-time domain decomposition methods.

## 1 Introduction

Space-time discretization methods became very popular in recent years, see for example the review article [12], and the references given therein. Applications in mind involve not only the direct simulation of time-dependent partial differential equations in fixed or moving domains, but also problems from optimization, optimal control, and inverse problems. The solution of the latter applications can be characterized by a coupled problem of a primal forward problem, and an adjoint backward problem, which motivates the use of space-time methods for the solution of the global problem in the space-time domain. As an example we mention a distributed control problem for the heat equation as considered in [6]. Space-time discretization methods also allow the use of general and unstructured finite elements, and therefore an adaptive resolution in space and time simultaneously. But the solution of

the overall global system in space and time requires the use of appropriate iterative solution strategies in parallel. Besides a pure parallelization strategy using distributed memory and matrix vector products in parallel, domain decomposition methods can be used for both the parallelization and the construction of suitable preconditioners. When doing a domain decomposition in space only, we may use the possibility to parallelize in time, where the latter can be done by using the parareal algorithm [8].

Following the well established approaches for domain decomposition methods for elliptic problems, e.g., [1, 5], we first consider the global space-time finite element discretization of the heat equation, using, e.g., lowest order piecewise linear continuous basis functions. Using a non-overlapping domain decomposition of the space-time domain, and reordering the global stiffness matrix accordingly, we end up with a block system of linear equations, where we can eliminate all local degrees of freedom, e.g., using direct solution methods locally. The resulting Schur complement system is then solved by a global GMRES iteration. In the case of a one-dimensional spatial domain we will consider different space-time domain decomposition methods, e.g., domain decompositions in space, in time, in space and time, and interfaces which are oblique in space and time. Although we will not consider preconditioning strategies in detail at this time, we will discuss possible preconditioners for the situations mentioned above. In the particular case of a domain decomposition into time slabs our approach is strongly related to the parareal algorithm. In any case, the numerical results as presented in this contribution indicate the great potential of space-time domain decomposition methods.

## 2 Space-time finite element methods

As a model problem we consider the Dirichlet boundary value problem for the heat equation,

$$\begin{aligned} \partial_t u(x, t) - \Delta_x u(x, t) &= f(x, t) && \text{for } (x, t) \in Q := \Omega \times (0, T), \\ u(x, t) &= 0 && \text{for } (x, t) \in \Sigma := \partial\Omega \times (0, T), \\ u(x, 0) &= 0 && \text{for } x \in \Omega, \end{aligned} \quad (1)$$

where  $\Omega \subset \mathbb{R}^n$ ,  $n = 1, 2, 3$ , is some bounded Lipschitz domain,  $T > 0$  is a finite time horizon, and  $f$  is some given source. For simplicity we only consider homogeneous boundary and initial conditions, but inhomogeneous data as well as other types of boundary conditions can be handled as well.

The variational formulation of (1) is to find  $u \in X$  such that

$$a(u, v) := \int_0^T \int_{\Omega} [\partial_t u v + \nabla_x u \cdot \nabla_x v] dx dt = \int_0^T \int_{\Omega} f v dx dt = \langle f, v \rangle_Q \quad (2)$$

is satisfied for all  $v \in Y$ . Here we use the standard Bochner spaces

$$X := L^2(0, T; H_0^1(\Omega)) \cap H_0^1(0, T; H^{-1}(\Omega)), \quad Y := L^2(0, T; H_0^1(\Omega)),$$

including zero boundary and initial conditions, with the norms

$$\|v\|_Y := \|\nabla_x v\|_{L^2(Q)}, \quad \|u\|_X := \sqrt{\|\partial_t u\|_{Y^*}^2 + \|u\|_Y^2} = \sqrt{\|w\|_Y^2 + \|u\|_Y^2},$$

where  $w \in Y$  is the unique solution of the variational problem

$$\int_0^T \int_{\Omega} \nabla_x w \cdot \nabla_x v dx dt = \int_0^T \int_{\Omega} \partial_t u v dx dt \quad \text{for all } v \in Y. \quad (3)$$

