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Technische Universität Graz
Institut für Angewandte Mathematik
Steyrergasse 30
A 8010 Graz

WWW: http://www.applied.math.tugraz.at

A unified framework for the analysis, numerical approximation and model reduction of linear operator equations, Part I: Well-posedness in space and time

Moritz Feuerle*, Richard Löscher†, Olaf Steinbach‡, Karsten Urban§

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Abstract

We present a unified framework to construct well-posed formulations for large classes of linear operator equations including elliptic, parabolic and hyperbolic partial differential equations. This general approach incorporates known weak variational formulations as well as novel space-time variational forms of the hyperbolic wave equation. The main concept is completion and extension of operators starting from the strong form of the problem.

This paper lays the theoretical foundation for a unified approach towards numerical approximation methods and also model reduction of parameterized linear operator equations which will be the subject of the following parts.

1 Introduction

Linear operator equations can, e.g., be derived from partial differential and integral equations. Typical examples include elliptic, parabolic and hyperbolic second order partial differential equations (PDEs) as well as boundary integral equations. The analysis of well-posedness (existence, uniqueness and stability of solutions) as well as the construction and investigation of numerical approximation methods are usually done problem-specific and different for the above mentioned classes of problems.

The aim of this paper is to provide a unified framework for the analysis, numerical approximation and model reduction of linear operator equations. In particular we aim at

*Institute for Numerical Mathematics, Ulm University, Germany (moritz.feuerle@uni-ulm.de)
†Institute of Applied Mathematics, TU Graz, Austria (loescher@math.tugraz.at)
‡Institute of Applied Mathematics, TU Graz, Austria (o.steinbach@tugraz.at)
§Institute for Numerical Mathematics, Ulm University, Germany (karsten.urban@uni-ulm.de)

deriving a general approach which allows us to construct well-posed and optimally stable[1] variational formulations for rather general linear operator equations. This is the scope of part I of this paper series. The forthcoming part will then concentrate on deriving corresponding discretizations and numerical approximations which will benefit from the general framework constructed in this first part. Finally, the close relation between the approximation error and the residual will then also allow us to construct a quite general approach towards model reduction of *parameterized* linear operator equations.

In such a general framework we are able to cover stationary elliptic as well as time-dependent parabolic and hyperbolic problems. Elliptic problems are formulated in Sobolev and the time-dependent problems in Lebesgue-Bochner spaces using space-time variational formulations. The latter ones are well established for parabolic problems, but less is known for transport, wave and Schrödinger-type problems. The framework presented in this paper allows for well-posed variational formulations for all those equations. We shall detail the general approach to several examples, which will also include known weak and ultra-weak formulations. To the very best of our knowledge, this is a novel framework. It is, however, related to [5], which focuses on ultra-weak formulations. We will comment on similarities and differences throughout the paper. Our approach also extends existing results in a unified manner, recovering also well-known formulations for elliptic and parabolic problems, while also allowing for well-posed formulations for hyperbolic problems.

The remainder of this paper is organized as follows. After collecting some notation and guiding examples in the sequel of this section, Section 2 is devoted to the presentation of the theoretical foundation of our general framework. The main technical ingredient is completion and extension of operators to be presented in §2.2. We show applications to the Poisson problem, the heat equation and the wave equation in Section 3. The paper ends with some conclusions and an outlook in Section 4.

1.1 Notation and basic facts

All spaces are assumed to be vector spaces over a common field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Thereby, for a normed vector space X, we denote by $\|\cdot\|_X$ its norm and by X' its dual space ($\mathbb{F} = \mathbb{R}$) or anti-dual space ($\mathbb{F} = \mathbb{C}$). If X is an inner product space, we denote its inner product by $(\cdot, \cdot)_X$. Hence, the inner product is a bilinear ($\mathbb{F} = \mathbb{R}$) or sesquilinear[2] form ($\mathbb{F} = \mathbb{C}$). Further, we denote by $\langle \cdot, \cdot \rangle_{X' \times X} : X' \times X \to \mathbb{F}$ the duality pairing (or evaluation map), given by $\langle f, x \rangle_{X' \times X} \coloneqq f(x)$ for $f \in X'$ and $x \in X$, which is also a bilinear or sesquilinear form, respectively. For two Banach spaces X, Y, we denote the space of all *linear and bounded operators* by $\mathcal{L}(X, Y') \coloneqq \{B : X \to Y' \,:\, B \text{ is linear and } \|B\|_{\mathcal{L}(X,Y')} < \infty\}$, with $\|B\|_{\mathcal{L}(X,Y')} \coloneqq \sup\limits_{x \in X \setminus \{0\}} \frac{\|Bx\|_{Y'}}{\|x\|_X}$, and the space of all *linear isomorphisms* by

$$\mathcal{L}_{\text{is}}(X, Y') \coloneqq \{B \in \mathcal{L}(X, Y') \,:\, B \text{ bijective and } B^{-1} \in \mathcal{L}(Y', X)\}.$$

[1] In the sense that the norms of the operator and its inverse are equal to 1.

[2] linear in the first and anti-linear in the second argument

As usual, we call $B \in \mathcal{L}(X, Y')$ *isometric*, if $\|B\|_{\mathcal{L}(X,Y')} = 1$. Given $B \in \mathcal{L}(X, Y')$ and $f \in Y'$, seeking for an unknown $x \in X$, we call $Bx = f$ in Y' interpreted as $\langle Bx, y\rangle_{Y'\times Y} = \langle f, y\rangle_{Y'\times Y}$ for all $y \in Y$, a *linear operator equation*. We call it *well-posed* (in the sense of Hadamard, [13]) if $B \in \mathcal{L}_{\text{is}}(X, Y')$ and *optimally stable* if B is isometric. Further, for a Hilbert space H, we denote by $R_H \in \mathcal{L}_{\text{is}}(H, H')$ the isometric *Riesz operator* given by

$$\langle R_H x, y\rangle_{H'\times H} \coloneqq (x, y)_H \quad \forall x, y \in H. \tag{1}$$

In addition, it holds $(\cdot,\cdot)_{H'} = \langle \cdot, R_H^{-1}\cdot\rangle_{H'\times H}$, $\|\cdot\|_H = \|R_H\cdot\|_{H'}$ and $\|\cdot\|_{H'} = \|R_H^{-1}\cdot\|_H$. For two normed vector spaces X, Y with $X \subseteq Y$, we call X *continuously embedded* in Y (abbreviated "$X \hookrightarrow Y$") if there exists an *embedding constant* $C < \infty$, such that $\|x\|_Y \le C\|x\|_X$ for all $x \in X$. Further, we denote by $X \subseteq_d Y$, that X is dense in Y, and by $X \hookrightarrow_d Y$, that $X \subseteq_d Y$ and $X \hookrightarrow Y$. We collect some well-known facts.

Remark 1.1. *If $X \subseteq_d Y \hookrightarrow_d Z$, then $X \subseteq_d Z$ and $X \hookrightarrow Z$ if $X \hookrightarrow Y \hookrightarrow Z$.*

Remark 1.2 ([3, §5 Remark 3])**.** *Let $X \hookrightarrow_d H$ be Banach spaces with embedding constant C.*

(i) It holds $H' \hookrightarrow X'$ with embedding constant C and $H' \hookrightarrow_d X'$, if X is reflexive.

(ii) Let H be a Hilbert space. Identifying $H \cong H'$ using the Riesz operator, we get

$$X \hookrightarrow_d H \cong H' \hookrightarrow X', \tag{2}$$

as well as $\|x\|_H \le C\|x\|_X$, $x \in X$, $\|x\|_{X'} \le C\|x\|_H$, $x \in H$ and for the duality paring holds

$$\langle f, x\rangle_{X'\times X} = (f, x)_H \qquad \forall f \in H,\ \forall x \in X. \tag{3}$$

Eq. (2) is usually called a Gelfand triple*, denoted by (X, H, X'). Note: If X is a Hilbert space itself, there exists the natural Riesz isomorphism between X and X', but it is not viewed as the identity map, i.e., $X \ncong X'$. Instead, we use the Riesz isomorphism between H and H' as the identity map to identify $H \cong H'$. To make things clear, for $x \in X$, the dual norm is to be understood as $\|x\|_{X'} = \|R_H x\|_{X'} = \sup\limits_{y\in X} \frac{(x,y)_H}{\|x\|_X}$ and not as $\|R_X x\|_{X'} = \sup\limits_{y\in X} \frac{(x,y)_X}{\|x\|_X}$.*

Function spaces

Let $I \coloneqq (0, T)$, $0 < T < \infty$ be a time interval, $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be an open bounded spatial domain with Lipschitz boundary for $d \ge 2$ and let $Q \coloneqq I \times \Omega$ denote the space-time domain. We denote by $C^{k;\ell}(Q)$, $k, \ell \in \mathbb{N}_0$, the set of all $C(Q)$ functions where all partial derivatives w.r.t. $t \in I$ up to order k and all partial derivatives w.r.t. $x \in \Omega$ up to order ℓ are in $C(Q)$, define $C_0(\overline{\Omega}) \coloneqq \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ and we denote the space of continuous functions with zero boundary in space and k^{th}-order zero initial conditions in time by

$$C^{(k)}_{0,;0}(\overline{Q}) \coloneqq \{u \in C(I \times \overline{\Omega}) \cap C^{k;0}([0, T) \times \Omega) \,:\, u|_{I\times\partial\Omega} = 0,\ \partial_t^\alpha u(0) = 0,\ \alpha = 0, ..., k\}.$$

For function spaces in time on $\bar{I}$, we shall always denote homogeneous initial conditions $u(0) = 0$ by the index "0," and homogeneous terminal conditions $u(T) = 0$ by ",0". As usual, $C_0^\infty(\Omega)$ is the space of compactly supported functions on $C^\infty(\Omega)$.

