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Technische Universität Graz
Institut für Angewandte Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.applied.math.tugraz.at`

Space-time finite element methods for distributed optimal control of the wave equation

Richard Löscher, Olaf Steinbach

Institut für Angewandte Mathematik, TU Graz,
Steyrergasse 30, 8010 Graz, Austria

Abstract

We consider space-time tracking type distributed optimal control problems for the wave equation in the space-time domain $Q := \Omega \times (0,T) \subset \mathbb{R}^{n+1}$, where the control is assumed to be in the energy space $[H^{1,1}_{0;,0}(Q)]^*$, rather than in $L^2(Q)$ which is more common. While the latter ensures a unique state in the Sobolev space $H^{1,1}_{0;0,}(Q)$, this does not define a solution isomorphism. Hence we use an appropriate state space X such that the wave operator becomes an isomorphism from X onto $[H^{1,1}_{0;,0}(Q)]^*$. Using space-time finite element spaces of piecewise linear continuous basis functions on completely unstructured but shape regular simplicial meshes, we derive a priori estimates for the error $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(Q)}$ between the computed space-time finite element solution $\widetilde{u}_{\varrho h}$ and the target function $\overline{u}$ with respect to the regularization parameter ϱ, and the space-time finite element mesh-size h, depending on the regularity of the desired state $\overline{u}$. These estimates lead to the optimal choice $\varrho = h^2$ in order to define the regularization parameter ϱ for a given space-time finite element mesh size h, or to determine the required mesh size h when ϱ is a given constant representing the costs of the control. The theoretical results will be supported by numerical examples with targets of different regularities, including discontinuous targets. Furthermore, an adaptive space-time finite element scheme is proposed and numerically analyzed.

Keywords: Distributed optimal control problem, wave equation, space-time finite element methods, a priori error estimates, adaptivity.
2010 MSC: 49M41, 35L05, 65M15, 65M60

1 Introduction

We consider a distributed optimal control problem to minimize a tracking type functional to reach a given target $\overline{u} \in L^2(Q)$ subject to the initial boundary value problem for the wave equation with zero initial and boundary conditions in the space-time domain Q. The standard setting of such kind of optimal control problems assumes the control to be in

$L^2(Q)$, see, e.g., [24, 31, 35]. In this case, the wave equation admits a unique solution in the Sobolev space $H^{1,1}_{0;0,}(Q)$, see [17, 33]. For our analysis though, we will use a regularization in the (energy) space $[H^{1,1}_{0;,0}(Q)]^*$ which is the dual of the test space for the variational formulation of the wave equation. To ensure unique solvability of the wave equation also in this case, we use a generalized variational formulation of the wave equation as recently discussed in [34]. Similar investigations using the energy norm for the control were already done for distributed optimal control problems subject to elliptic [23, 29] and parabolic partial differential equations [20, 22], and, as it turns out, our analysis fits into the same framework.

In this paper, our main interest will be in proving estimates for the error $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(Q)}$ for the computed space-time finite element solution $\widetilde{u}_{\varrho h}$, depending on the regularity of the target function $\overline{u}$ and on the regularization parameter ϱ. In particular, in the discrete setting, we will allow ϱ to depend on the mesh size h and we derive an optimal choice $\varrho = h^2$ in the sense, that we can achieve optimal orders of convergence with respect to the regularity of $\overline{u}$. This is of particular interest when the regularization parameter ϱ is required to ensure solvability of the unconstrained optimal control problem, i.e., the costs are not of practical interest, see, e.g., [32]. In this case, the minimization problem is closely related to the Tikhonov regularization in inverse problems, where the parameter dependent convergence as $\varrho \to 0$ is well-studied, see, e.g., [2, 8, 15]. On the other hand, when ϱ is a given constant representing the costs of the control, one can determine the required space-time finite element mesh size h in order to reach the minimum of the functional to be minimized. The optimal relation between the regularization parameter ϱ and the finite element mesh size h is also important for the design of preconditioned iterative solution methods for the discrete optimality system, see, e.g., [22, 23] for the elliptic and the parabolic case, respectively. To ease the presentation, at this time, we will not consider any control or state constraints, see, e.g., [13, 16]. However, state or control constraints can be considered within the abstract framework as given in [11].

When choosing an appropriate state space X as introduced in [34], the state equation, i.e., the Dirichlet problem for the wave equation, admits a unique solution $u_\varrho \in X$, for each right hand side $z_\varrho \in Z = Y^* = [H^{1,1}_{0;,0}(Q)]^*$, i.e., the wave operator $B : X \to Y^*$ is an isomorphism. In view of the Nečas–Babuška theorem, e.g., [3, 30], B is, in particular, inf-sup stable. Furthermore, when introducing a self-adjoint, elliptic and bounded operator $A : Y \to Y^*$, which gives raise to an equivalent norm in Y, and $p_\varrho \in Y$ as the solution of the adjoint wave equation $B^* p_\varrho = u_\varrho - \overline{u}$, we can eliminate the control z_ϱ by the gradient equation $p_\varrho + \varrho A^{-1} z_\varrho = 0$. Then, the unique solution of the optimal control problem can be computed by solving the reduced first order optimality system

$$\begin{pmatrix} \varrho^{-1} A & B \\ -B^* & I \end{pmatrix} \begin{pmatrix} p_\varrho \\ u_\varrho \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{u} \end{pmatrix},$$

for any given target function $\overline{u} \in L^2(Q)$, which can be interpreted as a stabilized saddle point formulation. This specific form arises also in boundary optimal control problems for the wave equation, see, e.g., [28], and, undoubtely, in many applications.

For the numerical treatment of the above considered optimal control problem, there are a myriad of methods available, e.g., [12, 14, 16, 31, 36], just to mention a few. Mostly space and time are treated separately, using, e.g., finite difference methods, mixed and discontinuous Galerkin finite element methods, finite volume methods, and time stepping schemes or variational in time methods. Here, we will consider a real space-time finite element method on completely unstructured, but shape regular, simplicial space-time finite element meshes decomposing the space-time domain Q, see also [7, 9] where such methods are given for the direct solution of the wave equation. Introducing conforming finite element spaces $X_h \subset X$ and $Y_h \subset Y$ with appropriate approximation properties, the discrete reduced optimality system admits again the form of a stabilized saddle point formulation. Though, at this point it is worth stressing, that the assumptions on the discrete operator $B_h : X_h \to Y_h^*$ are vastly weakened, i.e., we do not need a discrete inf-sup stability condition and not even a CFL-condition to be fulfilled. Therefore, this method directly allows for an adaptive finite element scheme, see, e.g., [27] in the case of a parabolic optimal control problem, and [4] for adaptive schemes for the wave equation, which we will also address in our numerical investigations.

The remainder of this paper is structured as follows: In Section 2 we will state the model problem and introduce the appropriate functional analytical setting required for the solution of the wave equation. In Section 3 we present the main result for the regularization error estimates which depend on the regularity of the target $\overline{u}$, and on the regularization parameter ϱ. The space-time finite element discretization and related a priori error estimates are investigated in Section 4, where we will conclude the optimal choice $\varrho = h^2$ for the regularization parameter. Numerical tests will confirm our theory in Section 5. Furthermore, we will compare the proposed energy regularization approach with the more standard L^2 regularization in the same setting, as well as propose an adaptive refinement strategy. In Section 6, we draw some conclusions and give an outlook on ongoing work.

2 Distributed optimal control problems

Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, be a bounded convex domain with, for $n = 2, 3$, Lipschitz boundary $\Gamma = \partial\Omega$, and let $T > 0$ be a given finite time horizon. Then we introduce the space-time domain $Q := \Omega \times (0, T)$ and the lateral boundary $\Sigma := \Gamma \times (0, T)$. For a given target $\overline{u} \in L^2(Q)$ and a regularization parameter $\varrho > 0$, we consider the minimization of the cost functional

$$\mathcal{J}(u_\varrho, z_\varrho) := \frac{1}{2} \int_0^T \int_\Omega [u_\varrho(x,t) - \overline{u}(x,t)]^2 \, dx \, dt + \frac{1}{2} \, \varrho \, \|z_\varrho\|_Z^2 \qquad (2.1)$$

subject to the initial boundary value problem for the wave equation with homogeneous Dirichlet boundary conditions,

$$\begin{aligned} \Box u_\varrho(x,t) := \partial_{tt} u_\varrho(x,t) - \Delta_x u_\varrho(x,t) &= z_\varrho(x,t) && \text{for } (x,t) \in Q, \\ u_\varrho(x,t) &= 0 && \text{for } (x,t) \in \Sigma, \\ u_\varrho(x,0) = \partial_t u_\varrho(x,t)_{|t=0} &= 0 && \text{for } x \in \Omega. \end{aligned} \qquad (2.2)$$

Our particular interest is in the numerical solution of the constrained minimization problem (2.1) and (2.2) by using a space-time finite element approach on simplicial meshes which are completely unstructured in space and time. For the error $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(Q)}$ of the computed numerical solution $\widetilde{u}_{\varrho h}$ we will provide estimates in the space-time finite element mesh size h, and in the regularization parameter ϱ from which we will derive an optimal choice for ϱ, which will depend on the choice of the regularization space Z.

First we consider $z_\varrho \in Z = L^2(Q)$. Following [33], the space-time variational formulation of the state equation (2.2) is to find $u_\varrho \in H^{1,1}_{0;0,}(Q)$ such that

$$b(u_\varrho, q) := -\langle \partial_t u_\varrho, \partial_t q\rangle_{L^2(Q)} + \langle \nabla_x u_\varrho, \nabla_x q\rangle_{L^2(Q)} = \langle z_\varrho, q\rangle_{L^2(Q)} \tag{2.3}$$

is satisfied for all $q \in H^{1,1}_{0;,0}(Q)$. Here we use the anisotropic Sobolev space

$$H^{1,1}_{0;0,}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;L^2(\Omega)),$$

where $H^1_{0,}(0,T;L^2(\Omega))$ covers the zero initial condition $u(x,0) = 0$ for $x \in \Omega$, while $L^2(0,T;H^1_0(\Omega))$ includes the homogeneous Dirichlet boundary condition on Σ. Note that the second initial condition $\partial_t u(x,t)_{|t=0} = 0$ for $x \in \Omega$ enters the variational formulation (2.3) in a natural way. A norm in $H^{1,1}_{0;0,}(Q)$ is given by the graph norm

$$\|u\|_{H^{1,1}_{0;0,}(Q)} := \sqrt{\|\partial_t u\|^2_{L^2(Q)} + \|\nabla_x u\|^2_{L^2(Q)}} \,=\, |u|_{H^1(Q)}.$$