Unique solvability of the variational problem (2) is based on the inf-sup stability condition [10, 11]

$$\frac{1}{\sqrt{2}} \|u\|_X \leq \sup_{0 \neq v \in Y} \frac{a(u, v)}{\|v\|_Y} \quad \text{for all } u \in X. \quad (4)$$

For the discretization of the variational formulation (2) we introduce conforming space-time finite element spaces  $X_h$  and  $Y_h$ , where we assume  $X_h \subset Y_h$ , i.e., we use the finite element spaces  $X_h = Y_h$  of piecewise linear and continuous basis functions, defined with respect to some admissible decomposition of the space-time domain  $Q$  into shape regular simplicial finite elements. Then, the Galerkin formulation of (2) is to find  $u_h \in X_h$  such that

$$a(u_h, v_h) = \langle f, v_h \rangle_Q \quad \text{for all } v_h \in Y_h, \quad (5)$$

and unique solvability of (5) follows from the discrete inf-sup stability condition

$$\frac{1}{\sqrt{2}} \|u_h\|_{X_h} \leq \sup_{0 \neq v_h \in Y_h} \frac{a(u_h, v_h)}{\|v_h\|_Y} \quad \text{for all } u_h \in X_h. \quad (6)$$

Note that in (6) we use the discrete norm

$$\|u\|_{X_h} := \sqrt{\|w_h\|_Y^2 + \|u\|_Y^2} \quad \text{for } u \in X,$$

where  $w_h \in Y_h$  is the unique solution of the Galerkin variational formulation

$$\int_0^T \int_{\Omega} \nabla_x w_h \cdot \nabla_x v_h \, dx \, dt = \int_0^T \int_{\Omega} \partial_t u v_h \, dx \, dt \quad \text{for all } v_h \in Y_h. \quad (7)$$

From (6) we then conclude the quasi-optimal a priori error estimate

$$\|u - u_h\|_{X_h} \leq 3 \inf_{v_h \in X_h} \|u - v_h\|_X,$$

and when assuming  $u \in H^2(Q)$  we finally obtain, when using piecewise linear continuous basis functions, i.e.,  $X_h = \text{span}\{\varphi_k\}_{k=1}^M$ ,

$$\|\nabla_x(u - u_h)\|_{L^2(Q)} \leq c h |u|_{H^2(Q)}. \quad (8)$$

The Galerkin variational formulation (5) is equivalent to a linear system of algebraic equations,  $K_h \underline{u} = \underline{f}$ , where the stiffness matrix  $K_h$  and the load vector  $\underline{f}$  are given as, for  $k, \ell = 1, \dots, M$ ,

$$\begin{aligned} K_h[\ell, k] &= \int_Q \left[ \partial_t \varphi_k(x, t) \varphi_\ell(x, t) + \nabla_x \varphi_k(x, t) \cdot \nabla_x \varphi_\ell(x, t) \right] dx \, dt, \quad (9) \\ f_\ell &= \int_Q f(x, t) \varphi_\ell(x, t) \, dx \, dt. \end{aligned}$$

For a more detailed numerical analysis of this space-time finite element method we refer to [11], and the review article [12].

### 3 A space-time domain decomposition method

Although the finite element stiffness matrix  $K_h$  as defined in (9) is invertible, it is neither symmetric nor definite. Hence we will use the GMRES method as iterative solver. But since we are discretizing the problem in the whole space-time domain  $Q$ , preconditioning and parallelization both in space and time are mandatory.