For Hilbert spaces X and a domain Ω (which could also be a time interval I or a space-time domain Q), we denote by $L^2(\Omega; X)$, $H^1(\Omega; X)$ and $H_0^1(\Omega; X)$ the Lebesgue-Bochner, Sobolev-Bochner and zero-trace Sobolev-Bochner Hilbert space (see, e.g., [14] for an introduction to Bochner spaces), endowed with the inner product $(u, v)_{L^2(\Omega;X)} := \int_\Omega (u(x), v(x))_X \,\mathrm{d}x$, $(u, v)_{H^1(\Omega;X)} := (u, v)_{L^2(\Omega;X)} + (\nabla u, \nabla v)_{L^2(\Omega;X)}$ and $(u, v)_{H_0^1(\Omega;X)} := (\nabla u, \nabla v)_{L^2(\Omega;X)}$, respectively. Further, we denote by $H^{-1}(\Omega; X) \equiv [H_0^1(\Omega; X)]'$ the dual space of $H_0^1(\Omega; X)$ and define the Hilbert space $H^\Delta(\Omega; X) := \{u \in H^1(\Omega; X) : \Delta u \in L^2(\Omega; X)\}$ endowed with $(u, v)_{H^\Delta(\Omega;X)} := (u, v)_{H^1(\Omega;X)} + (\Delta u, \Delta v)_{L^2(\Omega;X)}$. In addition, for a time interval I, it holds $H^1(I; X) \hookrightarrow C(\bar{I}; X)$ by [2, Proposition II.5.11], i.e., point evaluations $u(t) \in X$, $t \in \bar{I}$ are well-defined for $u \in H^1(I; X)$, thus we can define the Hilbert spaces

$$H_{0,}^1(I; X) := \{u \in H^1(I; X) : u(0) = 0\}, \quad H_{,0}^1(I; X) := \{u \in H^1(I; X) : u(T) = 0\},$$

endowed with the inner product $(\cdot, \cdot)_{H_0^1(I;X)}$. We write $L^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$, $H^{-1}(\Omega)$, $H^\Delta(\Omega)$, $H_{0,}^1(I)$ and $H_{,0}^1(I)$ for $L^2(\Omega; \mathbb{R})$, $H^1(\Omega; \mathbb{R})$, $H_0^1(\Omega; \mathbb{R})$, $H^{-1}(\Omega; \mathbb{R})$, $H^\Delta(\Omega; \mathbb{R})$, $H_{0,}^1(I; \mathbb{R})$ and $H_{,0}^1(I; \mathbb{R})$, respectively. Finally, for a second Hilbert space Y, we equip the vector space $X \times Y$ with the inner product $(\vec{u}, \vec{v})_{X\times Y} := (u_1, v_1)_X + (u_2, v_2)_Y$, thus forming again a Hilbert space.

1.2 Examples

We collect examples which we will reconsider in the course of this paper. All these will be expressed (in strong form) as $B_\circ u = f$ (or $B_\circ \vec{u} = \vec{f}$), including initial and/or boundary conditions.

Example 1.1 (Poisson equation)**.** *For $f : \Omega \to \mathbb{R}$ and $u : \overline{\Omega} \to \mathbb{R}$, the Poisson equation takes the form $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$.*

Example 1.2 (Heat equation)**.** *For $f : Q \to \mathbb{R}$ and $u : \overline{Q} \to \mathbb{R}$, the heat equation is expressed by $u_t - \Delta_x u = f$ in Q, $u(0) = 0$ in Ω and $u = 0$ on $I \times \partial\Omega$.*

Example 1.3 (Wave equation)**.** *For $f : Q \to \mathbb{R}$ and $u : \overline{Q} \to \mathbb{R}$, the wave equation reads $u_{tt} - \Delta_x u = f$ in Q, $u(0) = \partial_t u(0) = 0$ in Ω and $u = 0$ on $I \times \partial\Omega$.*

Example 1.4 (First-order in time wave equation)**.** *For $\vec{f} : Q \to \mathbb{R}^2$ and $\vec{u} : \overline{Q} \to \mathbb{R}^2$, the first-order in time formulation of the wave equation reads $\partial_t \vec{u} + A_\circ \vec{u} = \vec{f}$ in Q, $\vec{u}(0) = 0$ in Ω and $\vec{u} = 0$ on $I \times \partial\Omega$, where $A_\circ := \begin{pmatrix} 0 & -Id \\ -\Delta_x & 0 \end{pmatrix}$ and Id denoting the identity operator. For $\vec{f} = (0, f)$, this is equivalent to example 1.3 with $\vec{u}$ and u being related by $\vec{u} = (u, \partial_t u)$.*

Remark 1.3. *For the sake of simplicity, we only consider homogeneous initial and boundary conditions and the Laplace operator $-\Delta_x$ in space, but the above examples can be* *extend directly to (i) inhomogeneous initial and/or boundary conditions $u(0) = g_0$ in Ω, $\partial_t u(0) = g_2$ in Ω and $u = g_3$ on $\partial\Omega$ or $u = g_3$ on $I \times \partial\Omega$ using standard homogenization techniques; and (ii) a uniformly elliptic and bounded (time variant) spatial differential operator $A_\circ(t)u(x) := -\nabla_x \cdot \big(\underline{A}(t,x)\nabla_x u(x)\big) + \underline{b}(t,x) \cdot \nabla_x u(x) + \underline{c}(t,x)u(x)$ replacing $-\Delta_x$, with $\underline{A} : Q \to \mathbb{R}^{d\times d}$, $\underline{b} : Q \to \mathbb{R}^d$ and $\underline{c} : Q \to \mathbb{R}$ sufficiently smooth.*

2 Well-posed optimally stable weak formulations

We present the announced general framework for optimally stable weak formulations for linear operator equations, which we will then apply to the above mentioned example problems.

2.1 Operator equations

We are now going to formulate the class of linear operator equations that we will consider in the sequel.

Classical form

We consider linear operator equations on function spaces. Typically, those operators are defined pointwise by their mapping properties (i.e., a differential or integral operator) and corresponding initial and/or boundary conditions. This means in particular that those kind of conditions are included in the definition of the operator. In order to distinguish between the pointwise interpretation of the operator and the subsequent variational form, we add the subscript "$\circ$" to indicate the pointwise form. Hence, we start by $B_\circ : D(B_\circ) \to C(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is the domain of the primitive variables on which the seeked function is defined (which may be replaced by a space-time cylinder Q, see §1.2). Here,

$$D(B_\circ) := \{u \in C(\Omega) : \ B_\circ u \in C(\Omega)\}$$

is the *classical domain* of the operator. Then, for $f \in C(\Omega)$, the classical/pointwise formulation of an operator equation amounts seeking $u_\circ^* \in D(B_\circ)$ satisfying

$$B_\circ u_\circ^* = f \text{ in } \Omega, \quad \text{i.e.,} \quad B_\circ u_\circ^*(x) = f(x)\ \forall x \in \Omega. \tag{4}$$

Example 2.1. *With the operator equations in §1.2 their classical domains read:*

(i) $D(B_\circ) = C^2(\Omega) \cap C_0(\overline{\Omega})$ *for the Poisson equation in example 1.1;*

(ii) $D(B_\circ) = C^{1;2}(Q) \cap C^{(0)}_{0,;0}(\overline{Q})$ *for the heat equation in example 1.2;*

(iii) $D(B_\circ) = C^{2;2}(Q) \cap C^{(1)}_{0,;0}(\overline{Q})$ *for the wave equation in example 1.3;*

(iv) $D(B_\circ) = C^{1;2}(Q;\mathbb{R}^2) \cap C^{(0)}_{0,;0}(\overline{Q};\mathbb{R}^2)$ *for the first-order in time wave equation in example 1.4.*

Variational formulation

The classical form of an operator equation can only be expected to be well-posed in exceptional cases (depending on Ω and the data). Hence, we consider a variational formulation and stress the fact that such a variational form is not unique. Thus, we introduce an abstract framework for variational formulations and detail it for the examples mentioned above.

For a general framework, let U be a Banach space (called the *trial* or *ansatz space*), V be a reflexive Banach space (called the *test space*) and H be a Hilbert space, such that (V, H, V') forms a Gelfand triple, i.e. $V \hookrightarrow_d H \cong H' \hookrightarrow_d V'$. Then, a variational operator reads $B : D(B) \subseteq U \to V'$, where B is a (possibly unbounded) linear operator defined on a linear subspace $D(B) \subseteq U$, called the *domain* of B, see [3, §2.6]. The operator B is called *bounded*, if $D(B) = U$ and $B \in \mathcal{L}(U, V')$. For given $f \in V'$, an abstract variational formulation amounts seeking $u^* \in D(B)$ satisfying the operator equation

$$Bu^* = f \text{ in } V', \quad \text{i.e.,} \quad \langle Bu^*, v\rangle_{V'\times V} = \langle f, v\rangle_{V'\times V} \quad \forall v \in V. \tag{5}$$

As B is neither assumed to be bounded nor bijective, (5) is in general not well-posed.

Examples

We indicate some possible variational formulations for the examples introduced in §1.2. Generally speaking, to derive a variational formulation (5) of the classical form (4), select a Hilbert space H, typically $H := L^2(\Omega)$, multiply by a test function $v \in C_0^\infty(\Omega)$ and, in case of $H = L^2(\Omega)$, integrate over Ω, leading to

$$(B_\circ u, v)_H = (f, v)_H \qquad \forall u \in D(B_\circ),\ \forall v \in C_0^\infty(\Omega).$$

After possibly applying integration by parts on the left-hand side, this gives rise to a variational operator B, for wich the domain $D(B)$ and codomain V' are now to be determined. Starting by the codomain, V has to be a reflexive Banach space such that $V \hookrightarrow_d H$ while the relation $(B_\circ u, v)_H = \langle Bu, v\rangle_{V'\times V}$ has to hold for all $v \in V$ instead of for all $v \in C_0^\infty(\Omega)$ (and all $u \in D(B_\circ)$). Of course, this leaves some room for the choice of V, although V is typically chosen as the largest of these spaces. Next, for the domain $D(B)$ we need $D(B_\circ) \subseteq D(B)$ and it has to impose all initial and boundary conditions that have not already been imposed implicitly during the construction of B using integration by parts. Although $D(B_\circ)$ does not need to be a Banach space itself, it has to be embedded (densely for §2.3) in some Banach space U. By these assumptions, it becomes evident, that every solution $u_\circ^*$ of the classical formulation (4) also solves the variational formulation (5). Regarding our naming convention, we call a variational formulation (of a 2nd order operator) *strong*, *weak* or *ultra-weak* (in a variable), if it was derived by applying none, one or two integrations by parts (with respect to this variable).