Note that $H^{1,1}_{0;,0}(Q)$ is defined accordingly, but with a zero terminal condition $q(x,T) = 0$ for $x \in \Omega$. Then we have

$$|b(u,q)| \,\le\, |u|_{H^1(Q)}|q|_{H^1(Q)} \quad \text{for all } u \in H^{1,1}_{0;0,}(Q), q \in H^{1,1}_{0;,0}(Q). \tag{2.4}$$

For $z_\varrho \in L^2(Q)$ there exists a unique solution $u_\varrho \in H^{1,1}_{0;0,}(Q)$ of the variational formulation (2.3) satisfying, see, e.g., [33, Theorem 5.1], and [17],

$$\|u_\varrho\|_{H^{1,1}_{0;0,}(Q)} \le \frac{1}{\sqrt{2}}\, T\, \|z_\varrho\|_{L^2(Q)}\,.$$

Hence we can write $u_\varrho = \mathcal{S} z_\varrho$ with the solution operator $\mathcal{S} : L^2(Q) \to H^{1,1}_{0;0,}(Q) \subset L^2(Q)$, and we can introduce the reduced cost functional

$$\widetilde{\mathcal{J}}(z_\varrho) := \frac{1}{2}\,\|\mathcal{S} z_\varrho - \overline{u}\|^2_{L^2(Q)} + \frac{1}{2}\,\varrho\,\|z\|^2_{L^2(Q)},$$

whose minimizer is given by the gradient equation

$$p_\varrho(x,t) + \varrho\, z_\varrho(x,t) = 0 \quad \text{for } (x,t) \in Q, \tag{2.5}$$

and where $p_\varrho \in H^{1,1}_{0;,0}(Q)$ is the weak solution of the adjoint problem

$$\begin{aligned} \partial_{tt} p_\varrho(x,t) - \Delta_x p_\varrho(x,t) &= u_\varrho(x,t) - \overline{u}(x,t) && \text{for } (x,t) \in Q, \\ p_\varrho(x,t) &= 0 && \text{for } (x,t) \in \Sigma, \\ p_\varrho(x,T) = \partial_t p_\varrho(x,t)_{|t=T} &= 0 && \text{for } x \in \Omega. \end{aligned} \tag{2.6}$$

Similar as in [20] for the heat equation we can apply a space-time finite element method on completely unstructured simplicial meshes to discretize the optimality system (2.2) and (2.6) after eliminating the control z_ϱ from the gradient equation (2.5). Although we will consider this approach for a numerical comparison, at this time we are not able to provide a complete numerical analysis for this approach. As already seen in the elliptic case [23, 29], and in the parabolic case [20, 21, 22], there are differences both in the numerical analysis and in the properties of the numerical solutions when considering the regularization in $L^2(Q)$, and in the related energy space, which is the dual of the test space.

A direct space-time finite element discretization of the variational formulation (2.3) on space-time tensor product meshes using piecewise linear continuous basis functions requires an appropriate stability condition $h_t \leq h_x/\sqrt{n}$ where h_t and h_x are the temporal and spatial mesh sizes, respectively, see [33]. Moreover, the associated operator B to the variational formulation (2.3) does not define an isomorphism between $L^2(Q)$ and $H^{1,1}_{0;0,}(Q)$, see Theorem 2.1. Although the variational formulation (2.3) is well defined also for $z_\varrho \in [H^{1,1}_{0;,0}(Q)]^*$, it does not ensure unique solvability in $H^{1,1}_{0;0,}(Q)$ in this case. Instead we have to enlarge the ansatz space in order to incorporate the second initial condition $\partial_t u_\varrho(x,t)_{|t=0} = 0$ in an appropriate way. In what follows we will consider a generalized variational formulation of the wave equation, see [34]. When using a distributional definition of the wave operator, we consider an ultra-weak variational formulation of (2.2) to find $u \in L^2(Q)$ which is extended by zero to an enlarged domain to cover the initial conditions. This approach will allow us to define the regularization in a suitable energy norm. In this case we choose $Z = [H^{1,1}_{0;,0}(Q)]^*$ as the dual of the test space. A norm in this space is given as

$$\|z\|_{[H^{1,1}_{0;,0}(Q)]^*} := \sup_{0\neq q \in H^{1,1}_{0;,0}(Q)} \frac{\langle z, q\rangle_Q}{\|q\|_{H^{1,1}_{0;,0}(Q)}},$$

where $\langle \cdot,\cdot\rangle_Q$ is an extension of the inner product in $L^2(Q)$. For $z \in [H^{1,1}_{0;,0}(Q)]^*$, and using the Riesz isomorphism, there exists a unique $w_z \in H^{1,1}_{0;,0}(Q)$ such that

$$\langle Aw_z, q\rangle_Q := \langle \partial_t w_z, \partial_t q\rangle_{L^2(Q)} + \langle \nabla_x w_z, \nabla_x q\rangle_{L^2(Q)} = \langle z, q\rangle_Q \quad \text{for all } q \in H^{1,1}_{0;,0}(Q).$$

With this choice, A is self-adjoint, elliptic and bounded, i.e.,

$$\langle Aw, q\rangle_Q \leq |w|_{H^1(Q)} |q|_{H^1(Q)}, \quad \langle Aq, q\rangle_Q = |q|^2_{H^1(Q)} \quad \text{for all } w, q \in H^{1,1}_{0;,0}(Q),$$

and, hence, invertible. Thus, we can write

$$\|w_z\|^2_{H^{1,1}_{0;,0}(Q)} = \langle Aw_z, w_z\rangle_Q = \|z\|^2_{[H^{1,1}_{0;,0}(Q)]^*}$$

as well as

$$\|z\|^2_{[H^{1,1}_{0;,0}(Q)]^*} = \langle z, w_z\rangle_Q = \langle z, A^{-1}z\rangle_Q \quad \text{for all } z \in [H^{1,1}_{0;,0}(Q)]^*.$$

We proceed with stating some preliminaries. First, let us give a result concerning the boundedness of the solution $u_\varrho \in H^{1,1}_{0;0,}(Q)$ of (2.3) when considering the norm of z_ϱ in $[H^{1,1}_{0;,0}(Q)]^*$.

Theorem 2.1 [34, Theorem 1.1] *There does not exist a constant $c > 0$ such that each right-hand side $z_\varrho \in L^2(Q)$ and the corresponding solution $u_\varrho \in H^{1,1}_{0;0,}(Q)$ of (2.3) satisfy*

$$\|u_\varrho\|_{H^{1,1}_{0;0,}(Q)} \leq c\, \|z_\varrho\|_{[H^{1,1}_{0;,0}(Q)]^*}.$$

In particular, the inf-sup condition

$$c_S\, \|u\|_{H^{1,1}_{0;0,}(Q)} \leq \sup_{0 \neq q \in H^{1,1}_{0;,0}(Q)} \frac{b(u,q)}{\|q\|_{H^{1,1}_{0;,0}(Q)}} \quad \textit{for all } u \in H^{1,1}_{0;0,}(Q)$$

with a constant $c_S > 0$ does not hold true.

The issue to overcome is the handling of the initial condition $\partial_t u_\varrho(x,t)_{|t=0} = 0$ for $x \in \Omega$ for which we will proceed as in [34]. For the enlarged space-time domain $Q_- := \Omega \times (-T, T)$, and for $u \in L^2(Q)$ we define the zero extension

$$\widetilde{u}(x,t) := \begin{cases} u(x,t) & \text{for } (x,t) \in Q, \\ 0, & \text{else.} \end{cases}$$

The application of the wave operator $\Box \widetilde{u}$ on Q_- will be formulated as a distribution, i.e., for $\varphi \in C_0^\infty(Q_-)$ we define

$$\langle \Box \widetilde{u}, \varphi \rangle_{Q_-} := \int_{Q_-} \widetilde{u}(x,t)\, \Box\varphi(x,t)\, dx\, dt = \int_Q u(x,t)\, \Box\varphi(x,t)\, dx\, dt.$$

Now we are in the position to introduce the space

$$\mathcal{H}(Q) := \Big\{ u = \widetilde{u}_{|Q} : \widetilde{u} \in L^2(Q_-),\, \widetilde{u}_{|\Omega \times (-T,0)} = 0,\, \Box\widetilde{u} \in [H^1_0(Q_-)]^* \Big\},$$

with the graph norm

$$\|u\|_{\mathcal{H}(Q)} := \sqrt{\|u\|^2_{L^2(Q)} + \|\Box\widetilde{u}\|^2_{[H^1_0(Q_-)]^*}}\,.$$

The normed vector space $(\mathcal{H}(Q), \|\cdot\|_{\mathcal{H}(Q)})$ is a Banach space, and it holds true that, see [34, Lemma 3.5], $H^{1,1}_{0;0,}(Q) \subset \mathcal{H}(Q)$ i.e.,

$$\|\Box\widetilde{u}\|_{[H^1_0(Q_-)]^*} \leq \|u\|_{H^{1,1}_{0;0,}(Q)} \quad \text{for all } u \in H^{1,1}_{0;0,}(Q). \tag{2.7}$$

Therefore, we can consider the space

$$\mathcal{H}_{0;0,}(Q) := \overline{H^{1,1}_{0;0,}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}(Q)$$

which will serve as ansatz space. For $u \in \mathcal{H}_{0;0,}(Q)$, an equivalent norm is given as, see [34, Lemma 3.6],

$$\|u\|_{\mathcal{H}_{0;0,}(Q)} = \|\Box\widetilde{u}\|_{[H^1_0(Q_-)]^*}.$$

For given $z_\varrho \in [H^{1,1}_{0;,0}(Q)]^*$ we consider the variational formulation to find $u_\varrho \in \mathcal{H}_{0;0,}(Q)$ such that

$$\langle \Box \widetilde{u}_\varrho, \mathcal{E}q\rangle_{Q_-} = \langle z_\varrho, q\rangle_Q \quad \text{for all } q \in H^{1,1}_{0;,0}(Q), \tag{2.8}$$

where $\mathcal{E} : H^{1,1}_{0;,0}(Q) \to H^1_0(Q_-)$ is a suitable extension operator, e.g., reflection in time with respect to $t = 0$, satisfying

$$\|\mathcal{E}q\|_{H^1_0(Q_-)} \leq 2\, \|q\|_{H^{1,1}_{0;,0}(Q)} \quad \text{for all } q \in H^{1,1}_{0;,0}(Q).$$

We conclude that the bilinear form within the variational formulation (2.8) is bounded, i.e., for all $u \in \mathcal{H}_{0;0,}(Q)$ and $q \in H^{1,1}_{0;,0}(Q)$ we have

$$\left|\langle \Box \widetilde{u}_\varrho, \mathcal{E}q\rangle_{Q_-}\right| \;\leq\; \|\Box \widetilde{u}\|_{[H^1_0(Q_-)]^*} \|\mathcal{E}q\|_{H^1_0(Q_-)} \leq 2\, \|u\|_{\mathcal{H}_{0;0,}(Q)} \|q\|_{H^{1,1}_{0;,0}(Q)}. \tag{2.9}$$

Moreover, we have the following result.