One possible approach is to use a non-overlapping domain decomposition method as originally designed for elliptic problems, see, e.g., [1, 5]. For the space-time domain  $Q := \Omega \times (0, T) \subset \mathbb{R}^{n+1}$  we consider a domain decomposition into  $p$  non-overlapping subdomains,

$$\overline{Q} = \bigcup_{i=1}^p \overline{Q}_i, \quad Q_i \cap Q_j = \emptyset \quad \text{for } i \neq j.$$



We do not necessarily assume any tensor-product structure of the subdomains  $Q_i$  as shown in Fig. 1 a), b), d), f), we may also consider subdomains arbitrary in space and time, see, e.g., Fig. 1 c), e). We assume that the underlying space-time finite element mesh is resolved by the domain decomposition. In this case we can rearrange all global degrees of freedom  $\underline{u}$  into local interior ones per subdomain, i.e.,

$$\underline{u}_I = \left( \underline{u}_{I,1}^\top, \dots, \underline{u}_{I,p}^\top \right)^\top,$$

and the remaining global degrees of freedom  $\underline{u}_C$  on the coupling boundaries. Hence we can rewrite the linear system  $K_h \underline{u} = \underline{f}$  as

$$\begin{pmatrix} K_{II} & K_{CI} \\ K_{IC} & K_{CC} \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{u}_C \end{pmatrix} = \begin{pmatrix} \underline{f}_I \\ \underline{f}_C \end{pmatrix}, \quad (10)$$

with the block-diagonal matrix

$$K_{II} = \text{diag}(K_{11} \dots K_{pp}),$$

where the block matrices  $K_{ii}$  correspond to the interior degrees of freedom in the subdomain  $Q_i$ . Instead of (10) we now consider the Schur complement system

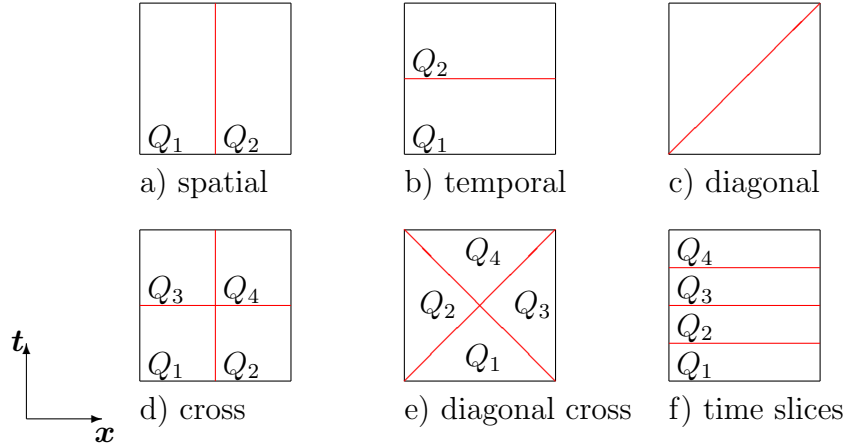
$$S_C \underline{u}_C := (K_{CC} - K_{IC} K_{II}^{-1} K_{CI}) \underline{u}_C = \underline{f}_C - K_{IC} K_{II}^{-1} \underline{f}_I =: \underline{f}, \quad (11)$$

which will be solved by using some global iterative solver such as GMRES. At this time we will not focus on preconditioning the Schur complement system (11), but we will consider different cases of possible space-time domain decompositions as given in Fig. 1 and the influence of the resulting interface in the space-time domain.

## 4 Numerical results

In this section we present first numerical results for the space-time finite element domain decomposition method in the case of a one-dimensional spatial domain  $\Omega = (0, 1)$  and the time horizon  $T = 1$ , i.e., we have  $Q = (0, 1)^2$ . In all examples we consider the smooth function  $u(x, t) = \cos \pi t \sin \pi x$  to ensure optimal linear convergence in  $L^2(0, T; H_0^1(\Omega))$  as expected from the

Figure 1: Domain decomposition of the space-time domain  $Q = \Omega \times (0, T) \subset \mathbb{R}^2$ .



a priori error estimate (8). In Table 1 we also present the error in  $L^2(Q)$  where we observe second order convergence in  $h$ . Note that DoF denotes the global number of degrees of freedom, Iter is the number of GMRES iterations without preconditioning to reach a relative accuracy of  $\varepsilon = 10^{-7}$ .