To get a better hold on this procedure, consider Poisson's equation in example 1.1 and let $u \in D(B_\circ)$ with $D(B_\circ)$ given in example 2.1. Multiplying by $v \in C_0^\infty(\Omega)$ and integrating over Ω gives $(B_\circ u, v)_{L^2(\Omega)} = (-\Delta_x u, v)_{L^2(\Omega)}$ while further applying one or two integrations by parts (recalling that $u_{|\partial\Omega} = v_{|\partial\Omega} = 0$) yield $(B_\circ u, v)_H = (\nabla_x u, \nabla_x v)_{L^2(\Omega)} = (u, -\Delta_x v)_{L^2(\Omega)}$ for all $u \in D(B_\circ)$ and $v \in C_0^\infty(\Omega)$. This gives rise to the following strong, weak and ultra-weak operator

$$\begin{aligned} -\Delta_x^{\mathrm{st}} &: H^\Delta(\Omega) \to L^2(\Omega), & (-\Delta_x^{\mathrm{st}} u, v)_{L^2(\Omega)} &:= (-\Delta_x u, v)_{L^2(\Omega)}, \\ -\Delta_x^{\mathrm{we}} &: H^1(\Omega) \to [H^1(\Omega)]', & \langle -\Delta_x^{\mathrm{we}} u, v\rangle_{[H^1(\Omega)]' \times H^1(\Omega)} &:= (\nabla_x u, \nabla_x v)_{L^2(\Omega)}, \\ -\Delta_x^{\mathrm{uw}} &: L^2(\Omega) \to [H^\Delta(\Omega)]', & \langle -\Delta_x^{\mathrm{uw}} u, v\rangle_{[H^\Delta(\Omega)]' \times H^\Delta(\Omega)} &:= (u, -\Delta_x v)_{L^2(\Omega)}, \end{aligned} \tag{6}$$

representing the starting point for a variational operator B^{st}, B^{we} and B^{uw}, respectively. Now, we have to select a suitable domain $D(B^{\mathrm{st}}), D(B^{\mathrm{we}})$ and $D(B^{\mathrm{uw}})$ as well as a suitable codomain $[V^{\mathrm{st}}]', [V^{\mathrm{we}}]'$ and $[V^{\mathrm{uw}}]'$. Choosing the largest possible test space V while still ensuring $(B_\circ u, v)_H = \langle Bu, v\rangle_{V' \times V}$ for all $v \in V$, we end up with $V^{\mathrm{st}} := L^2(\Omega)$, $V^{\mathrm{we}} := H_0^1(\Omega)$ and $V^{\mathrm{uw}} := H^\Delta(\Omega) \cap H_0^1(\Omega)$, which are – of course – not the only possible options, but the most natural choices. Thereby, we note, that $V^{\mathrm{st}}, V^{\mathrm{we}}, V^{\mathrm{uw}} \hookrightarrow_d L^2(\Omega)$. Now, regarding the domains, we first note, that $D(B_\circ) \subseteq D(B^{\mathrm{st}}), D(B^{\mathrm{we}}), D(B^{\mathrm{uw}})$ provides a lower bound. Second, regarding the boundary conditions, we note that in context of B^{uw}, the boundary condition is already implicitly imposed by the construction of the operator in form of a vanishing boundary integral, while for B^{st} and B^{we}, the boundary condition needs to be imposed by the domain, i.e., $D(B^{\mathrm{st}}), D(B^{\mathrm{we}}) \subseteq H_0^1(\Omega)$. Hence, as an upper bound, we get $D(B^{\mathrm{st}}) \subseteq U^{\mathrm{st}} := H^\Delta(\Omega) \cap H_0^1(\Omega)$ as well as $D(B^{\mathrm{we}}) \subseteq U^{\mathrm{we}} := H_0^1(\Omega)$ and $D(B^{\mathrm{uw}}) \subseteq U^{\mathrm{uw}} := L^2(\Omega)$. Thereby, we note, that each domain is a dense subspace of the Banach space U^{st}, U^{we} and U^{uw}, respectively, as $D(B_\circ)$ is already dense in each of them. In a similar fashion, we can derive several variational formulations for each example given in §1.2, for which the following formulations will be considered in the scope of this paper.

Remark 2.1. *We stress, that $H^\Delta(\Omega) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$ if Ω is quasi-convex [11, Def. 8.9, Lem. 8.11], in particular if Ω is convex or has a smooth boundary. In this case $(\|\nabla \cdot\|_{L^2(\Omega)}^2 + \|\Delta \cdot\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$ is a norm on $H^\Delta(\Omega) \cap H_0^1(\Omega)$ equivalent to the standard $H^2(\Omega)$ norm, e.g. [12, Thm. 8.12].*

Example 2.2 (Poisson equation)**.** *For the Poisson equation in example 1.1, set $H := L^2(\Omega)$, let $D(B_\circ)$ as given in example 2.1 and consider the formulations*

(i) strong*: $V := L^2(\Omega)$ and $U := H^\Delta(\Omega) \cap H_0^1(\Omega)$ with $B := -\Delta_x^{\mathrm{st}}|_{D(B)} : D(B) \subseteq U \to V'$ for any domain $D(B)$ such that $D(B_\circ) \subseteq D(B) \subseteq U$;*

(ii) weak*: $V := H_0^1(\Omega)$ and $U := H_0^1(\Omega)$ with $B := -\Delta_x^{\mathrm{we}}|_{D(B)} : D(B) \subseteq U \to V'$ for any domain $D(B)$ such that $D(B_\circ) \subseteq D(B) \subseteq U$;*

(iii) ultra-weak*: $V := H^\Delta(\Omega) \cap H_0^1(\Omega)$ and $U := L^2(\Omega)$ with $B := -\Delta_x^{\mathrm{uw}}|_{D(B)} : D(B) \subseteq U \to V'$ for any domain $D(B)$ such that $D(B_\circ) \subseteq D(B) \subseteq U$.*

Thereby, we equip $H^\Delta(\Omega) \cap H_0^1(\Omega)$ with $(\|\nabla \cdot\|_{L^2(\Omega)}^2 + \|\Delta \cdot\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$ as norm. Further holds in all three cases, that $D(B) \subseteq_d U$ as $D(B_\circ)$ is already dense in U.

Example 2.3 (Heat equation – strong in time)**.** *For the heat equation given in example 1.2, set $H := L^2(Q)$ and define the Hilbert spaces $V := L^2(I; H_0^1(\Omega))$ and $U := L^2(I; H_0^1(\Omega)) \cap$*

$H^1_{0,}(I; H^{-1}(\Omega))$ *with their corresponding norms* $\|\cdot\|_V := \|\nabla_x \cdot\|_{L^2(Q)}$ *and* $\|\cdot\|_U^2 := \|\cdot\|_V^2 + \|\partial_t \cdot\|_{V'}^2$*, respectively. We consider the variational operator* $B : U \to V'$ *arising from* $B_\circ$ *in example 1.2 by one integration by parts in space, for all* $u \in U$ *and* $v \in V$ *given by* $\langle Bu, v\rangle_{V'\times V} := (\partial_t u, v)_{L^2(Q)} + \langle -\Delta_x^{\mathrm{we}} u, v\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H^1_0(\Omega))}$*, see also [6, 17].*

Example 2.4 (Heat equation – weak in time)**.** *For the heat equation given in example 1.2, set* $H := L^2(Q)$ *and define the Hilbert spaces* $U := L^2(I; H^1_0(\Omega))$ *and* $V := L^2(I; H^1_0(\Omega)) \cap H^1_{,0}(I; H^{-1}(\Omega))$ *with their corresponding norms* $\|\cdot\|_U := \|\nabla_x \cdot\|_{L^2(Q)}$ *and* $\|\cdot\|_V^2 := \|\cdot\|_U^2 + \|\partial_t \cdot\|_{U'}^2$*, respectively. We consider the variational operator* $B : U \to V'$ *arising from* $B_\circ$ *in example 1.2 by one integration by parts in time and one in space, for all* $u \in U$ *and* $v \in V$ *given by* $\langle Bu, v\rangle_{V'\times V} := -(u, \partial_t v)_{L^2(Q)} + \langle -\Delta_x^{\mathrm{we}} u, v\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H^1_0(\Omega))}$*, see also [1, 4].*

Example 2.5 (Wave equation – strong in time)**.** *For the first order in time reformulation of the wave equation given in example 1.4, set* $H := L^2(Q; \mathbb{R}^2)$ *and define the Hilbert spaces* $U := H^1_{0,}(I; L^2(\Omega) \times H^{-1}(\Omega)) \cap L^2(I; H^1_0(\Omega) \times L^2(\Omega))$ *and* $V := L^2(I; L^2(\Omega) \times H^1_0(\Omega))$ *with their corresponding norms* $\|\cdot\|_V := \|\cdot\|_{L^2(I;L^2(\Omega)\times H^1_0(\Omega))}$ *and* $\|\cdot\|_U^2 := \|\partial_t \cdot\|^2_{L^2(I;L^2(\Omega)\times H^{-1}(\Omega))} + \|\cdot\|^2_{L^2(I;H^1_0(\Omega)\times L^2(\Omega))}$*, respectively. Introducing the operator* $A := \left(\begin{smallmatrix} 0 & -Id \\ -\Delta_x^{\mathrm{we}} & 0 \end{smallmatrix}\right) : H^1_0(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times H^{-1}(\Omega)$*, we consider the variational operator* $B : U \to V'$ *araising from* $B_\circ$ *in example 1.4 by one integration by parts in space, for all* $\vec{u} \in U$ *and* $\vec{v} \in V$ *given by*

$$\begin{aligned}\langle B\vec{u}, \vec{v}\rangle_{V'\times V} &:= \langle \partial_t \vec{u} + A\vec{u}, \vec{v}\rangle_{V'\times V} \\ &= (\partial_t u_1 - u_2, v_1)_{L^2(Q)} + \langle \partial_t u_2 - \Delta_x^{\mathrm{we}} u_1, v_2\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H^1_0(\Omega))}.\end{aligned}$$

Example 2.6 (Wave equation – weak in time)**.** *For the wave equation given in example 1.3, set* $H := L^2(Q)$ *and define the Hilbert spaces* $U := H^1_{0,}(I; H^1_0(\Omega))$ *and* $V := H^1_{,0}(I; H^1_0(\Omega))$ *with norms* $\|\cdot\|_U^2 \equiv \|\cdot\|_V^2 := \|\partial_t \cdot\|^2_{L^2(Q)} + \|\nabla_x \cdot\|^2_{L^2(Q)}$*. We consider the variational operator* $B : U \to V'$ *arising from* $B_\circ$ *in example 1.3 by one integration by parts in time and one in space, for all* $u \in U$ *and* $v \in V$ *given by*

$$\langle Bu, v\rangle_{V'\times V} := -(\partial_t u, \partial_t v)_{L^2(Q)} + \langle -\Delta_x^{\mathrm{we}} u, v\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H^1_0(\Omega))}. \tag{7}$$