Theorem 2.2 [34, Theorem 3.9] *For each given $z_\varrho \in [H^{1,1}_{0;,0}(Q)]^*$, there exists a unique solution $u_\varrho \in \mathcal{H}_{0;0,}(Q)$ of the variational formulation* (2.8) *satisfying*

$$\|u_\varrho\|_{\mathcal{H}_{0;0,}(Q)} = \|\Box \widetilde{u}_\varrho\|_{[H^1_0(Q_-)]^*} = \|z_\varrho\|_{[H^{1,1}_{0;,0}(Q)]^*}.$$

In particular, there holds the inf-sup stability condition

$$\|u\|_{\mathcal{H}_{0;0,}(Q)} \leq \sup_{0 \neq q \in H^{1,1}_{0;,0}(Q)} \frac{\langle \Box \widetilde{u}, \mathcal{E}q\rangle_{Q_-}}{\|q\|_{H^{1,1}_{0;,0}(Q)}} \quad \textit{for all } u \in \mathcal{H}_{0;0,}(Q). \tag{2.10}$$

Remark 2.1 *The use of the bilinear form $\langle \Box \widetilde{u}, \mathcal{E}q\rangle_{Q_-}$ might seem cumbersome. But we have, see* [34, Lemma 3.5]*,*

$$\langle \Box \widetilde{u}, \mathcal{E}q\rangle_{Q_-} = -\langle \partial_t u, \partial_t q\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x q\rangle_{L^2(Q)} \textit{ for all } u \in H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0;0,}(Q), q \in H^{1,1}_{0;,0}(Q).$$

This is of particular interest when considering the discrete setting, as piecewise linear continuous functions are in $H^1(Q)$.

In view of Theorem 2.2 we have a solution operator $\mathcal{S} : [H^{1,1}_{0;,0}(Q)]^* \to \mathcal{H}_{0;0,}(Q) \subset L^2(Q)$. So, we can write the reduced cost functional

$$\begin{aligned} \widetilde{\mathcal{J}}(z_\varrho) \;&:=\; \frac{1}{2}\, \|\mathcal{S} z_\varrho - \overline{u}\|^2_{L^2(Q)} + \frac{1}{2}\, \varrho\, \|z_\varrho\|^2_{[H^{1,1}_{0;,0}(Q)]^*} \\ &=\; \frac{1}{2}\, \langle \mathcal{S}^*\mathcal{S} z_\varrho, z_\varrho\rangle_Q - \langle \mathcal{S}^*\overline{u}, z_\varrho\rangle_Q + \frac{1}{2}\, \|\overline{u}\|^2_{L^2(Q)} + \frac{1}{2}\, \varrho\, \langle A^{-1} z_\varrho, z_\varrho\rangle_Q, \end{aligned}$$

where $\mathcal{S}^* : [\mathcal{H}_{0;0,}(Q)]^* \to H^{1,1}_{0;,0}(Q)$ denotes the dual of the solution operator. The minimizer of the reduced cost functional is the unique solution of the gradient equation

$$\mathcal{S}^*(\mathcal{S} z_\varrho - \overline{u}) + \varrho\, A^{-1} z_\varrho = 0, \tag{2.11}$$

i.e., we have to find $z_\varrho \in [H^{1,1}_{0;,0}(Q)]^*$ as solution of

$$\varrho\, A^{-1} z_\varrho + \mathcal{S}^* \mathcal{S} z_\varrho = \mathcal{S}^* \overline{u} \quad \text{in } H^{1,1}_{0;,0}(Q).$$

Note that $T_\varrho := \varrho\, A^{-1} + \mathcal{S}^* \mathcal{S} : [H^{1,1}_{0;,0}(Q)]^* \to H^{1,1}_{0;,0}(Q)$ is bounded and elliptic, thus unique solvability of the operator equation (2.11) follows immediately. When introducing the adjoint state $p_\varrho = \mathcal{S}^*(u_\varrho - \overline{u})$, and $w_{z_\varrho} \in H^{1,1}_{0;,0}(Q)$ as solution of $Aw_{z_\varrho} = z_\varrho$, we can write the gradient equation (2.11) as

$$p_\varrho + \varrho\, w_{z_\varrho} = 0, \tag{2.12}$$

where $p_\varrho \in H^{1,1}_{0;,0}(Q)$ is the unique solution of the adjoint generalized wave equation

$$\langle \Box \widetilde{v}, \mathcal{E} p_\varrho \rangle_{Q_-} = \langle u_\varrho - \overline{u}, v \rangle_{L^2(Q)} \quad \text{for all } v \in \mathcal{H}_{0;0,}(Q). \tag{2.13}$$

The optimality system to be solved covers the forward (generalized) wave equation (2.8), the adjoint backward (generalized) wave equation (2.13), and the gradient equation (2.12). When considering $w_{z_\varrho} = A^{-1} z_\varrho = -\varrho^{-1} p_\varrho$ we can eliminate the control by $z_\varrho = -\varrho^{-1} A p_\varrho$ to end up with the system to find $(u_\varrho, p_\varrho) \in \mathcal{H}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q)$ such that

$$\varrho^{-1} \langle A p_\varrho, q \rangle_Q + \langle \Box \widetilde{u}_\varrho, \mathcal{E} q \rangle_{Q_-} = 0, \quad -\langle \Box \widetilde{v}, \mathcal{E} p_\varrho \rangle + \langle u_\varrho, v \rangle_{L^2(Q)} = \langle \overline{u}, v \rangle_{L^2(Q)} \tag{2.14}$$

is satisfied for all $(v, q) \in \mathcal{H}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q)$.

When the state $u_\varrho \in \mathcal{H}_{0;0,}(Q)$ is known, and since we are interested in the reconstruction of the control, we can compute $z_\varrho \in [H^{1,1}_{0;,0}(Q)]^*$ as unique solution of the variational formulation

$$\langle z_\varrho, q \rangle_Q = \langle \Box \widetilde{u}_\varrho, \mathcal{E} q \rangle_{Q_-} \quad \text{for all } q \in H^{1,1}_{0;,0}(Q). \tag{2.15}$$

3 Regularization error estimates

We introduce $X := \mathcal{H}_{0;0,}(Q)$ and $Y := H^{1,1}_{0;,0}(Q)$ with norms

$$\|u\|_X = \|u\|_{\mathcal{H}_{0;0,}(Q)}, \quad \|q\|_Y = \|q\|_{H^{1,1}_{0;,0}(Q)} = |q|_{H^1(Q)},$$

and we can write the optimality system (2.14) as operator equation to find $(u_\varrho, p_\varrho) \in X \times Y$ such that

$$\begin{pmatrix} \varrho^{-1} A & B \\ -B^* & I \end{pmatrix} \begin{pmatrix} p_\varrho \\ u_\varrho \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{u} \end{pmatrix}, \tag{3.1}$$

where $B : X \to Y^*$ is defined via

$$\langle Bv, q \rangle_Q = \langle \Box \widetilde{v}, \mathcal{E} q \rangle_{Q_-} \quad \text{for all } (v, q) \in X \times Y.$$

Note that, using (2.9), we have

$$\|Bu\|_{Y^*} = \sup_{0 \neq q \in Y} \frac{\langle Bu, q \rangle_Q}{\|q\|_Y} = \sup_{0 \neq q \in H^{1,1}_{0;,0}(Q)} \frac{\langle \Box \widetilde{u}, \mathcal{E} q \rangle_{Q_-}}{\|q\|_{H^{1,1}_{0;,0}(Q)}} \leq 2\, \|u\|_{\mathcal{H}_{0;0,}(Q)} = 2\, \|u\|_X$$

for all $u \in X$, i.e., $B : X \to Y^*$ is bounded. Since A is invertible, we can eliminate $p_\varrho = -\varrho A^{-1} B u_\varrho$ to end up with the Schur complement equation to find $u_\varrho \in X$ such that

$$\big[I + \varrho B^* A^{-1} B\big] u_\varrho = \overline{u} \quad \text{in } X^*. \tag{3.2}$$

Lemma 3.1 *The operator $S := B^* A^{-1} B : X \to X^*$ is bounded and elliptic, i.e.,*

$$\|Su\|_{X^*} \leq 4\,\|u\|_X, \quad \langle Su, u\rangle_Q \geq \|u\|_X^2 \quad \textit{for all } u \in X.$$

Moreover, $\|u\|_S := \langle Su, u\rangle_Q^{1/2}$, $u \in X$, defines an equivalent norm on X,

$$\|u\|_X \leq \|u\|_S \leq 2\,\|u\|_X \quad \textit{for all } u \in X. \tag{3.3}$$

Proof. The boundedness results from the boundedness of $B : X \to Y^*$, and from the invertibility of $A : Y \to Y^*$, i.e., for $u \in X$ we have

$$\|Su\|_{X^*} = \sup_{0 \neq v \in X} \frac{\langle Su, v\rangle_Q}{\|v\|_X} = \sup_{0 \neq v \in X} \frac{\langle A^{-1}Bu, Bv\rangle_Q}{\|v\|_X} \leq 4\,\|u\|_X.$$

Further, the inf-sup stability condition (2.10) implies

$$\|u\|_X = \|u\|_{\mathcal{H}_{0;0,}(Q)} \leq \sup_{0 \neq q \in H^{1,1}_{0;0,}(Q)} \frac{\langle \Box \widetilde{u}, \mathcal{E} q\rangle_{Q_-}}{\|q\|_{H^{1,1}_{0;,0}(Q)}} = \sup_{0 \neq q \in Y} \frac{\langle Bu, q\rangle_Q}{\|q\|_Y} = \|Bu\|_{Y^*}$$

for all $u \in X$. When introducing, for $u \in X$, the auxiliary variable $p_u = A^{-1}Bu \in Y$, we first have

$$\langle Su, u\rangle_Q = \langle A^{-1}Bu, Bu\rangle_Q = \langle p_u, Bu\rangle_Q = \langle p_u, Ap_u\rangle_Q = \|p_u\|_Y^2\,.$$

Moreover, we have that

$$\|u\|_X \leq \|Bu\|_{Y^*} = \|Ap_u\|_{Y^*} \leq \|p_u\|_Y,$$

and we conclude

$$\|u\|_X^2 \leq \|p_u\|_Y^2 = \langle Su, u\rangle_Q.$$

This also shows that

$$\|u\|_X^2 \leq \|u\|_S^2 = \langle Su, u\rangle_Q \leq \|Su\|_{X^*}\|u\|_X \leq 4\,\|u\|_X^2 \quad \text{for all } u \in X,$$

which gives the desired equivalence of norms. □

The variational formulation of the Schur complement equation (3.2) is to find $u_\varrho \in X$ such that

$$\varrho\,\langle Su_\varrho, v\rangle_Q + \langle u_\varrho, v\rangle_{L^2(Q)} = \langle \overline{u}, v\rangle_{L^2(Q)} \quad \text{for all } v \in X. \tag{3.4}$$

Unique solvability of (3.4) immediately follows from the properties of S for all $\varrho \in \mathbb{R}_+$. In particular for $v = u_\varrho$ this gives

$$\varrho\,\|u_\varrho\|_S^2 + \|u_\varrho\|_{L^2(Q)}^2 = \langle \overline{u}, u_\varrho\rangle_{L^2(Q)} \leq \|\overline{u}\|_{L^2(Q)}\|u_\varrho\|_{L^2(Q)},$$

and hence,

$$\|u_\varrho\|_{L^2(Q)} \leq \|\overline{u}\|_{L^2(Q)}, \quad \sqrt{\varrho}\,\|u_\varrho\|_S \leq \|\overline{u}\|_{L^2(Q)} \tag{3.5}$$

follow. As in [22, Lemma 2.3] we can prove the following regularization error estimates.