In Table 1 we first present the results for the case without domain decomposition, and for the domain decompositions a)-d) as depicted in Fig. 1. We observe that the spatial domain decomposition a) and the diagonal decomposition c) give rather good and comparable results, while the results for the other two cases show a more significant dependence on the mesh size  $h$ . When considering the Schur complement system (11) in the case of the spatial decomposition a) we note that the Schur complement matrix  $S_C$  is the finite element approximation of an continuous operator  $S : H^{1/4}(\Gamma_C) \rightarrow H^{-1/4}(\Gamma_C)$ , representing the interface conditions along the coupling boundary  $\Gamma_C$  in time only. Since  $S$  behaves like the heat potential hypersingular boundary integral operator, e.g., [3], in particular it is an operator of order  $\frac{1}{2}$ , the spectral condition number of  $S_C$  behaves like  $h^{-1/2}$ , and hence the number of iterations to reach a given accuracy grows as  $h^{-1/4}$ , which corresponds to a factor of 1.19 in the case of a uniform mesh refinement. This behavior is clearly seen in the last three refinement steps. To bound the number of required iterations independent of the mesh level, we can use a suitable preconditioning strategy. One possible option is the use of operator preconditioning,

i.e., the Galerkin discretization of the single layer heat potential as described in [4]. The situation is similar in the case c) of a diagonal domain decomposition. Here, the interface  $\Gamma_C$  is the diagonal  $t = x$ , and so the Schur complement matrix  $S_C$  is the finite element approximation of a continuous operator  $S : H^{1/2,1/4}(\Gamma_C) \rightarrow H^{-1/2,-1/4}(\Gamma)$ , using boundary trace spaces of anisotropic Sobolev spaces in the domain. Note that the mapping properties of related boundary integral operators remain true, and so operator preconditioning can be used also in this case, as well as in the higher dimensional case  $\Omega \subset \mathbb{R}^n$ ,  $n = 2, 3$ .

Table 1: Numerical results for different space-time domain decompositions.

| DoF    | $\ u - u_h\ _{L^2(Q)}$ |    |      |  | GMRES iterations |    |      |      |    |     |    |     |
|--------|------------------------|----|------|--|------------------|----|------|------|----|-----|----|-----|
|        |                        |    |      |  | no               | a) | b)   | c)   | d) |     |    |     |
| 12     | 7.072                  | -2 |      |  | 5.969            | -1 | 0.97 | 12   | 4  | 3   | 3  | 6   |
| 56     | 1.912                  | -2 | 1.89 |  | 3.057            | -1 | 0.97 | 44   | 8  | 7   | 7  | 14  |
| 240    | 4.863                  | -3 | 1.98 |  | 1.538            | -1 | 0.99 | 135  | 16 | 15  | 15 | 30  |
| 992    | 1.219                  | -3 | 2.00 |  | 7.705            | -2 | 1.00 | 363  | 26 | 31  | 23 | 56  |
| 4.032  | 3.048                  | -4 | 2.00 |  | 3.855            | -2 | 1.00 | 928  | 33 | 63  | 29 | 95  |
| 16.256 | 7.618                  | -5 | 2.00 |  | 1.928            | -2 | 1.00 | 2414 | 39 | 127 | 38 | 161 |
| 65.280 | 1.907                  | -5 | 2.00 |  | 9.638            | -3 | 1.00 | 6536 | 46 | 201 | 48 | 285 |

As the uniform finite element meshes used for the domain decompositions a)-d) can be described within time-slabs, the proposed space-time domain decomposition method can be applied also in the case of general space-time finite element meshes. In the case e) of a diagonal cross domain decomposition we apply a recursive red refinement within the subdomains, as depicted in Table 2. We observe similar results as in the case d) of a cross decomposition. Again we may use a suitable preconditioning strategy which has to take care of the coarse grid involved. A possible approach is the combination of opposite operator preconditioning locally, with global preconditioning using BDDC [7].