Example 2.7 (Wave equation – ultra-weak in time)**.** *For the first order in time reformulation of the wave equation given in example 1.4, set* $H := L^2(Q; \mathbb{R}^2)$ *and define the Hilbert spaces* $U := L^2(I; H^1_0(\Omega) \times L^2(\Omega))$ *and* $V := H^1_{,0}(I; H^{-1}(\Omega) \times L^2(\Omega)) \cap L^2(I; L^2(\Omega) \times H^1_0(\Omega))$ *with their corresponding norms* $\|\cdot\|_U := \|\cdot\|_{L^2(I;H^1_0(\Omega)\times L^2(\Omega))}$ *and* $\|\cdot\|_V^2 := \|\partial_t \cdot\|^2_{L^2(I;H^{-1}(\Omega)\times L^2(\Omega))} + \|\cdot\|^2_{L^2(I;L^2(\Omega)\times H^1_0(\Omega))}$*, respectively. Using, that* $-\Delta_x^{\mathrm{we}}$ *and* Id *are self-adjoint, the adjoint of* A *given in example 2.5 reads* $A^* = \left(\begin{smallmatrix} 0 & -\Delta_x^{\mathrm{we}} \\ -Id & 0 \end{smallmatrix}\right) : L^2(\Omega) \times H^1_0(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$ *and we consider the variational operator* $B : U \to V'$ *araising from* $B_\circ$ *in example 1.4 by one integration by parts in time (due to the first order reformulation, this effectively corresponds to two integration by parts in time, thus the name* ultra-weak in time*) and one in space, for all* $\vec{u} \in U$ *and* $\vec{v} \in V$ *given by*

$$\begin{aligned}\langle B\vec{u}, \vec{v}\rangle_{V'\times V} &:= \langle \vec{u}, -\partial_t \vec{v} + A^*\vec{v}\rangle_{U\times U'} \\ &= \langle u_1, -\partial_t v_1 - \Delta_x^{\mathrm{we}} v_2\rangle_{L^2(I;H^1_0(\Omega))\times L^2(I;H^{-1}(\Omega))} + (u_2, -\partial_t v_2 - v_1)_{L^2(Q)}.\end{aligned}$$

Remark 2.2. *The above variational formulations involving $-\Delta_x^{\text{we}}$ can be extended directly to an arbitrary bounded and coercive (time variant) operator $A(t) : H_0^1(\Omega) \to H^{-1}(\Omega)$ instead of $-\Delta_x^{\text{we}}$. In particular, for the elliptic operator $A_\circ$ defined in remark 1.3 (omitting the time variance for brevity), we end up with the strong, weak and ultra weak operator*

$$\begin{aligned}
(A^{\text{st}}u, v)_{L^2(\Omega)} &\coloneqq (-\nabla_x \cdot (\underline{A}\nabla_x u) + \underline{b} \cdot \nabla_x u + \underline{c}u, v)_{L^2(\Omega)}, \\
\langle A^{\text{we}}u, v\rangle_{[H^1(\Omega)]' \times H^1(\Omega)} &\coloneqq (\underline{A}\nabla_x u, \nabla_x v)_{L^2(\Omega)} + (\underline{b} \cdot \nabla_x u, v)_{L^2(\Omega)} + (\underline{c}u, v)_{L^2(\Omega)}, \\
\langle A^{\text{uw}}u, v\rangle_{[H^2(\Omega)]' \times H^2(\Omega)} &\coloneqq (u, -\nabla_x \cdot (\underline{A}^\top \nabla_x v) - \underline{b} \cdot \nabla_x v + (\underline{c} - \nabla_x \cdot \underline{b})v)_{L^2(\Omega)},
\end{aligned}$$

respectively, replacing (6) *– assuming that $\underline{A}$, $\underline{b}$ and $\underline{c}$ are sufficiently smooth.*

These examples already show that there is a huge variety regarding the choice of the test space V and the variational formulation (5), recalling that well-posedness is not required as (5) only serves as a starting point.

2.2 Completion and extension

Our goal is to *construct a well-posed and optimally stable extension* $\overline{B} : \overline{U} \to V'$ of (5). Thereby, an extension of (5) means that $D(B) \subseteq_d \overline{U}$ and $\overline{B}|_{D(B)} = B$. To this end, it turns out that the following two properties are crucial:

(B1) B is injective on $D(B)$;

(B2) the range $R(B)$ of B is dense in V' (weak surjectivity of B).

We first note a sufficient condition for (B1) and (B2), namely that (5) is well-posed and stable on a dense subspace $Y \subseteq_d V'$ of right-hand sides.

Lemma 2.1. *Let $Y \subseteq_d V'$ be a normed subspace and $C < \infty$. If* (5) *admits a solution $u^* \in D(B)$ for each $f \in Y$ satisfying the stability estimate $\|u^*\|_U \leq C\|f\|_Y$, then (B1) and (B2) are valid.*

Proof. Consider the operator equation (5) for $f = 0$. Since Y is a subspace, we have $0 \in Y$, i.e., there exists some $u^* \in D(B)$ such that $Bu^* = 0$ in V' and $\|u^*\|_U \leq C\|f\|_Y = 0$ by assumption. Thus, if $Bu^* = 0$ then $u^* = 0$, i.e., the solution of the homogeneous problem is unique and hence $\ker(B) = \{0\}$, i.e., B is injective on $D(B)$, i,e., (B1). Since Y is dense in V' and $Y \subseteq R(B) \subseteq V'$, we conclude that $R(B)$ is dense in V', i.e., (B2). □

Remark 2.3. *Using that (V, H, V') forms a Gelfand triple together with remark 1.1, possible choices for Y include in particular $Y \subseteq_d H$ and $Y \subseteq_d V$. In fact, this is the main reason, why the Gelfand triple (V, H, V') is important as it is often sufficient to consider the operator equation for smooth right-hand sides, where existence results are typically easier to derive.*

If (B1) holds, then it is easy to see, that $\|\cdot\|_{D(B)} := \|B\cdot\|_{V'}$ defines a norm on $D(B)$. Thus, there exists a (up to isometric isomorphisms) uniquely determined completion $(\overline{U}, \|\cdot\|_{\overline{U}})$ of $(D(B), \|\cdot\|_{D(B)})$, e.g. [7, Theorem 2.32 & Corollary 2.60]. Furthermore, there exists a uniquely determined continuous extension $\overline{B} \in \mathcal{L}(\overline{U}, V')$ of B from $D(B)$ to $\overline{U}$, e.g. [7, Proposition 2.59]. By definition, $\overline{U}$ is a Banach space, $D(B) \subseteq_d \overline{U}$ and $\overline{B}|_{D(B)} = B$. Furthermore, it is easy to see[3], that

$$\|\bar{u}\|_{\overline{U}} = \|\overline{B}\bar{u}\|_{V'},\ \bar{u} \in \overline{U} \quad \text{and} \quad \|u\|_{\overline{U}} = \|u\|_{D(B)} = \|Bu\|_{V'},\ u \in D(B). \tag{8}$$

Now, for $f \in V'$, we seek the solution $\bar{u}^* \in \overline{U}$ of the extended operator equation

$$\overline{B}\bar{u}^* = f \text{ in } V', \quad \text{i.e.,} \quad \langle \overline{B}\bar{u}^*, v\rangle_{V'\times V} = \langle f, v\rangle_{V'\times V} \quad \forall v \in V. \tag{9}$$

By $\overline{B}|_{D(B)} = B$ and $D(B) \subseteq \overline{U}$, it is easy to see, that every solution u^* of (5) also solves (9), but not the other way around. Hence, the extended problem (9) is a weaker form of (5).

Remark 2.4. *(i) If B is bounded (i.e., $D(B) = U$ and $B \in \mathcal{L}(U, V')$), it holds*

$$U \hookrightarrow_d \overline{U}, \qquad \text{with} \qquad \|u\|_{\overline{U}} \leq \|B\|_{\mathcal{L}(U,V')}\|u\|_U \quad \forall u \in U. \tag{10}$$

(ii) The space $\overline{U}$ consists of the limits of all Cauchy sequences in $(D(B), \|\cdot\|_{D(B)})$. The extension $\overline{B}$ reads $\overline{B}\bar{u} = \lim_{n\to\infty} Bu_n$ for the limit $\bar{u} := \lim_{n\to\infty} u_n$ of a Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset D(B)$. Note, that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $D(B)$ if and only if $(Bu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V', i.e., $\overline{B}$ is well-defined, since V' is complete.

(iii) The extended problem (9) is independent of the chosen variational formulation (5) for the classical problem (4), as long as the test space V is fixed. In fact, the completion and the continuous extension are both unique. Thus, for any other operator $B^\star : D(B^\star) \subseteq U^\star \to V'$ satisfying (B1) with $D(B^\star) \subseteq_d \overline{U}$ and $\overline{B}|_{D(B^\star)} = B^\star$ we have $\overline{U^\star} \equiv \overline{U}$ and $\overline{B^\star} \equiv \overline{B}$, with $\overline{U^\star}$ and $\overline{B^\star}$ denoting the completion and continuous extension of $D(B^\star)$ and $B^\star$, respectively. See also remark 3.1 below.

We now state our main result that the extended operator $\overline{B}$ is an isometric isomorphism, i.e., the operator equation (9) is well-posed and optimally stable.

Theorem 2.2. *Let (B1) and (B2) hold. The operator $\overline{B}$ from (9) is*
(a) an isomorphism, i.e., $\overline{B} \in \mathcal{L}_{\text{is}}(\overline{U}, V')$ (well-posedness) and
(b) isometric, i.e., $\|\overline{B}\|_{\mathcal{L}(\overline{U},V')} = \|\overline{B}^{-1}\|_{\mathcal{L}(V',\overline{U})} = 1$ (optimal stability).