Theorem 3.2 *For $\overline{u} \in L^2(Q)$ let $u_\varrho \in X$ be the unique solution of the variational formulation* (3.4) *where $\varrho \in \mathbb{R}_+$. Then the following estimate holds true*

$$\|u_\varrho - \overline{u}\|_{L^2(Q)} \leq \|\overline{u}\|_{L^2(Q)}. \tag{3.6}$$

Moreover, for $\overline{u} \in X$ we have

$$\|u_\varrho - \overline{u}\|_{L^2(Q)} \leq \sqrt{\varrho}\, \|\overline{u}\|_S\,, \tag{3.7}$$

as well as

$$\|u_\varrho - \overline{u}\|_S \leq \|\overline{u}\|_S\,. \tag{3.8}$$

If in addition $\overline{u} \in X$ is such that $S\overline{u} \in L^2(Q)$ is satisfied, then

$$\|u_\varrho - \overline{u}\|_{L^2(Q)} \leq \varrho\, \|S\overline{u}\|_{L^2(Q)}, \tag{3.9}$$

and

$$\|u_\varrho - \overline{u}\|_S \leq \sqrt{\varrho}\, \|S\overline{u}\|_{L^2(Q)}. \tag{3.10}$$

Proof. Let us first consider the case $\overline{u} \in L^2(Q)$. Then, when choosing $v = u_\varrho \in X$ within the variational formulation (3.4), this gives

$$\varrho\, \langle S u_\varrho, u_\varrho \rangle_Q = \langle \overline{u} - u_\varrho, u_\varrho \rangle_{L^2(Q)} = -\langle \overline{u} - u_\varrho, \overline{u} - u_\varrho \rangle_{L^2(Q)} + \langle \overline{u} - u_\varrho, \overline{u} \rangle_{L^2(Q)},$$

i.e.,

$$\|u_\varrho - \overline{u}\|^2_{L^2(Q)} + \varrho\, \|u_\varrho\|^2_S = \langle \overline{u} - u_\varrho, u_\varrho \rangle_{L^2(Q)} \leq \|u_\varrho - \overline{u}\|_{L^2(Q)} \|\overline{u}\|_{L^2(Q)},$$

and (3.6) follows.

For $\overline{u} \in X$ we can consider the variational formulation (3.4) for $v = \overline{u} - u_\varrho \in X$ to obtain

$$\begin{aligned} \|\overline{u} - u_\varrho\|^2_{L^2(Q)} &= \langle \overline{u} - u_\varrho, \overline{u} - u_\varrho \rangle_{L^2(Q)} = \varrho\, \langle S u_\varrho, \overline{u} - u_\varrho \rangle_Q \\ &= \varrho\, \langle S\overline{u}, \overline{u} - u_\varrho \rangle_Q - \varrho\, \langle S(\overline{u} - u_\varrho), \overline{u} - u_\varrho \rangle_Q, \end{aligned}$$

i.e.,

$$\|u_\varrho - \overline{u}\|^2_{L^2(Q)} + \varrho\, \|u_\varrho - \overline{u}\|^2_S \leq \varrho\, \langle S\overline{u}, \overline{u} - u_\varrho \rangle_Q \leq \varrho\, \|\overline{u}\|_S \|u_\varrho - \overline{u}\|_S,$$

and hence, (3.8) and (3.7) follow.

If $\overline{u} \in X$ is such that $S\overline{u} \in L^2(Q)$ is satisfied, we also have

$$\|u_\varrho - \overline{u}\|^2_{L^2(Q)} + \varrho\, \|u_\varrho - \overline{u}\|^2_S \leq \varrho\, \langle S\overline{u}, \overline{u} - u_\varrho \rangle_Q \leq \varrho\, \|S\overline{u}\|_{L^2(Q)} \|u_\varrho - \overline{u}\|_{L^2(Q)},$$

from which (3.9) and (3.10) follow. $\square$

Corollary 3.3 *For* $\overline{u} \in H^{1,1}_{0;0,}(Q) \subset X = \mathcal{H}_{0;0,}(Q)$ *we conclude from* (3.7), (3.3), *and* (2.7),

$$\|u_\varrho - \overline{u}\|_{L^2(Q)} \leq \sqrt{\varrho}\, \|\overline{u}\|_S \, \leq 2\sqrt{\varrho}\, \|\overline{u}\|_X \leq 2\sqrt{\varrho}\, \|\overline{u}\|_{H^{1,1}_{0;0,}(Q)},$$

and using a space interpolation argument, see, e.g., [1, 25, 26], this gives

$$\|u_\varrho - \overline{u}\|_{L^2(Q)} \leq c\, \varrho^{s/2}\, \|\overline{u}\|_{H^{s,s}_{0;0,}(Q)} \tag{3.11}$$

when assuming $\overline{u} \in H^{s,s}_{0;0,}(Q) := [L^2(Q), H^{1,1}_{0;0,}(Q)]_s$ *for some* $s \in [0,1]$, *and where the positive constant* c *is independent of* ϱ.

Next we consider $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^2(Q)$ *and assume that* $\overline{u}$ *is such that* $A^{-1}B\overline{u} \in H^2(Q)$. *Note that* A *is related to the space-time Laplacian, but with mixed Dirichlet and Neumann boundary conditions which may reduce the regularity of its solution. The application of the adjoint wave operator then finally gives* $S\overline{u} = B^*A^{-1}B\overline{u} \in L^2(Q)$. *Then the error estimate* (3.9) *implies*

$$\|u_\varrho - \overline{u}\|_{L^2(Q)} \leq c\, \varrho\, \|\overline{u}\|_{H^2(Q)},$$

and using an interpolation argument finally gives

$$\|u_\varrho - \overline{u}\|_{L^2(Q)} \leq c\, \varrho^{s/2}\, \|\overline{u}\|_{H^s(Q)} \tag{3.12}$$

when assuming $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^s(Q)$ *for some* $s \in (1,2]$.

4 Space-time finite element methods

For the Galerkin discretization of the Schur complement variational formulation (3.4) we introduce the conforming finite element space $X_h := S_h^1(Q) \cap \mathcal{H}_{0;0,}(Q) = \text{span}\{\varphi_k\}_{k=1}^{M_X} \subset X$ of piecewise linear and continuous basis functions φ_k which are defined with respect to some admissible globally quasi-uniform decomposition $\mathcal{T}_h = \{\tau_\ell\}_{\ell=1}^N$ of the space-time domain Q into shape-regular simplicial finite elements τ_ℓ of mesh size h_ℓ, see, e.g., [5]. Then the finite element approximation of (3.4) is to find $u_{\varrho h} \in X_h$ such that

$$\varrho\, \langle S u_{\varrho h}, v_h \rangle_Q + \langle u_{\varrho h}, v_h \rangle_{L^2(Q)} = \langle \overline{u}, v_h \rangle_{L^2(Q)} \tag{4.1}$$

ist satisfied for all $v_h \in X_h$. Using standard arguments, we conclude unique solvability of (4.1), and the following Cea type a priori error estimate

$$\varrho\, \|u_\varrho - u_{\varrho h}\|_S^2 + \|u_\varrho - u_{\varrho h}\|^2_{L^2(Q)} \leq \inf_{v_h \in X_h} \Big[\varrho\, \|u_\varrho - v_h\|_S^2 + \|u_\varrho - v_h\|^2_{L^2(Q)} \Big]. \tag{4.2}$$

Theorem 4.1 *Assume* $\overline{u} \in [L^2(Q), H^{1,1}_{0;0,}(Q)]_s$ *for* $s \in [0,1]$ *or* $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^s(Q)$ *for* $s \in (1,2]$. *For the unique solution* $u_{\varrho h} \in X_h$ *of* (4.1) *there holds the finite element error estimate*

$$\|u_{\varrho h} - \overline{u}\|_{L^2(Q)} \leq c\, h^s\, \|\overline{u}\|_{H^s(Q)}, \tag{4.3}$$

provided that $\varrho = h^2$. *For* $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^s(Q)$ *and* $s \in [1,2]$ *we also have the error estimate*

$$\|u_{\varrho h} - \overline{u}\|_S \leq c\, h^{s-1}\, \|\overline{u}\|_{H^s(Q)}. \tag{4.4}$$

Proof. We first consider the error estimate (4.2) for the particular function $v_h \equiv 0$, and using (3.5) this gives

$$\|u_\varrho - u_{\varrho h}\|^2_{L^2(Q)} \leq \varrho \, \|u_\varrho\|^2_S + \|u_\varrho\|^2_{L^2(Q)} \leq 2 \, \|\overline{u}\|^2_{L^2(Q)} \, .$$

Hence we conclude

$$\|u_{\varrho h} - \overline{u}\|_{L^2(Q)} \leq \|u_\varrho - \overline{u}\|_{L^2(Q)} + \|u_\varrho - u_{\varrho h}\|_{L^2(Q)} \leq (1+\sqrt{2}) \, \|\overline{u}\|_{L^2(Q)} \, . \qquad (4.5)$$