The last example covers case f) of a domain decomposition into time slabs, where we consider up to  $p = 16$  temporal subdomains, see Table 3. Even without preconditioning of the global Schur complement system (11) we observe a rather good behavior in the number of required iterations. It is obvious that this approach is strongly related to the parareal algorithm [8] where the coarse grid corresponds to the time slabs of the domain decomposition.

Table 2: Space-time finite element mesh and numerical results for case e) of a diagonal cross domain decomposition with 4 subdomains.

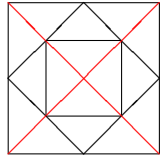
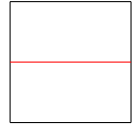

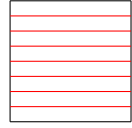

|   | DoF    | $\ u - u_h\ _{L^2(Q)}$ |    | $\ \nabla_x(u - u_h)\ _{L^2(Q)}$ |         | Iter |     |
|---|--------|------------------------|----|----------------------------------|---------|------|-----|
|  | 27     | 2.57 -2                |    | 3.52 -1                          |         | 6    |     |
|   | 119    | 6.93                   | -3 | 1.89                             | 1.77 -1 | 0.99 | 14  |
|   | 495    | 1.82                   | -3 | 1.93                             | 8.89 -2 | 1.00 | 28  |
|   | 2.015  | 4.64                   | -4 | 1.97                             | 4.45 -2 | 1.00 | 51  |
|   | 8.127  | 1.17                   | -4 | 1.99                             | 2.23 -2 | 1.00 | 85  |
|   | 32.639 | 2.92                   | -5 | 2.00                             | 1.11 -2 | 0.99 | 138 |

Table 3: Numerical results for a domain decomposition into time slabs.

|   | DoF     | $p = 2$ | $p = 4$ | $p = 8$ | $p = 16$ |
|---|---------|---------|---------|---------|----------|
|   | 12      | 3       |         |         |          |
|   | 56      | 7       | 17      |         |          |
|   | 240     | 15      | 29      | 49      |          |
|   | 992     | 31      | 36      | 65      | 101      |
|  | 4.032   | 63      | 65      | 91      | 132      |
|   | 16.256  | 127     | 128     | 133     | 198      |
|   | 65.280  | 201     | 226     | 255     | 280      |
|  | 261.632 | 281     | 311     | 383     | 497      |

## 5 Conclusions

In this note we have presented first numerical results for the numerical solution of the heat equation by using standard domain decomposition methods. This approach is based on a space-time finite element discretization, where the resulting global stiffness matrix is, as in standard domain decomposition methods for elliptic problems, reordered with respect to some non-overlapping domain decomposition of the space-time domain. Eliminating the local degrees of freedom, we finally solve the resulting Schur complement system by a GMRES method without preconditioning. In the case of rather simple domain decompositions of the space-time domain for a one-dimensional spatial domain we discuss the influence of the choice of the inter-

face in the space-time setting. Since the single layer heat potential boundary integral operator can be defined for any manifold in the space-time domain, it can be used for operator preconditioning of the global Schur complement system, in combination with some coarse grid preconditioning as in BDDC [7], or in space-time FETI methods [9]. On the other hand, the global Schur complement matrix is spectrally equivalent to the global Galerkin matrix of the hypersingular heat potential boundary integral operator, which then allows the use of multigrid methods for an iterative solution, see [2] for a related discussion in the case of boundary element domain decomposition methods for elliptic problems. In the case of a space-time domain decomposition into time slabs the proposed approach is obviously related to the parareal algorithm [8].

This contribution only gives first numerical results for space-time finite element domain decomposition methods for parabolic problems, and there are many open problems to be resolved in future work. In addition to different preconditioning strategies as already discussed this covers the parallel implementation in the case of two- or three-dimensional spatial domains, and the application to more complex parabolic equations including problems from fluid mechanics. Some of these topics are already ongoing work, and related results will be published elsewhere.

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