Proof. First, note that $\overline{U}$ and $\overline{B}$ are well-defined thanks to (B1). By (8) we have that the operator $\overline{B} : \overline{U} \to R(\overline{B}) \subseteq V'$ is an isometry, thus injective and $\|\overline{B}\|_{\mathcal{L}(\overline{U},V')} = 1$. It remains

[3] By the representations of $\|\cdot\|_{\overline{U}}$ and $\overline{B}$ given in the proofs of [7, Theorem 2.32 & Proposition 2.59], respectively, it holds for all $\bar{u} \in \overline{U}$, that $\|\bar{u}\|_{\overline{U}} = \lim_{n\to\infty}\|Bu_n\|_{V'} = \|\lim_{n\to\infty} Bu_n\|_{V'} = \|\overline{B}\bar{u}\|_{V'}$, with $(u_n)_{n\in\mathbb{N}} \subset D(B)$ denoting any sequence with $\lim_{n\to\infty} u_n = \bar{u}$ in $\overline{U}$ (note that $D(B) \subseteq_d \overline{U}$).

to shown, that (i) $\overline{B}$ is surjective, i.e., $R(\overline{B}) = V'$, and (ii) $\|\overline{B}^{-1}\|_{\mathcal{L}(V',\overline{U})} = 1$. To show the surjectivity, let $f \in V'$ be arbitrary but fixed. By (B2), the range of B is dense in V'. Hence, there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset D(B)$, such that $\lim_{n\to\infty}\|f - Bu_n\|_{V'} = 0$. Thus, for all $n, m \in \mathbb{N}$

$$\|u_n - u_m\|_{\overline{U}} \overset{(8)}{=} \|u_n - u_m\|_{D(B)} = \|B(u_n - u_m)\|_{V'} \leq \|Bu_n - f\|_{V'} + \|Bu_m - f\|_{V'} \;\to\; 0$$

as $n, m \to \infty$, i.e., $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\overline{U}$. Hence, by the completeness of $\overline{U}$, there exists $\bar{u} \in \overline{U}$ with $\lim_{n\to\infty}\|\bar{u} - u_n\|_{\overline{U}} = 0$, and it holds

$$\|f - \overline{B}\bar{u}\|_{V'} \leq \|f - Bu_n\|_{V'} + \|Bu_n - \overline{B}\bar{u}\|_{V'} = \|f - Bu_n\|_{V'} + \|u_n - \bar{u}\|_{\overline{U}} \xrightarrow{n\to\infty} 0,$$

i.e., $f = \overline{B}\bar{u} \in R(\overline{B})$ so that $V' \subseteq R(\overline{B}) \subseteq V'$, where the second inclusion follows from (B2). Hence we conclude $V' = R(\overline{B})$, i.e., (i). In order to show (ii), we have

$$\|\overline{B}^{-1}\|_{\mathcal{L}(V',\overline{U})} = \sup_{f\in V'\setminus\{0\}} \frac{\|\overline{B}^{-1}f\|_{\overline{U}}}{\|f\|_{V'}} = \sup_{f\in V'\setminus\{0\}} \frac{\|\overline{B}\,\overline{B}^{-1}f\|_{V'}}{\|f\|_{V'}} = \sup_{f\in V'\setminus\{0\}} \frac{\|f\|_{V'}}{\|f\|_{V'}} = 1.$$

In particular, the inverse $\overline{B}^{-1} : V' \to \overline{U}$ is bounded (even isometric) and thus $\overline{B} \in \mathcal{L}_{\text{is}}(\overline{U}, V')$, which concludes the proof. □

Corollary 2.3. *Under the assumptions of theorem 2.2, we have, that*
(a) the trial space $\overline{U}$ is reflexive as the test space V is reflexive;
(b) the trial space $\overline{U}$ is a Hilbert space w.r.t. the inner product $(\cdot,\cdot)_{\overline{U}} := (\overline{B}\cdot, \overline{B}\cdot)_{V'}$ if and only if the test space V is a Hilbert space.

Proof. theorem 2.2 yields that $\overline{U} \cong V' \cong V$ are isomorphic. □

Remark 2.5. *(i) The identity $\|\bar{u}^*\|_{\overline{U}} = \|f\|_{V'}$ is of particular interest in numerical applications as it leads to the error-residual identity $\|\bar{u} - \bar{u}^*\|_{\overline{U}} = \|\overline{B}\bar{u} - f\|_{V'}$ for all $\bar{u} \in \overline{U}$. This will be explored in Part II.*

(ii) The operator B and hence also $\overline{B}$ are often defined in terms of a bilinear ($\mathbb{F} = \mathbb{R}$) or sesquilinear ($\mathbb{F} = \mathbb{C}$) form $b : D(B) \times V \to \mathbb{F}$ and $\bar{b} : \overline{U} \times V \to \mathbb{F}$, respectively. Then, theorem 2.2 states, that $\bar{b}$ is continuous and inf-sup stable, with

$$\left.\begin{array}{c}\bar{\gamma}\\[2ex] \bar{\beta}\\[2ex] \bar{\beta}^*\end{array}\right\} := \left\{\begin{array}{c}\displaystyle\sup_{\bar{u}\in\overline{U}\setminus\{0\}}\;\sup_{v\in V\setminus\{0\}}\\ \displaystyle\inf_{\bar{u}\in\overline{U}\setminus\{0\}}\;\sup_{v\in V\setminus\{0\}}\\ \displaystyle\inf_{v\in V\setminus\{0\}}\;\sup_{\bar{u}\in\overline{U}\setminus\{0\}}\end{array}\right\} \frac{|\bar{b}(\bar{u}, v)|}{\|\bar{u}\|_{\overline{U}}\|v\|_V} = \left\{\begin{array}{c}\|\overline{B}\|_{\mathcal{L}(\overline{U},V')}\\[2ex] \|\overline{B}^{-1}\|^{-1}_{\mathcal{L}(V',\overline{U})}\\[2ex] \|\overline{B}^{-*}\|^{-1}_{\mathcal{L}(\overline{U}',V)}\end{array}\right\} = 1,$$

where $\overline{B}^{-*} := (\overline{B}^*)^{-1}$ denotes the inverse of the adjoint of $\overline{B}$.

Let us add a note regarding the norm $\|\cdot\|_{\overline{U}} = \|\overline{B}\cdot\|_{V'}$ on $\overline{U}$. To this end, assume that V is a Hilbert space and denote the Riesz operator (1) by $R_V : V \to V'$. Now, for $\bar{u} \in \overline{U}$, defining $v_{\bar{u}} := R_V^{-1}\overline{B}\bar{u} \in V$, it holds $\|v_{\bar{u}}\|_V = \|R_V^{-1}\overline{B}\bar{u}\|_V = \|\overline{B}\bar{u}\|_{V'}$ and

$$\|\bar{u}\|_{\overline{U}}^2 = \|\overline{B}\bar{u}\|_{V'}^2 = (\overline{B}\bar{u}, \overline{B}\bar{u})_{V'} = \langle \overline{B}\bar{u}, R_V^{-1}\overline{B}\bar{u}\rangle_{V'\times V} = \langle \overline{B}\bar{u}, v_{\bar{u}}\rangle_{V'\times V}. \tag{11}$$

Thus, $v_{\bar{u}}$ is a supremizer of $\|\overline{B}\bar{u}\|_{V'}$. Before we continue, let us summarize the procedure how to derive an optimally stable well-posed formulation:

1. Starting from the classical formulation $B_\circ : D(B_\circ) \to C(\Omega)$, find a Gelfand triple $V \hookrightarrow_d H \cong H' \hookrightarrow_d V'$ for the test space and a variational form $Bu = f$ in V' for $B : D(B) \subseteq U \to V'$, being a (possibly unbounded) linear operator defined on a linear subspace $D(B)$ of a Banach space U.
2. Either show (B1) and (B2) or find a normed subspace $Y \subseteq_d V'$ such that $Bu = f$ is well-posed and stable on Y (i.e., only for $f \in Y$).
3. Build the unique completion $(\overline{U}, \|\cdot\|_{\overline{U}})$ of $D(B)$ w.r.t. $\|\cdot\|_{D(B)} := \|B\cdot\|_{V'}$.
4. Build the unique continuous extension $\overline{B} \in \mathcal{L}(\overline{U}, V')$ of B form $D(B)$ to $\overline{U}$.

$\Rightarrow$ $\overline{B} \in \mathcal{L}_{\text{is}}(\overline{U}, V')$ (well-posedness)
and $\|\overline{B}\|_{\mathcal{L}(\overline{U},V')} = \|\overline{B}^{-1}\|_{\mathcal{L}(V',\overline{U})} = 1$ (optimal stability).

2.3 Gelfand triple for the trial space

The sufficient condition formulated in lemma 2.1 and required for theorem 2.2 relies on (V, H, V'), i.e., a Gelfand triple for the test space. Even more can be said, if also the trial space admits a Gelfand triple structure, which is in fact the case for all examples 2.2 to 2.7, see example 2.8 below. To fix the notation, let G be a Hilbert space, such that (U, G, U') forms a Gelfand triple. It would be beneficial, if the Gelfand triple structure would carry over from U to $\overline{U}$. It turns out that this can be achieved by an assumption similar to lemma 2.1 (B is well-posed and stable on a dense subspace $Y \subseteq_d V'$), but for the adjoint operator B^*. In order to define B^*, B needs to be densely defined, i.e., $D(B) \subseteq_d U$. Thus, we consider the following embeddings

$$D(B) \subseteq_d U \hookrightarrow_d G \cong G' \hookrightarrow U'. \tag{12}$$

Then, following e.g. [3, §2.6], there exists a unique adjoint operator $B^* : D(B^*) \subseteq_d V \to U'$ of the (possibly unbounded) operator $B : D(B) \subseteq_d U \to V'$, with its domain

$$D(B^*) := \left\{ v \in V \ : \sup_{u \in D(B)\setminus\{0\}} \frac{|\langle Bu, v\rangle_{V'\times V}|}{\|u\|_U} < \infty \right\} \tag{13}$$

being dense in V (even $D(B^*) = V$ if B is bounded), and

$$\langle B^*v, u\rangle_{U'\times U} = \langle Bu, v\rangle_{V'\times V} \qquad \forall u \in D(B),\ \forall v \in D(B^*). \tag{14}$$

Theorem 2.4. *Let (B1), (B2) and* (12) *hold,* $X \subseteq_d G$ *be a subset and* $C^* < \infty$*. If the adjoint problem of* (5)*, namely to find* $v^* \in D(B^*)$ *such that* $B^* v^* = g$ *in* U'*, admits a solution for each* $g \in X$ *and the solution satisfies the stability estimate* $\|v^*\|_V \leq C^* \|g\|_G$*, then*

$$\overline{U} \hookrightarrow_d G \cong G' \hookrightarrow_d \overline{U}' \qquad \text{with} \qquad \|\bar{u}\|_G \leq C^* \|\bar{u}\|_{\overline{U}} \quad \forall \bar{u} \in \overline{U}. \tag{15}$$