We now assume $\overline{u} \in H^{1,1}_{0;0,}(Q) \subset X$, and from (4.2) we obtain, using the triangle inequality, (3.8) and (3.7), the inclusion $H^{1,1}_{0;0,}(Q) \subset X$, and standard approximation properties of piecewise linear finite element functions, e.g., Scott–Zhang interpolation [5],

$$\begin{aligned}
\varrho \, \|u_\varrho - u_{\varrho h}\|^2_S + \|u_\varrho - u_{\varrho h}\|^2_{L^2(Q)} \;\; &\leq \;\; \inf_{v_h \in X_h} \Big[\varrho \, \|u_\varrho - v_h\|^2_S + \|u_\varrho - v_h\|^2_{L^2(Q)} \Big] \\
&\leq 2 \left[\varrho \, \|u_\varrho - \overline{u}\|^2_S + \|u_\varrho - \overline{u}\|^2_{L^2(Q)} + \inf_{v_h \in X_h} \Big[\varrho \, \|\overline{u} - v_h\|^2_S + \|\overline{u} - v_h\|^2_{L^2(Q)} \Big] \right] \\
&\leq 2 \left[2 \, \varrho \, \|\overline{u}\|^2_S + \inf_{v_h \in X_h} \Big[c \, \varrho \, \|\overline{u} - v_h\|^2_{H^{1,1}_{0;0,}(Q)} + \|\overline{u} - v_h\|^2_{L^2(Q)} \Big] \right] \\
&\leq c \Big[\varrho + h^2 \Big] \, \|\overline{u}\|^2_{H^1(Q)} \, .
\end{aligned}$$

In particular for $\varrho = h^2$ this gives

$$h^2 \, \|u_\varrho - u_{\varrho h}\|^2_S + \|u_\varrho - u_{\varrho h}\|^2_{L^2(Q)} \leq c \, h^2 \, \|\overline{u}\|^2_{H^1(Q)} \, .$$

Hence, using the triangle inequality and (3.11),

$$\|u_{\varrho h} - \overline{u}\|_{L^2(Q)} \leq \|u_{\varrho h} - u_\varrho\|_{L^2(Q)} + \|u_\varrho - \overline{u}\|_{L^2(Q)} \leq c \, h \, \|\overline{u}\|_{H^1(Q)} \qquad (4.6)$$

follows, while with (3.8) we obtain

$$\|u_{\varrho h} - \overline{u}\|_S \leq \|u_{\varrho h} - u_\varrho\|_S + \|u_\varrho - \overline{u}\|_S \leq c \, \|\overline{u}\|_{H^1(Q)}. \qquad (4.7)$$

For $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^2(Q) \subset X$, using (3.10) and (3.9), we can prove in the same way

$$\varrho \, \|u_\varrho - u_{\varrho h}\|^2_S + \|u_\varrho - u_{\varrho h}\|^2_{L^2(Q)} \leq c \Big[\varrho^2 + \varrho \, h^2 + h^4 \Big] \, \|\overline{u}\|^2_{H^2(Q)} = c \, h^4 \, \|\overline{u}\|^2_{H^2(Q)},$$

provided that $\varrho = h^2$. Now, using (3.9), (3.10) and Corollary 3.3, we obtain

$$\|u_{\varrho h} - \overline{u}\|_{L^2(Q)} \leq c \, h^2 \, \|\overline{u}\|_{H^2(Q)},$$

and

$$\|u_{\varrho h} - \overline{u}\|_S \leq c \, h \, \|\overline{u}\|_{H^2(Q)}.$$

The general estimates for $s \in (0,1]$ and $s \in (1,2)$ now follow again from a space interpolation argument. □

Corollary 4.2 *As already given in the previous proof, there hold the error estimates*

$$\varrho \, \|u_\varrho - u_{\varrho h}\|_S^2 + \|u_\varrho - u_{\varrho h}\|^2_{L^2(Q)} \leq c \left[\varrho + h^2 \right] \|\overline{u}\|^2_{H^1(Q)}$$

when assuming $\overline{u} \in H^{1,1}_{0;0,}(Q)$, *and*

$$\varrho \, \|u_\varrho - u_{\varrho h}\|_S^2 + \|u_\varrho - u_{\varrho h}\|^2_{L^2(Q)} \leq c \left[\varrho^2 + \varrho \, h^2 + h^4 \right] \|\overline{u}\|^2_{H^2(Q)}$$

when assuming $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^2(Q)$.

Next we are going to define a computable approximation of $Su = B^* A^{-1} Bu$. For $u \in X$, let $p_u = A^{-1} Bu \in Y$ be the unique solution of the variational formulation

$$\langle A p_u, q \rangle_Q = \langle Bu, q \rangle_Q \quad \text{for all } q \in Y,$$

and hence, $Su = B^* p_u$. Let $Y_h := S_h^1(Q) \cap H^{1,1}_{0;,0}(Q) = \text{span}\{\psi_i\}_{i=1}^{M_Y}$ be a second finite element space of piecewise linear continuous basis functions, which, for simplicity, are defined with respect to the same decomposition of the space-time domain Q into finite elements as X_h. Let now $p_{uh} \in Y_h$ solve

$$\langle A p_{uh}, q_h \rangle_Q = \langle Bu, q_h \rangle_Q \quad \text{for all } q_h \in Y_h,$$

and define $\widetilde{S}u := B^* p_{uh}$, where $\widetilde{S} : X \to X^*$ is bounded due to the properties of $A : Y \to Y^*$ and $B : X \to Y^*$, respectively. Instead of (4.1), we now consider the perturbed variational formulation to find $\widetilde{u}_{\varrho h} \in X_h$ such that

$$\varrho \, \langle \widetilde{S} \widetilde{u}_{\varrho h}, v_h \rangle_Q + \langle \widetilde{u}_{\varrho h}, v_h \rangle_{L^2(Q)} = \langle \overline{u}, v_h \rangle_{L^2(Q)} \tag{4.8}$$

is satisfied for all $v_h \in X_h$. Unique solvability of (4.8) follows since the matrix realization of $\widetilde{S}$ is positive semi-definite, while the mass matrix, which is related to the inner product in $L^2(Q)$, is positive definite.

Lemma 4.3 *Let* $u_{\varrho h}, \widetilde{u}_{\varrho h} \in X_h$ *be the unique solutions of the variational formulations* (4.1) *and* (4.8), *respectively. For* $\overline{u} \in L^2(Q)$ *there holds the error estimate*

$$\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(Q)} \leq \|\overline{u}\|_{L^2(Q)}. \tag{4.9}$$

For $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^2(Q) \subset X$ *we have*

$$\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(Q)} \leq c \, h^2 \, \|\overline{u}\|_{H^2(Q)}. \tag{4.10}$$

Proof. The estimate (4.9) follows when considering the perturbed variational formulation (4.8) for $v_h = \widetilde{u}_{\varrho h}$, i.e.,

$$\varrho \, \langle \widetilde{S} \widetilde{u}_{\varrho h}, \widetilde{u}_{\varrho h} \rangle_Q = \langle \overline{u} - \widetilde{u}_{\varrho h}, \widetilde{u}_{\varrho h} \rangle_{L^2(Q)} = -\langle \overline{u} - \widetilde{u}_{\varrho h}, \overline{u} - \widetilde{u}_{\varrho h} \rangle_{L^2(Q)} + \langle \overline{u} - \widetilde{u}_{\varrho h}, \overline{u} \rangle_{L^2(Q)},$$

and

$$\varrho\,\langle \widetilde{S}\widetilde{u}_{\varrho h}, \widetilde{u}_{\varrho h}\rangle_Q + \|\overline{u} - \widetilde{u}_{\varrho h}\|^2_{L^2(Q)} = \langle \overline{u} - \widetilde{u}_{\varrho h}, \overline{u}\rangle_{L^2(Q)} \leq \|\overline{u} - \widetilde{u}_{\varrho h}\|_{L^2(Q)}\|\overline{u}\|_{L^2(Q)}.$$

When subtracting the perturbed variational formulation (4.8) from (4.1), this gives

$$\varrho\,\langle S u_{\varrho h} - \widetilde{S}\widetilde{u}_{\varrho h}, v_h\rangle_Q + \langle u_{\varrho h} - \widetilde{u}_{\varrho h}, v_h\rangle_{L^2(Q)} = 0 \quad \text{for all } v_h \in X_h,$$

i.e.,

$$\varrho\,\langle (S - \widetilde{S})u_{\varrho h}, v_h\rangle_Q + \langle u_{\varrho h} - \widetilde{u}_{\varrho h}, v_h\rangle_{L^2(Q)} = \varrho\,\langle \widetilde{S}(\widetilde{u}_{\varrho h} - u_{\varrho h}), v_h\rangle_Q \quad \text{for all } v_h \in X_h.$$

In particular for $v_h = \widetilde{u}_{\varrho h} - u_{\varrho h}$ we further conclude

$$\begin{aligned} 0 &\leq \varrho\,\langle \widetilde{S}(\widetilde{u}_{\varrho h} - u_{\varrho h}), \widetilde{u}_{\varrho h} - u_{\varrho h}\rangle_Q \\ &= \varrho\,\langle (S - \widetilde{S})u_{\varrho h}, \widetilde{u}_{\varrho h} - u_{\varrho h}\rangle_Q + \langle u_{\varrho h} - \widetilde{u}_{\varrho h}, \widetilde{u}_{\varrho h} - u_{\varrho h}\rangle_{L^2(Q)}, \end{aligned}$$

i.e., using an inverse inequality in X_h,

$$\begin{aligned} \|\widetilde{u}_{\varrho h} - u_{\varrho h}\|^2_{L^2(Q)} &\leq \varrho\,\langle (S - \widetilde{S})u_{\varrho h}, \widetilde{u}_{\varrho h} - u_{\varrho h}\rangle_Q \\ &= \varrho\,\langle B^*(p_{u_{\varrho h}} - p_{u_{\varrho h}h}), \widetilde{u}_{\varrho h} - u_{\varrho h}\rangle_Q \\ &= \varrho\,\langle p_{u_{\varrho h}} - p_{u_{\varrho h}h}, B(\widetilde{u}_{\varrho h} - u_{\varrho h})\rangle_Q \\ &= \varrho\,\langle \Box(\widetilde{u}_{\varrho h} - u_{\varrho h}), \mathcal{E}(p_{u_{\varrho h}} - p_{u_{\varrho h}h})\rangle_{Q_-} \\ &= \varrho\, b(\widetilde{u}_{\varrho h} - u_{\varrho h}, p_{u_{\varrho h}} - p_{u_{\varrho h}h}) \\ &\leq \varrho\,|\widetilde{u}_{\varrho h} - u_{\varrho h}|_{H^1(Q)}|p_{u_{\varrho h}} - p_{u_{\varrho h}h}|_{H^1(Q)} \\ &\leq c\,\varrho\, h^{-1}\,\|\widetilde{u}_{\varrho h} - u_{\varrho h}\|_{L^2(Q)}|p_{u_{\varrho h}} - p_{u_{\varrho h}h}|_{H^1(Q)}. \end{aligned}$$

Hence, using $\varrho = h^2$ and the triangle inequality, this gives

$$\begin{aligned} \|\widetilde{u}_{\varrho h} - u_{\varrho h}\|_{L^2(Q)} &\leq c\,h\,|p_{u_{\varrho h}} - p_{u_{\varrho h}h}|_{H^1(Q)} \\ &\leq c\,h\left[|p_{u_{\varrho h}} - p_{\overline{u}}|_{H^1(Q)} + |p_{\overline{u}} - p_{\overline{u}h}|_{H^1(Q)} + |p_{\overline{u}h} - p_{u_{\varrho h}h}|_{H^1(Q)}\right]. \end{aligned}$$