Proof. First, note that B^* is well-defined by $D(B) \subseteq_d U$, while $\overline{U}$ and $\overline{B}$ are well-defined by (B1). We denote by $\mathcal{V} := \{v \in D(B^*) : B^* v \in X\} \subseteq D(B^*) \subseteq V$ the set of all solutions of the adjoint problem with right-hand sides in X. Thus, in particular $\|v\|_V \leq C^* \|B^* v\|_G$ for all $v \in \mathcal{V}$ by the stability of the adjoint problem and $B^*(\mathcal{V}) = X$ as there exists a solution for all right-hand sides in X. Thus, we get for all $u \in D(B) \subseteq G$, that

$$\begin{aligned}
\|u\|_{D(B)} &\overset{(8)}{=} \|Bu\|_{V'} = \sup_{v \in V} \frac{|\langle Bu, v\rangle_{V' \times V}|}{\|v\|_V} \geq \sup_{v \in \mathcal{V}} \frac{|\langle Bu, v\rangle_{V' \times V}|}{\|v\|_V} \geq \sup_{v \in \mathcal{V}} \frac{|\langle Bu, v\rangle_{V' \times V}|}{C^* \|B^* v\|_G} \\
&\overset{(14)}{=} \sup_{v \in \mathcal{V}} \frac{|\langle B^* v, u\rangle_{U' \times U}|}{C^* \|B^* v\|_G} = \sup_{g \in X} \frac{|\langle g, u\rangle_{U' \times U}|}{C^* \|g\|_G} \overset{(3)}{=} \sup_{g \in X} \frac{|(g, u)_G|}{C^* \|g\|_G} \overset{(*)}{=} \sup_{g \in G} \frac{|(g, u)_G|}{C^* \|g\|_G} \\
&\geq \frac{|(u, u)_G|}{C^* \|u\|_G} = \frac{1}{C^*} \|u\|_G,
\end{aligned}$$

where we used in $(*)$, that X is dense in G (w.r.t. $\|\cdot\|_G$). Thus, the graph-type norm defined by $\|\cdot\|_B^2 := \|\cdot\|_G^2 + \|B\cdot\|_{V'}^2$ of B is equivalent to $\|\cdot\|_{D(B)}$ on $D(B)$ since for all $u \in D(B)$

$$\|u\|_{D(B)}^2 := \|Bu\|_{V'}^2 \leq \|u\|_B^2 = \|u\|_G^2 + \|Bu\|_{V'}^2 \leq (1 + C^{*2}) \|Bu\|_{V'}^2 = (1 + C^{*2}) \|u\|_{D(B)}^2.$$

Denoting by $(\widetilde{U}, \|\cdot\|_{\widetilde{U}})$ the completion of $D(B)$ w.r.t. $\|\cdot\|_B$ instead of $\|\cdot\|_{D(B)}$, we have $\widetilde{U} \subseteq G$ by the definition of $\|\cdot\|_B$ and the completeness of G. Moreover, $\widetilde{U} = \overline{U}$ (as sets) by the equivalence of $\|\cdot\|_B$ and $\|\cdot\|_{D(B)}$. Due to the definitions of $\|\cdot\|_B$ and $\|\cdot\|_{D(B)}$, we obtain that $\|\cdot\|_{\widetilde{U}}^2 = \|\cdot\|_G^2 + \|\overline{B}\cdot\|_{V'}^2$. Now, let $\bar{u} \in \overline{U}$ be arbitrary but fixed. By $\bar{u} \in \widetilde{U}$ and $D(B) \subseteq_d \widetilde{U}$ there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset D(B)$ with $0 = \lim_{n \to \infty} \|\bar{u} - u_n\|_{\widetilde{U}}^2 = \lim_{n \to \infty} (\|\bar{u} - u_n\|_G^2 + \|\bar{u} - u_n\|_{\overline{U}}^2)$, i.e., $\lim_{n \to \infty} u_n = \bar{u}$ w.r.t. $\|\cdot\|_G$ and $\|\cdot\|_{\overline{U}}$. Thus, $\|\bar{u}\|_{\overline{U}} = \lim_{n \to \infty} \|u_n\|_{\overline{U}} = \lim_{n \to \infty} \|u_n\|_{D(B)} \geq \lim_{n \to \infty} \frac{\|u_n\|_G}{C^*} = \frac{\|\bar{u}\|_G}{C^*}$, and we conclude $\overline{U} \hookrightarrow G$. Finally, $\overline{U} \subseteq_d G$ follows by $D(B) \subseteq \overline{U} \subseteq G$, and the fact that $D(B)$ is dense in G by (12) and remark 1.1. Finally, $G' \hookrightarrow_d \overline{U}'$ holds by remark 1.2 and the fact that $\overline{U}$ is reflexive by corollary 2.3. □

Remark 2.6. *By* (12) *and remark 1.1, possible choices for* X *include* $X \subseteq_d G$*,* $X \subseteq_d U$ *and* $X \subseteq_d D(B)$*, where the density in the latter is taken w.r.t.* $\|\cdot\|_U$ *or* $\|\cdot\|_G$ *but not* $\|\cdot\|_{D(B)}$*. Note that the stability estimate for the adjoint problem has to hold w.r.t.* $\|\cdot\|_G$*, unlike to lemma 2.1, where the norm could be chosen arbitrary.*

Remark 2.7. *(i) If* B *is bounded, it holds* $U \hookrightarrow_d \overline{U} \hookrightarrow_d G \cong G' \hookrightarrow_d \overline{U}' \hookrightarrow_d U'$*, using* (10) *and* (15)*.*

(ii) By (15)*, the graph-type norm* $\|\cdot\|_{\overline{B}}^2 := \|\cdot\|_G^2 + \|\overline{B}\cdot\|_{V'}^2$ *is equivalent to* $\|\cdot\|_{\overline{U}}$*.*

Let us summarize the procedure.

5. Check if the trial space admits a Gelfand structure $U \hookrightarrow_d G \cong G' \hookrightarrow U'$.
6. Find a dense subset $X \subseteq_d G$ such that the adjoint problem $B^* v = g$ in U' is well-posed and stable on X (i.e. only for $g \in X$).

$\Rightarrow$ The extended trial space admits a Gelfand structure $\overline{U} \hookrightarrow_d G \cong G' \hookrightarrow_d \overline{U}'$.

Remark 2.8. *Note that theorem 2.4 implies (B1) and (B2) but for B^* instead of B by a similar argument to that in lemma 2.1. These are precisely the assumption (A^*1) and (A^*2) in [5]. There, (A^*1) and (A^*2) are used to construct a well-posed extension of B^*, while we use these assumptions to add additional structure to an existing extension instead.*

Example 2.8. *For the variational formulations in examples 2.2 to 2.7, we have that $G := L^2(\Omega)$ for the Poisson equation, $G = L^2(Q)$ for the heat equation and the weak in time form of the wave equation as well as $G = L^2(Q, \mathbb{R}^2)$ for the strong and ultra-weak in time form of the wave equation.*

3 Applications

Now, with the abstract operator framework at hand, let us consider the examples introduced in §1.2, in particular, their variational formulations as given in examples 2.2 to 2.7 (together with the pivot space G as given in example 2.8). For each of these formulations, we are now going to build their well-posed and optimally stable extension (9) and characterize the extended trial space $\overline{U}$ and the extended operator $\overline{B}$ as far as possible.

3.1 The Poisson equation

It is well known by Kellogg's theorem, e.g. [12, Theorem 6.14], that – under suitable regularity assumptions on the domain Ω – the classical formulation of the Poisson problem given in example 1.1 has a unique solution[4] $u^* \in C_0^{2,\alpha}(\overline{\Omega})$ satisfying $\|u^*\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{0,\alpha}(\overline{\Omega})}$ for all right-hand sides $f \in C^{0,\alpha}(\overline{\Omega})$ and some $C < \infty$. As it holds $C_0^\infty(\Omega) \subset C_0^{k,\alpha}(\overline{\Omega}) \subset C_0^k(\Omega) \subset L^2(\Omega)$, we conclude, that $u^* \in D(B_\circ)$ and, by the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, that $C^{0,\alpha}(\overline{\Omega})$ is dense in $L^2(\Omega)$. Thus, u^* is a solution for each variational formulation given in example 2.2 and for each formulation holds theorem 2.2 by lemma 2.1 and remark 2.3. We will now see, that the abstract framework introduced in §2 recovers the well-known formulations for the Poisson problem.

3.1.1 Strong formulation

Consider the operator $B := -\Delta_x^{\mathrm{st}}|_{D(B)} : D(B) \subseteq_d H^\Delta(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ given in example 2.2 (i). Then, writing $-\Delta \equiv -\Delta_x^{\mathrm{st}}$, it holds for all $u \in D(B)$, that $\|Bu\|_{V'} = \|-\Delta_x^{\mathrm{st}}|_{D(B)} u\|_{L^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)}$, while the norm on $U := H^\Delta(\Omega) \cap H_0^1(\Omega)$ reads $\|\cdot\|_U^2 =$

[4] $C^{k,\alpha}(\overline{\Omega})$, $k \in \mathbb{N}$, is the Hölder space with Hölder exponent $\alpha \in (0, 1]$; $C_0^{k,\alpha}(\overline{\Omega}) = C^{k,\alpha}(\overline{\Omega}) \cap C_0(\overline{\Omega})$.

$\|\nabla\cdot\|^2_{L^2(\Omega)} + \|\Delta\cdot\|^2_{L^2(\Omega)}$. Note, that the norm induced by B and the norm on U are equivalent, as it holds by integration by parts, the Cauchy–Schwarz inequality and Poincaré's inequality, that

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega)} = (\nabla u, \nabla u)_{L^2(\Omega)} &= (u, -\Delta u)_{L^2(\Omega)} + \underbrace{(u, \vec{n}\cdot\nabla u)_{L^2(\partial\Omega)}}_{=0 \text{ as } u=0 \text{ on } \partial\Omega} \\ &\le \|u\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} \le C_\Omega\|\nabla u\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)},\end{aligned}$$

for all $u \in H^\Delta(\Omega) \cap H^1_0(\Omega)$, where $C_\Omega < \infty$ denotes the Poincaré constant on Ω. Dividing by $\|\nabla u\|_{L^2(\Omega)}$ then gives the norm equivalence as it holds for all $u \in H^\Delta(\Omega) \cap H^1_0(\Omega)$, that

$$\|\Delta u\|^2_{L^2(\Omega)} \le \|\nabla u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \le (1 + C^2_\Omega)\|\Delta u\|^2_{L^2(\Omega)}. \tag{16}$$

As $D(B) \subseteq_d H^\Delta(\Omega) \cap H^1_0(\Omega)$ and $H^\Delta(\Omega) \cap H^1_0(\Omega)$ is complete with respect to $\|\nabla\cdot\|_{L^2(\Omega)} + \|\Delta\cdot\|_{L^2(\Omega)}$, it holds for the well-posed and optimally stable completion (9), that $\overline{U} = H^\Delta(\Omega) \cap H^1_0(\Omega)$ and $\overline{B} = -\Delta^{\mathrm{st}}_x$.