For the first term we further have

$$\begin{aligned} |p_{u_{\varrho h}} - p_{\overline{u}}|^2_{H^1(Q)} &= \langle A(p_{u_{\varrho h}} - p_{\overline{u}}), p_{u_{\varrho h}} - p_{\overline{u}}\rangle_Q \\ &= \langle B(u_{\varrho h} - \overline{u}), p_{u_{\varrho h}} - p_{\overline{u}}\rangle_Q \\ &\leq \|B(u_{\varrho h} - \overline{u})\|_{Y^*}\|p_{u_{\varrho h}} - p_{\overline{u}}\|_Y \\ &\leq 2\,\|u_{\varrho h} - \overline{u}\|_X\|p_{u_{\varrho h}} - p_{\overline{u}}\|_Y, \end{aligned}$$

i.e.,

$$|p_{u_{\varrho h}} - p_{\overline{u}}|_{H^1(Q)} \leq 2\,\|u_{\varrho h} - \overline{u}\|_X \leq 2\,\|u_{\varrho h} - \overline{u}\|_S \leq c\,h\,\|\overline{u}\|_{H^2(Q)}.$$

Following the same lines we can also estimate the third term by

$$|p_{\overline{u}h} - p_{u_{\varrho h}h}|_{H^1(Q)} \leq c\,h\,\|\overline{u}\|_{H^2(Q)}.$$

To estimate the second term, let us first recall that $p_{\overline{u}} \in Y = H^{1,1}_{0;,0}(Q)$ solves

$$\langle A p_{\overline{u}}, q \rangle_Q = \langle B\overline{u}, q \rangle_Q \quad \text{for all } q \in Y,$$

while $p_{\overline{u}h} \in Y_h$ solves

$$\langle A p_{\overline{u}h}, q_h \rangle_Q = \langle B\overline{u}, q_h \rangle_Q \quad \text{for all } q_h \in Y_h.$$

Thus, we conclude the Galerkin orthogonality

$$\langle A(p_{\overline{u}} - p_{\overline{u}h}), q_h \rangle_Q = 0 \quad \text{for all } q_h \in Y_h,$$

and Cea's lemma,

$$|p_{\overline{u}} - p_{\overline{u}h}|_{H^1(Q)} \le \inf_{q_h \in Y_h} |p_{\overline{u}} - q_h|_{H^1(Q)} \le c\, h\, |p_{\overline{u}}|_{H^2(Q)},$$

when assuming $p_{\overline{u}} = A^{-1}B\overline{u} \in H^2(Q)$. Indeed, for a convex space-time domain Q we have

$$|p_{\overline{u}}|_{H^2(Q)} \le c\, \|A p_{\overline{u}}\|_{L^2(Q)} = c\, \|B\overline{u}\|_{L^2(Q)} \le c\, \|\overline{u}\|_{H^2(Q)}.$$

This concludes the proof. □

Remark 4.1 *In the proof of Lemma 4.3 we have used an inverse inequality which in general assumes a globally quasi-uniform finite element mesh. However, when using a variable regularization function $\varrho(x,t) = h_\ell^2$ for $(x,t) \in \tau_\ell$, it is sufficient to use the inverse inequality locally, allowing adaptively refined and locally quasi-uniform finite element meshes. For a related approach for a distributed optimal control problem with variable regularization subject to the Poisson equation, see [18].*

Corollary 4.4 *When using a space interpolation argument, from* (4.9) *and* (4.10) *we now conclude the final error estimate*

$$\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(Q)} \le c\, h^s \;\; \|\overline{u}\|_{H^s(Q)} \tag{4.11}$$

when assuming $\overline{u} \in [L^2(Q), H^{1,1}_{0;0,}(Q)]_s$ for $s \in [0,1]$ or $\overline{u} \in H^{1,1}_{0;0,}(Q) \cap H^s(Q)$ for $s \in (1,2]$.

When the approximate state $\widetilde{u}_{\varrho h}$ is known, as in (2.15) we can compute the associate control $\widetilde{z}_\varrho = B\widetilde{u}_{\varrho h} \in [H^{1,1}_{0;,0}(Q)]^*$ as unique solution of the variational formulation

$$\langle \widetilde{z}_\varrho, q \rangle_Q = \langle \Box \widetilde{u}_{\varrho h}, \mathcal{E} q \rangle_{Q_-} = \langle B\widetilde{u}_{\varrho h}, q \rangle_Q \quad \text{for all } q \in H^{1,1}_{0;,0}(Q).$$

With this we conclude that $\widetilde{z}_\varrho$ is the minimizer of the functional

$$\mathcal{F}(z) := \frac{1}{2}\, \|z - B\widetilde{u}_{\varrho h}\|^2_{[H^{1,1}_{0;,0}(Q)]^*} = \frac{1}{2}\, \langle A^{-1}(z - B\widetilde{u}_{\varrho h}), z - B\widetilde{u}_{\varrho h} \rangle_Q,$$

i.e., $\widetilde{z}_\varrho \in [H^{1,1}_{0;,0}(Q)]^*$ is the unique solution of the gradient equation

$$A^{-1}(\widetilde{z}_\varrho - B\widetilde{u}_{\varrho h}) = 0.$$

This is equivalent to the coupled system to find $(\psi, \widetilde{z}_\varrho) \in H^{1,1}_{0;,0}(Q) \times [H^{1,1}_{0;,0}(Q)]^*$ such that

$$A\psi + \widetilde{z}_\varrho = B\widetilde{u}_{\varrho h}, \quad \psi = 0. \tag{4.12}$$

Let $Z_H \subset [H^{1,1}_{0;,0}(Q)]^*$ be a suitable finite element space, then we consider the Galerkin variational formulation to find $(\psi_h, \widetilde{z}_{\varrho H}) \in Y_h \times Z_H$ such that

$$\langle A\psi_h, \phi_h\rangle_{L^2(Q)} + \langle \widetilde{z}_{\varrho H}, \phi_h\rangle_{L^2(Q)} = \langle B\widetilde{u}_{\varrho h}, \phi_h\rangle_{L^2(Q)}, \quad \langle \psi_h, \eta_H\rangle_{L^2(Q)} = 0 \tag{4.13}$$

is satisfied for all $(\phi_h, \eta_H) \in Y_h \times Z_H$. Unique solvability of (4.13) follows when the discrete inf-sup stability condition

$$c_S \, \|z_H\|_{[H^{1,1}_{0;,0}(Q)]^*} \leq \sup_{0 \neq \phi_h \in Y_h} \frac{\langle z_H, \phi_h\rangle_{L^2(Q)}}{\|\phi_h\|_{H^{1,1}_{0;,0}(Q)}} \quad \text{for all } z_H \in Z_H$$

is satisfied, i.e., when Y_h is defined with respect to a space-time finite element mesh size h which is sufficiently small compared to the mesh size H of Z_H. From a practical point of view it is sufficient to consider one additional refinement when defining first Z_H, and afterwards Y_h, i.e., $h = H/2$. As in mixed finite element methods and using the Strang lemma we can then derive related error estimates for the Galerkin solution $\widetilde{z}_{\varrho H}$.

5 Numerical results

The perturbed variational formulation (4.8) corresponds to the Galerkin discretization of the coupled variational formulation (2.14). With the finite element spaces

$$X_h := S^1_h(\mathcal{T}_h) \cap H^{1,1}_{0;0,}(Q) = \text{span}\{\varphi_k\}_{k=1}^{M_X}$$

and

$$Y_h := S^1_h(\mathcal{T}_h) \cap H^{1,1}_{0;,0}(Q) = \text{span}\{\psi_i\}_{i=1}^{M_Y}$$

as already used in Section 4, the equivalent linear system of algebraic equations reads

$$\begin{pmatrix} \varrho^{-1} A_h & B_h \\ -B_h^\top & M_h \end{pmatrix} \begin{pmatrix} \underline{p} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \end{pmatrix}, \tag{5.1}$$

where the system matrix is positive definite but skew-symmetric, and where the matrix entries are given as, for $k, \ell = 1, \ldots, M_X$, $i, j = 1, \ldots, M_Y$,

$$\begin{aligned} A_h[j,i] &= \langle \nabla_{(x,t)}\psi_i, \nabla_{(x,t)}\psi_j\rangle_{L^2(Q)}, \\ M_h[\ell,k] &= \langle \varphi_k, \varphi_\ell\rangle_{L^2(Q)}, \\ B_h[j,k] &= -\langle \partial_t\varphi_k, \partial_t\psi_j\rangle_{L^2(Q)} + \langle \nabla_x\varphi_k, \nabla_x\psi_j\rangle_{L^2(Q)}, \end{aligned}$$

and with the load vector

$$f_\ell = \langle \overline{u}, \varphi_\ell \rangle_{L^2(Q)}.$$

In addition to the energy regularization we will also consider the control $z_\varrho \in L^2(Q)$, where the arising matrix system is given as

$$\begin{pmatrix} \varrho^{-1}\overline{M}_h & B_h \\ -B_h^\top & M_h \end{pmatrix} \begin{pmatrix} \underline{p} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{f} \end{pmatrix}, \tag{5.2}$$

with the related mass matrix

$$\overline{M}_h[j,i] := \langle \psi_i, \psi_j \rangle_{L^2(Q)} \quad \text{for } i,j = 1, \dots, M_Y.$$

A similar analysis as for the energy regularization shows, that in this case, the optimal choice for the relaxation parameter is $\varrho = h^4$, see also [19] in the case of a distributed optimal control problem for the Poisson equation.