3.1.2 Weak formulation

Consider the operator $B \coloneqq -\Delta^{\mathrm{we}}_x|_{D(B)} : D(B) \subseteq_d H^1_0(\Omega) \to H^{-1}(\Omega)$ given in example 2.2 (ii). As the isometric Riesz operator (1) on $H^1_0(\Omega)$ reads $R_{H^1_0(\Omega)} \equiv -\Delta^{\mathrm{we}}_x$, it holds for all $u \in D(B)$, that

$$\|Bu\|_{V'} = \|-\Delta^{\mathrm{we}}_x|_{D(B)}u\|_{H^{-1}(\Omega)} = \|-\Delta^{\mathrm{we}}_x u\|_{H^{-1}(\Omega)} = \|u\|_{H^1_0(\Omega)} = \|\nabla u\|_{L^2(\Omega)}.$$

Hence, the norm induced by the operator B is the H^1-seminorm. As $D(B) \subseteq_d H^1_0(\Omega)$ and $H^1_0(\Omega)$ is complete with respect to $\|\nabla\cdot\|_{L^2(\Omega)}$, it holds for the well-posed and optimally stable completion (9), that $\overline{U} = H^1_0(\Omega)$ and $\overline{B} = -\Delta^{\mathrm{we}}_x$.

3.1.3 Ultra-weak formulation

Consider the operator $B \coloneqq -\Delta^{\mathrm{uw}}_x|_{D(B)} : D(B) \subseteq_d L^2(\Omega) \to [H^\Delta(\Omega) \cap H^1_0(\Omega)]'$ given in example 2.2 (iii). Let $u \in D(B)$ be arbitrary. We have already seen in context of the strong formulation, that $-\Delta \equiv -\Delta^{\mathrm{st}}_x : H^\Delta(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is an isomorphism and it holds (16), i.e., there exists a $v_u \in H^\Delta(\Omega) \cap H^1_0(\Omega)$ such that $-\Delta v_u = u$ in $L^2(\Omega)$ and $\|u\|_{L^2(\Omega)} \le \|v_u\|_V \le \sqrt{(1 + C^2_\Omega)}\|u\|_{L^2(\Omega)}$ with $\|\cdot\|^2_V \coloneqq \|\nabla\cdot\|^2_{L^2(\Omega)} + \|\Delta\cdot\|^2_{L^2(\Omega)}$. Thus, it holds for v_u, that

$$\frac{|\langle Bu, v_u\rangle_{V'\times V}|}{\|v_u\|_V} = \frac{|(u, \Delta v_u)_{L^2(\Omega)}|}{\|v_u\|_V} = \frac{|(u, u)_{L^2(\Omega)}|}{\|v_u\|_V} \ge \frac{1}{\sqrt{(1 + C^2_\Omega)}}\|u\|_{L^2(\Omega)},$$

and for all $v \in H^\Delta(\Omega) \cap H^1_0(\Omega)$, that

$$\frac{|\langle Bu, v\rangle_{V'\times V}|}{\|v\|_V} = \frac{|(u, \Delta v)_{L^2(\Omega)}|}{\|v\|_V} \le \frac{\|u\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)}}{\|v\|_V} \le \frac{\|u\|_{L^2(\Omega)}\|v\|_V}{\|v\|_V} = \|u\|_{L^2(\Omega)}.$$

Hence, we conclude, that the norm induced by the operator and the L^2-norm are equivalent as it holds $\frac{1}{\sqrt{(1+C_\Omega^2)}}\|u\|_{L^2(\Omega)} \leq \|Bu\|_{[H^\Delta(\Omega)\cap H_0^1(\Omega)]'} \leq \|u\|_{L^2(\Omega)}$. As $D(B) \subseteq_d L^2(\Omega)$ and $L^2(\Omega)$ is complete with respect to $\|\cdot\|_{L^2(\Omega)}$, it holds for the well-posed and optimally stable completion (9), that $\overline{U} = L^2(\Omega)$ and $\overline{B} = -\Delta_x^{\mathrm{uw}}$.

Remark 3.1. *In the three settings given in example 2.2, the domain $D(B)$ was not fixed and was allowd to range from $D(B_\circ)$ to U. Choosing $D(B) = D(B_\circ)$ we can replace B by $B_\circ$ as it holds $B|_{D(B_\circ)} = B_\circ$ in V' by the construction of B and V'. Further, we could also replace U by*[5] *$L^2(\Omega)$. Even after applying these changes (regarding B, $D(B)$ and U) to each of the three formulations – in particular, we can choose $B = B_\circ$, $D(B) = D(B_\circ)$ and $U = L^2(\Omega)$ for all three cases – they still lead to the same well-posed extensions $\overline{U}$ and $\overline{B}$ as presented above. Thus, V is the sole quantity that encodes the regularity of the resulting extension, i.e., the choice of V determines if we end up with $-\Delta_x^{\mathrm{st}}$, $-\Delta_x^{\mathrm{we}}$ or $-\Delta_x^{\mathrm{uw}}$.*

Remark 3.2. *The above results still hold for general bounded elliptic second-order differential operators as defined in remark 2.2 although the constants of the norm equivalence between $\|\cdot\|_U$ and $\|B\cdot\|_{V'}$ will change. In case of the weak formulation, this coincides with the well-known result that $\|B\cdot\|_{H^{-1}(\Omega)}$ defines an equivalent norm to $\|\cdot\|_{H_0^1(\Omega)}$ on $H_0^1(\Omega)$ for any bounded self-adjoint and coercive operator*[6] *$B : H_0^1(\Omega) \to H^{-1}(\Omega)$. In this case, the isometric Riesz operator (1) on $H_0^1(\Omega)$ equipped with the norm $\|B\cdot\|_{H^{-1}(\Omega)}$ reads $R_{H_0^1(\Omega)} \equiv B$ instead of $R_{H_0^1(\Omega)} \equiv -\Delta_x^{\mathrm{we}}$.*

3.2 The heat equation

Let us consider the heat equation in example 1.2.

3.2.1 Strong in time

Starting with the strong in time variational formulation given in example 2.3. It is well known, that $Bu = f$ in V' admits a unique solution $u^* \in U$ for each $f \in V'$, [10, 17, 19]. In particular, $B \in \mathcal{L}(U, V)$ and there holds the inf-sup stability

$$\|u\|_U \leq \sup_{0\neq v\in V} \frac{\langle Bu, v\rangle_{V'\times V}}{\|v\|_V} = \|Bu\|_{V'}, \tag{17}$$

for all $u \in U$. Thus, we have the norm equivalence

$$\|u\|_U \leq \|Bu\|_{V'} \leq \sqrt{2}\|u\|_U \quad \forall u \in U, \tag{18}$$

and we immediately see, that the norm induced by the operator is equivalent to the norm on U. Thus, applying our abstract framework to the heat equation does not do much, in

[5]Note that the upper bound of $D(B)$ is fixed and does not increase with U.

[6]By $R_{H_0^1(\Omega)} \equiv B$ holds $\|B\cdot\|^2_{H^{-1}(\Omega)} = (B\cdot, B\cdot)_{H^{-1}(\Omega)} = \langle B\cdot, \cdot\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}$. Then, the norm equivalence immediately follows by the boundedness and coercivity of B, while the self-adjointness gives rise not only to a norm but also to an inner product.

fact, as $\|\cdot\|_U$ and $\|B\cdot\|_{V'}$ are equivalent on U, and U is already complete with respect to $\|\cdot\|_U$, the completion of U with respect to $\|B\cdot\|_{V'}$ simply reads $\overline{U} \equiv U$ and $\|\cdot\|_{\overline{U}} = \|B\cdot\|_{V'}$ as it holds $\overline{B} \equiv B$. Hence, by going from U to $\overline{U}$, we effectively replaced the norm by an equivalent one. Thus, let us use our abstract framework to compute this norm $\|B\cdot\|_{V'}$. Therefore, we need the Riesz operator (1) on V given by $R_V \equiv -\Delta_x^{\mathrm{we}}$, i.e., $B = \partial_t + R_V$. Then using (11), we compute for $u \in U$, that

$$\begin{aligned} \|Bu\|_{V'}^2 &= \langle Bu, R_V^{-1}Bu\rangle_{V'\times V} = \langle \partial_t u + R_V u, R_V^{-1}(\partial_t u + R_V u)\rangle_{V'\times V} \\ &= \langle \partial_t u, R_V^{-1}\partial_t u\rangle_{V'\times V} + 2\langle \partial_t u, u\rangle_{V'\times V} + \langle R_V u, u\rangle_{V'\times V} \\ &= \|\partial_t u\|_{V'}^2 + \|u\|_V^2 + 2\langle \partial_t u, u\rangle_{V'\times V} = \|u\|_U^2 + 2\langle \partial_t u, u\rangle_{V'\times V}. \end{aligned} \tag{19}$$

The norm equivalence (18) thus relies on the fact that

$$\begin{aligned} 0 \le \|u(T)\|_{L^2(\Omega)}^2 &= \int_\Omega u(T,x)^2 \,\mathrm{d}x \overset{u(0)=0}{=} \int_\Omega \int_I \partial_t u(t,x)^2 \,\mathrm{d}t\,\mathrm{d}x = 2\langle \partial_t u, u\rangle_{V'\times V} \\ &\le 2\|\partial_t u\|_{V'}\|u\|_V \le \|\partial_t u\|_{V'}^2 + \|u\|_V^2 = \|u\|_U^2. \end{aligned}$$

Hence, we get $\overline{B} = B = \partial_t - \Delta_x^{\mathrm{we}}$ and $\overline{U} = U = L^2(I; H_0^1(\Omega)) \cap H_{0,}^1(I; H^{-1}(\Omega))$ with norm $\|\cdot\|_{\overline{U}} = \sqrt{\|\cdot\|_U^2 + 2\langle \partial_t \cdot, \cdot\rangle_{V'\times V}}$.