5.1 Uniform refinement

In order to check our theoretical findings, we consider three test examples of different regularity for the target function $\overline{u}$, in the space-time domain $Q := (0,1) \times (0,1) \subset \mathbb{R}^2$. First we consider a smooth function $\overline{u}_1 \in C^2(\overline{Q}) \cap H^{1,1}_{0;0,}(Q)$ given as

$$\overline{u}_1(x,t) = \begin{cases} \frac{1}{2}(6t - 3x - 2)^3(3x - 6t)^3, & x \le t \text{ and } t - x \le 2, \\ 0, & \text{else.} \end{cases} \tag{5.3}$$

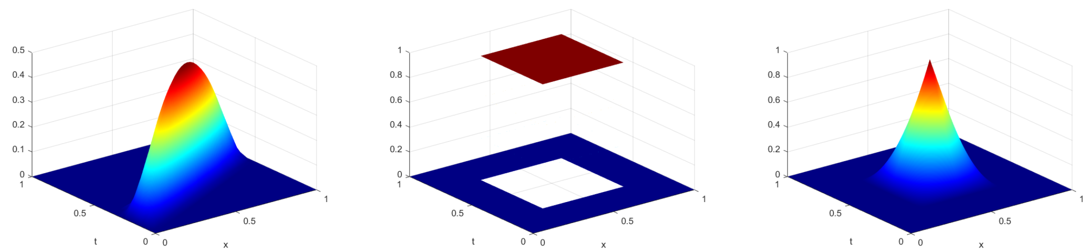
As a second target function we have the piecewiese constant function $\overline{u}_2 \in H^{1/2-\varepsilon}(Q)$, $\varepsilon > 0$, given as

$$\overline{u}_2(x,t) = \begin{cases} 1, & (x,t) \in (0.25, 0.75)^2 \subset Q, \\ 0, & \text{else.} \end{cases} \tag{5.4}$$

Finally, we consider a piecewise bilinear function $\overline{u}_3 \in H_0^{3/2-\varepsilon}(Q)$, $\varepsilon > 0$, defined as

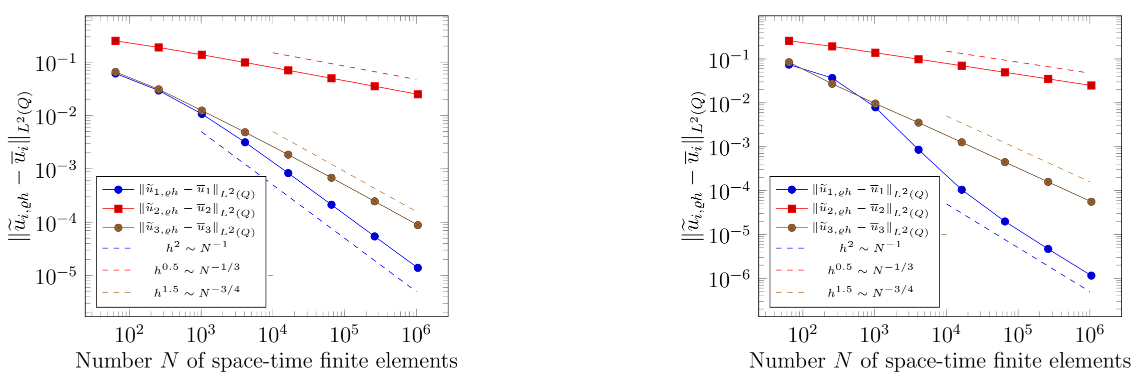
$$\overline{u}_3(x,t) = \phi(x)\phi(t), \quad \phi(s) = \begin{cases} 1, & s = 0.5, \\ 0, & s \notin [0.25, 0.75], \\ \text{linear}, & \text{else.} \end{cases} \tag{5.5}$$

The numerical results for the energy regularization (5.1) with the optimal regularization parameter $\varrho = h^2$, and for the L^2 regularization (5.2) with $\varrho = h^4$, are depicted in Fig. 2, where we observe optimal orders of convergence for each of the three examples, as predicted by the theory.
Although the convergence rates and errors for both approaches, the energy and the L^2 regularization method, seem to be comparable, we observe a difference in the behaviour of the discontinuous solution $\widetilde{u}_{2,\varrho h}$, see Fig. 3. This is due to the additional regularity $z_\varrho \in H^{1,1}_{0;,0}(Q)$ which we gain when considering the control in $Z = L^2(Q)$.



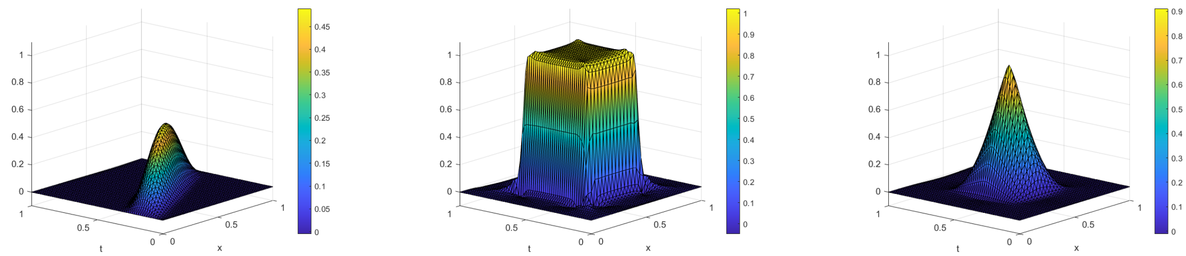
(a) $\overline{u}_1$ (b) $\overline{u}_2$ (c) $\overline{u}_3$

Figure 1: Target functions $\overline{u}_i$, $i = 1, 2, 3$.

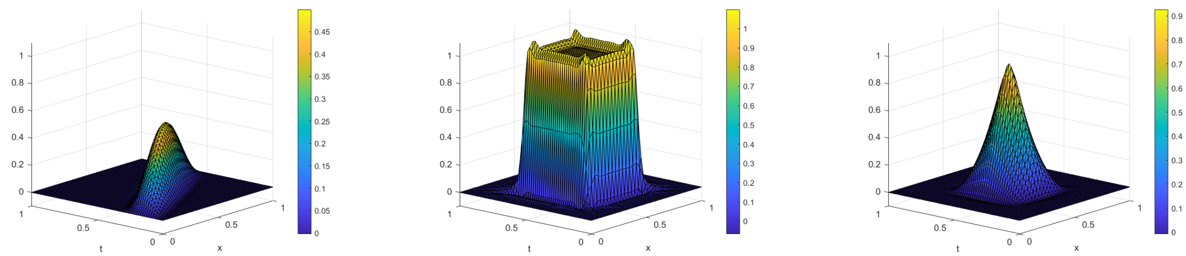


(a) Energy regularization, $\varrho = h^2$ (b) L^2 regularization, $\varrho = h^4$

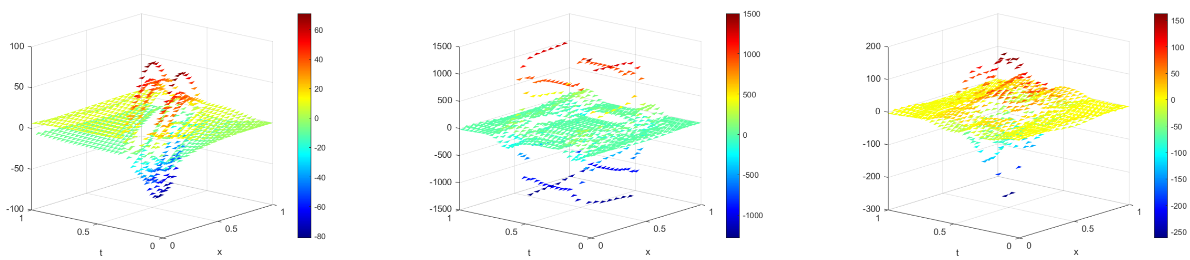
Figure 2: Convergence plots for the three different target functions $\overline{u}_i$, $i = 1, 2, 3$ for the energy and the L^2 regularization.



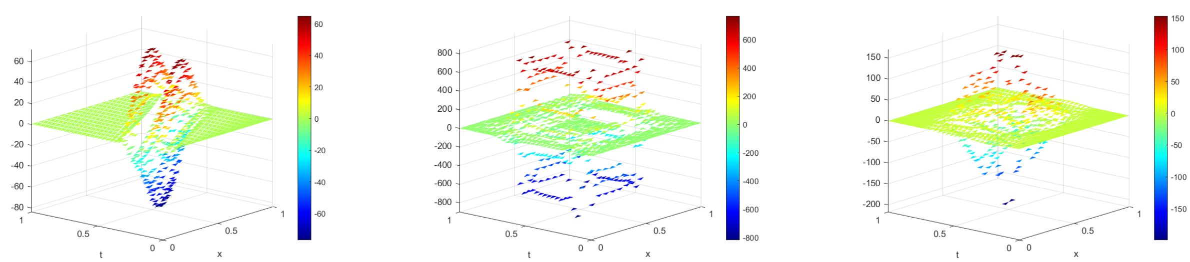
Computed states in the case of energy regularization



Computed states in the case of L^2 regularization



Reconstructed controls in the case of energy regularization



Reconstructed controls in the case of L^2 regularization

Figure 3: Comparison of the computed states and the reconstructed controls for energy and L^2 regularizations on level 3, with 4096 elements and 1984 degrees of freedom.

Remark 5.1 *The choice of A enforces homogeneous Neumann conditions at the origin $t = 0$, while we have homogeneous Dirichlet boundary conditions elsewhere. Due to this change in the boundary conditions we may have a reduced regularity for the solution of the space-time Poisson equation, see Corollary 3.3. This results in a reduced order of convergence, as observed for the target $\overline{u}_4(x,t) = t\sin(\pi t)\sin(\pi x)$, for $(x,t) \in (0,1)^2$, where the solution of the energy regularization (5.1) converges with a rate 1.5 instead of 2, see Table 1. To regain optimal rates, there are three possible remedies. First, one might choose $\varrho = h^3$ for the energy regularization. Then the term is penalized strong enough to ensure optimal orders of convergence. Second, one might compute the solution on an enlarged domain, embedding the target function such that in a neighborhood of $t = 0$ and $t = T$ the function is constant zero, as we have done in our examples $\overline{u}_i$, $i = 1, 2, 3$. Then p_ϱ will (approximately) fulfil the homogeneous Neumann condition. A third possibility is to adaptively refine the mesh and resolve the singularities. This will be discussed in the next section. Note, that for the L^2 regularization approach this effect does not occur, since the operator $A = id : L^2(Q) \to L^2(Q)$ does not enforce any initial condition.*

Level	DoFs	N	h	$\varrho\,(h^2)$	$\|\widetilde{u}_{4,\varrho h} - \overline{u}_4\|_{L^2(Q)}$	eoc
0	24	64	0.125	$1.56 \cdot 10^{-2}$	$2.46 \cdot 10^{-2}$	0.00
1	112	256	0.063	$3.91 \cdot 10^{-3}$	$7.75 \cdot 10^{-3}$	1.67
2	480	1,024	0.031	$9.77 \cdot 10^{-4}$	$2.70 \cdot 10^{-3}$	1.52
3	1,984	4,096	0.016	$2.44 \cdot 10^{-4}$	$9.63 \cdot 10^{-4}$	1.48
4	8,064	16,384	0.008	$6.10 \cdot 10^{-5}$	$3.44 \cdot 10^{-4}$	1.48
5	32,512	65,536	0.004	$1.53 \cdot 10^{-5}$	$1.23 \cdot 10^{-4}$	1.49
6	130,560	262,144	0.002	$3.81 \cdot 10^{-6}$	$4.35 \cdot 10^{-5}$	1.49
7	523,264	1,048,576	0.001	$9.54 \cdot 10^{-7}$	$1.54 \cdot 10^{-5}$	1.50

Table 1: Errors and orders of convergence for $\overline{u}_4(x,t) = t\sin(t\pi)\sin(x\pi)$ in the case of an uniform refinement strategy with $\varrho = h^2$.

Since in optimal control theory we are mainly interested in the control z, rather than in the computed state $\widetilde{u}_{\varrho h}$, we are going to reconstruct the control z_ϱ in a post-processing step when solving (4.13). We introduce the finite element space $Z_H := S_H^0(Q) = \text{span}\{\phi_r\}_{r=1}^{N_H} \subset [H^{1,1}_{0;,0}(Q)]^*$ of piecewise constant basis functions ϕ_r. When using $Y_h := S_h^1(Q) \cap Y$, (4.13) is equivalent to the linear system of algebraic equations to find $\underline{\psi} \in \mathbb{R}^{M_Y} \leftrightarrow \psi_h \in Y_h$ and $\underline{z} \in \mathbb{R}^{N_H} \leftrightarrow \widetilde{z}_{\varrho H} \in Z_H$ such that

$$\begin{pmatrix} A_h & P_{hH}^\top \\ P_{hH} & 0 \end{pmatrix} \begin{pmatrix} \underline{\psi} \\ \underline{z} \end{pmatrix} = \begin{pmatrix} B_h \underline{u} \\ 0 \end{pmatrix},$$

with matrices B_h as above and

$$P_{hH}[r,j] = \langle \psi_j, \phi_r \rangle_{L^2(Q)}, \quad r = 1, \dots, N_H,\ j = 1, \dots, M_Y.$$

Resolving the system for $\underline{z}$ gives

$$\underline{z} = (P_{hH} A_h^{-1} P_{hH}^\top)^{-1} P_{hH} A_h^{-1} B_h \underline{u}. \tag{5.6}$$

In Fig. 3 we also present the reconstructed controls for both the energy and the L^2 regularization approach.

5.2 Adaptive refinement

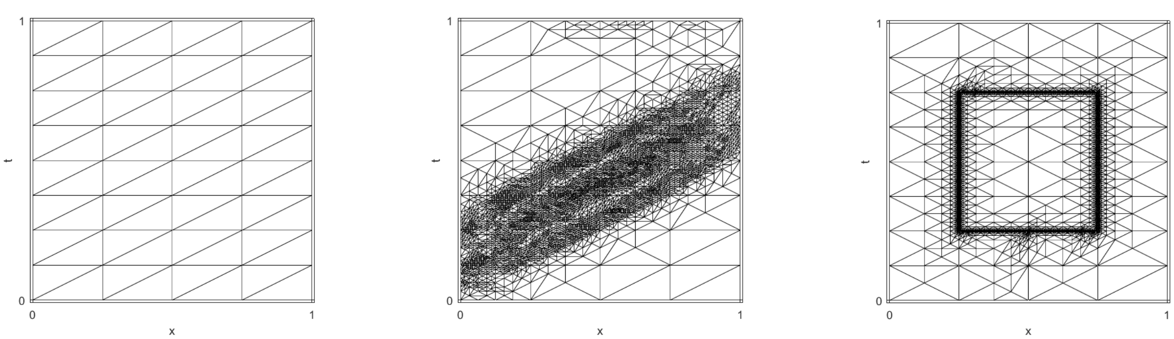
In this section we present some examples for an adaptive space-time refinement strategy for the energy regularization (5.1). We will apply an adaptive refinement strategy using Dörfler marking [6] with the refinement indicator $\eta_\ell = \|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\tau_\ell)}$ on each simplicial space-time finite element τ_ℓ, $\ell = 1, \dots, N$. With this choice we see that the approximation error fufills

$$\|\widetilde{u}_{\varrho h} - \overline{u}\|^2_{L^2(Q)} = \sum_{\ell=1}^N \eta_\ell^2.$$

We will refine all elements τ_k that satisfy

$$\eta_k \geq \theta \max_{\ell=1,\dots,N} \eta_\ell,$$

with $\theta = 0.5$. The initial mesh with 64 elements and 24 degrees of freedom (DoFs) and the resulting adaptively refined meshes for the target functions $\overline{u}_1$ at level 10 with 8824 elements and 4389 DoFs and for $\overline{u}_2$ at level 7 with 15159 elements and 7571 DoFs are shown in Fig. 4. In Tables 2 and 3 we present a comparison of the errors of the adaptive refinement strategy against the errors of the uniform refinement at levels with comparably many elements for both target functions. We clearly see, that in both cases considerably less elements are needed to achieve errors of the same order.



(a) Level 0 (b) $\overline{u}_1$, Level 10 (c) $\overline{u}_2$, Level 7

Figure 4: Initial mesh and adaptively refined meshes for the target functions $\overline{u}_i$, $i = 1, 2$.

All comutations were carried out with Matlab using a sparse direct solver. For the adaptive refinement strategy the package from [10] was adapted suitably.

L	Adaptive			L	Uniform		
	#DoFs	$\|\widetilde{u}_{1,\varrho h} - \overline{u}_1\|_{L^2(Q)}$	$\varrho = h^2_{min}$		#DoFs	$\|\widetilde{u}_{1,\varrho h} - \overline{u}_1\|_{L^2(Q)}$	$\varrho = h^2$
0	24	$6.12415 \cdot 10^{-2}$	$1.56 \cdot 10^{-2}$	0	24	$6.12415 \cdot 10^{-2}$	$1.56 \cdot 10^{-2}$
2	101	$1.26895 \cdot 10^{-2}$	$9.77 \cdot 10^{-4}$	1	112	$2.94242 \cdot 10^{-2}$	$3.91 \cdot 10^{-3}$
5	399	$2.05047 \cdot 10^{-3}$	$1.22 \cdot 10^{-4}$	2	480	$1.06888 \cdot 10^{-2}$	$9.77 \cdot 10^{-4}$
8	1654	$5.13791 \cdot 10^{-4}$	$3.05 \cdot 10^{-5}$	3	1984	$3.14290 \cdot 10^{-3}$	$2.44 \cdot 10^{-4}$
13	8266	$6.61367 \cdot 10^{-5}$	$1.91 \cdot 10^{-6}$	4	8064	$8.31332 \cdot 10^{-4}$	$6.10 \cdot 10^{-5}$
18	39821	$1.19888 \cdot 10^{-5}$	$1.19 \cdot 10^{-7}$	5	32512	$2.12588 \cdot 10^{-4}$	$1.53 \cdot 10^{-5}$
22	162774	$2.86782 \cdot 10^{-6}$	$2.98 \cdot 10^{-8}$	6	130560	$5.41480 \cdot 10^{-5}$	$3.82 \cdot 10^{-6}$
25	377896	$1.07848 \cdot 10^{-6}$	$7.45 \cdot 10^{-9}$	7	523264	$1.39319 \cdot 10^{-5}$	$9.54 \cdot 10^{-7}$
26	636878	$7.03678 \cdot 10^{-7}$	$1.86 \cdot 10^{-9}$				

Table 2: Comparison of the uniform refinement to the adaptive refinement strategy for levels (L) with comparably many elements for the target function $\overline{u}_1$ for the energy regularization (5.1).

L	Adaptive			L	Uniform		
	#DoFs	$\|\widetilde{u}_{2,\varrho h} - \overline{u}_2\|_{L^2(Q)}$	$\varrho = h^2_{min}$		#DoFs	$\|\widetilde{u}_{2,\varrho h} - \overline{u}_2\|_{L^2(Q)}$	$\varrho = h^2$
0	24	$2.50691 \cdot 10^{-1}$	$1.56 \cdot 10^{-2}$	0	24	$2.50691 \cdot 10^{-1}$	$1.56 \cdot 10^{-2}$
2	198	$1.36350 \cdot 10^{-1}$	$9.77 \cdot 10^{-4}$	1	112	$1.88590 \cdot 10^{-1}$	$3.91 \cdot 10^{-3}$
3	435	$9.74050 \cdot 10^{-2}$	$2.44 \cdot 10^{-4}$	2	480	$1.37373 \cdot 10^{-1}$	$9.77 \cdot 10^{-4}$
5	1895	$4.92039 \cdot 10^{-2}$	$1.53 \cdot 10^{-5}$	3	1984	$9.85712 \cdot 10^{-2}$	$2.44 \cdot 10^{-4}$
7	7571	$2.46665 \cdot 10^{-2}$	$9.54 \cdot 10^{-7}$	4	8064	$7.02300 \cdot 10^{-2}$	$6.10 \cdot 10^{-5}$
9	30027	$1.23436 \cdot 10^{-2}$	$5.96 \cdot 10^{-8}$	5	32512	$4.98503 \cdot 10^{-2}$	$1.53 \cdot 10^{-5}$
11	119554	$6.17867 \cdot 10^{-3}$	$3.73 \cdot 10^{-9}$	6	130560	$3.53171 \cdot 10^{-2}$	$3.82 \cdot 10^{-6}$
13	477542	$3.09069 \cdot 10^{-3}$	$2.3 \cdot 10^{-10}$	7	523264	$2.49969 \cdot 10^{-2}$	$9.54 \cdot 10^{-7}$
14	957389	$2.18324 \cdot 10^{-3}$	$5.8 \cdot 10^{-11}$				

Table 3: Comparison of the uniform refinement to the adaptive refinement strategy for levels (L) with comparably many elements for the target function $\overline{u}_2$ for the energy regularization (5.1).

6 Conclusions and outlook

We have introduced and investigated a space-time finite element method for distributed optimal control problems for the wave equation with energy regularization. In particular, we have shown $L^2(Q)$ error estimates between the desired state $\overline{u}$ and the computable discrete solution $\widetilde{u}_{\varrho h}$, with respect to the regularity of the target function. It has been proven that in this case the choice $\varrho = h^2$ delivers optimal orders of convergence, and the findings have been supported by several numerical examples. Moreover, we compared the results to the case where a $L^2(Q)$ regularization is used. Furthermore, we proposed an adaptive finite element strategy and presented its performance for target functions with different regularities, where we observed that considerably less elements are needed for a comparable error than in the case of an uniform refinement.

The system matrices, for both the L^2 and the energy regularization approach, are positive definite but skew-symmetric, or alternatively, symmetric but indefinite. Thus it is of highest interest to develop robust iterative solvers as already done in the elliptic and parabolic case [19, 22, 23]. This will then also allow an efficient solution of related optimal control problems in two and three space dimensions. Moreover, for discontinuous targets and an adaptive finite element scheme, it would be sensible to consider a relaxation parameter $\varrho = \varrho(x,t)$ that is locally varying, to have a better resolution of the control defined on the adaptive mesh. This has already been done for the optimal control problem subject to the Poisson equation [18]. In addition, in order to be of practical interest, the consideration of control and/or state constraints can be considered within the abstract framework as done in [11].

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