Remark 3.3. *As seen above, the mixed term in the norm can be written as* $2\langle \partial_t u, u\rangle_{V'\times V} = \|u(T)\|_{L^2(\Omega)}^2$, *i.e., the norm* $\|\cdot\|_{\overline{U}}$ *and therefore the well-posed formulation* (9) *of the heat equation corresponds exactly to the well-posed and optimally stable formulation introduced in [23].*

Remark 3.4. *Note, that the supremizer of* (17) *is given in* (11) *as* $v_u = R_V^{-1}Bu = R_V^{-1}\partial_t u + u$, *which was already used in [8, 17, 18, 21, 22, 23] to show the stability of the formulation.*

Remark 3.5. *The above results hold true when replacing* $-\Delta_x^{we}$ *by an arbitrary bounded self-adjoint coercive operator* $A_x : H_0^1(\Omega) \to H^{-1}(\Omega)$ *as we can replace the norm on* $H_0^1(\Omega)$ *by the equivalent norm* $\|A_x\cdot\|_{H^{-1}(\Omega)}$ *as mentioned in remark 3.2.*

3.2.2 Weak in time

Now, let us consider the weak in time variational formulation given in example 2.4, which is exactly the formulation used e.g. in [1, 4]. It was shown in [4, Thm. 2.2], that $B \in \mathcal{L}_{\mathrm{is}}(U, V')$, i.e. (5) is already well-posed and we immediately get $\overline{U} \equiv U$ with $\|\cdot\|_{\overline{U}}$ being equivalent to $\|\cdot\|_U$, and $\overline{B} \equiv B$. Hence, the application of our framework reduces to changing the norm on the trial space and thus adding optimal stability to the already given well-posedness. Unlike to the strong in time formulation above, there is no explicit representation for the Riesz operator R_V. Hence, we can not give an representation for $\|\cdot\|_{\overline{U}}$ by a similar calculation to (19).

3.3 The wave equation

Next, we consider the wave equation in example 1.3.

3.3.1 Strong in time

We start with the strong in time variational formulation of the wave equation as given in example 2.5. We denote by $\mathbb{F}$ the 2D flip operator, i.e. for two Banach spaces X_1 and X_2, their vector product $X := X_1 \times X_2$ and $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in X$, we have

$$\mathbb{F}\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_2 \\ x_1 \end{smallmatrix}\right), \qquad \mathbb{F}(X_1 \times X_2) = X_2 \times X_1, \qquad \|\cdot\|_{\mathbb{F}X} = \|\mathbb{F}\cdot\|_X,$$
$$(\mathbb{F}X)' = \mathbb{F}X', \qquad \mathbb{F}L^2(I;X) = L^2(I;\mathbb{F}X).$$

Thus, with $W := L^2(\Omega) \times H_0^1(\Omega)$, we get $U = H^1_{0,}(I;W') \cap L^2(I;\mathbb{F}W)$ and $V = L^2(I;W)$ and their norms are given by $\|\vec{w}\|_W^2 := \|w_1\|^2_{L^2(\Omega)} + \|w_2\|^2_{H_0^1(\Omega)}$, $\|\cdot\|_U^2 := \|\partial_t\cdot\|^2_{V'} + \|\cdot\|^2_{\mathbb{F}V}$ and $\|\cdot\|_V := \|\cdot\|_{L^2(I;W)}$. Now, for $\vec{f} \in V'$, the variational formulation in example 2.5 amounts to find $\vec{u}^* \in U$ such that

$$B\vec{u}^* = \vec{f} \text{ in } V' \qquad \text{with} \qquad \langle B\vec{u}, \vec{v}\rangle_{V'\times V} := \langle \partial_t \vec{u} + A\vec{u}, \vec{v}\rangle_{V'\times V}, \tag{20}$$

for all $\vec{u} \in U$ and all $\vec{v} \in V$ and $A := \left(\begin{smallmatrix} 0 & -Id \\ -\Delta_x^{\mathrm{we}} & 0 \end{smallmatrix}\right) : \mathbb{F}W \to W'$. Further, by $W' = L^2(\Omega) \times H^{-1}(\Omega)$, the dual space and the dual norm of V' read

$$V' = L^2(I;W'), \qquad \|\cdot\|^2_{V'} = \|\cdot\|^2_{L^2(Q)} + \|\cdot\|^2_{L^2(I;H^{-1}(\Omega))}. \tag{21}$$

Remark 3.6. *The operator B in* (20) *is bounded, i.e. $B \in \mathcal{L}(U,V')$, as it holds for $\vec{u} \in U$, using $R_{H_0^1(\Omega)} \equiv -\Delta_x^{\mathrm{we}}$ for the Riesz operator* (1) *on $H_0^1(\Omega)$, that*

$$\begin{aligned}
\|B\vec{u}\|^2_{V'} &\overset{(21)}{=} \|\partial_t u_1 - u_2\|^2_{L^2(Q)} + \|(\Delta_x^{\mathrm{we}})^{-1}(\partial_t u_2 - \Delta_x^{\mathrm{we}} u_1)\|^2_{L^2(I;H_0^1(\Omega))} \\
&\le 2\big(\|\partial_t u_1\|^2_{L^2(Q)} + \|u_2\|^2_{L^2(Q)} + \|\partial_t u_2\|^2_{L^2(I;H^{-1}(\Omega))} + \|u_1\|^2_{L^2(I;H_0^1(\Omega))}\big) \\
&= 2(\|\vec{u}\|^2_{\mathbb{F}V} + \|\partial_t \vec{u}\|^2_{V'}) = 2\|\vec{u}\|^2_U.
\end{aligned}$$

Theorem 3.1 ([16, Ch. 3 Thm. 8.1 & eq. (8.15)])**.** *Let $f \in L^2(Q)$, then there exists a unique $u^* \in H^2(I;H^{-1}(\Omega)) \cap H^1(I;L^2(\Omega)) \cap L^2(I;H_0^1(\Omega))$ such that*

$$\partial_{tt} u^* - \Delta_x^{\mathrm{we}} u^* = f \text{ in } L^2(I;H^{-1}(\Omega)), \quad u^*(0) = 0, \quad \partial_t u^*(0) = 0.$$

Further, there exists $\hat{C} < \infty$ independent of u^* and f such that $\|\partial_t u^*\|^2_{L^2(Q)} + \|u^*\|^2_{L^2(I;H_0^1(\Omega))} \le \hat{C}\|f\|^2_{L^2(Q)}$.

Remark 3.7. *Using the techniques of [19, Theorem 5.1, Remark 4.6], we can show that the constant in the preceding theorem equals $\hat{C} = \frac{T^2}{2}$ and the estimate is in fact sharp in the powers of T.*

Corollary 3.2. *The variational formulation* (20) *possesses a solution* $\vec{u}^* \in U$ *for each* $\vec{f} \in Y := H^1_{0,}(I; L^2(\Omega)) \times L^2(Q)$. *Furthermore, there exists* $C_Q < \infty$ *independent of* $\vec{u}^*$ *and* $\vec{f}$ *such that* $\|\vec{u}^*\|_U \leq C_Q \|\vec{f}\|_{L^2(Q;\mathbb{R}^2)}$. *In particular theorem 2.2 holds by lemma 2.1 and remark 2.3.*

Proof. For $\vec{f} = (f_1, f_2) \in Y$ define $f := \partial_t f_1 + f_2 \in L^2(Q)$ and denote by u^* and C the solution and the constant given by theorem 3.1 with respect to the right-hand side f. Now, defining $\vec{u}^* \equiv (u_1^*, u_2^*) := (u^*, \partial_t u^* - f_1)$, it holds $\vec{u}^* \in U$, $\partial_t u_1^* - u_2^* = f_1$ in $L^2(Q)$ and $\partial_t u_2^* - \Delta_x^{\text{we}} u_1^* = \partial_{tt} u^* - \partial_t f_1 - \Delta_x^{\text{we}} u^* = f_2$ in $L^2(I; H^{-1}(\Omega))$. Hence, we conclude, that $\vec{u}^*$ solves (20) as it holds for all $\vec{v} \in V$, that

$$\begin{aligned}\langle B\vec{u}^*, \vec{v}\rangle_{V'\times V} &= (\partial_t u_1^* - u_2^*, v_1)_{L^2(Q)} + \langle \partial_t u_2^* - \Delta_x^{\text{we}} u_1^*, v_2\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H_0^1(\Omega))} \\ &= (f_1, v_1)_{L^2(Q)} + \langle f_2, v_2\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H_0^1(\Omega))} = \langle \vec{f}, \vec{v}\rangle_{V'\times V}.\end{aligned}$$

Now, regarding the stability estimate, it first holds by theorem 3.1, together with the 1D Poincaré inequality $\|\partial_t f\|_{L^2(Q)} \leq \frac{T}{\sqrt{2}} \|f\|_{L^2(Q)}$ for all $f \in H^1_{0,}(I; L^2(\Omega))$, that

$$\begin{aligned}\|\partial_t u_1^*\|^2_{L^2(Q)} + \|u_1^*\|^2_{L^2(I;H_0^1(\Omega))} &\leq \hat{C}\|\partial_t f_1 + f_2\|^2_{L^2(Q)} \leq 2\hat{C}\big(\|\partial_t f_1\|^2_{L^2(Q)} + \|f_2\|^2_{L^2(Q)}\big) \\ &\leq 2\hat{C}\tilde{C}_T \|\vec{f}\|^2_{L^2(Q;\mathbb{R}^2)},\end{aligned}$$

with $\tilde{C}_T := \max\{1, \frac{T^2}{2}\}$. Next, using Poincaré's inequality in Ω, it holds $H_0^1(\Omega) \hookrightarrow_d L^2(\Omega)$ for some $C_\Omega < \infty$, in particular $\|f\|_{L^2(I;H^{-1}(\Omega))} \leq C_\Omega \|f\|_{L^2(Q)}$ for all $f \in L^2(Q)$ by remark 1.2, and thus

$$\begin{aligned}\|\partial_t u_2^*\|^2_{L^2(I;H^{-1}(\Omega))} + \|u_2^*\|^2_{L^2(Q)} &= \|f_2 + \Delta_x^{\text{we}} u_1^*\|^2_{L^2(I;H^{-1}(\Omega))} + \|\partial_t u_1^* - f_1\|^2_{L^2(Q)} \\ &\leq 2\big(\|f_2\|^2_{L^2(I;H^{-1}(\Omega))} + \|u_1^*\|^2_{L^2(I;H_0^1(\Omega))} + \|\partial_t u_1^*\|^2_{L^2(Q)} + \|f_1\|^2_{L^2(Q)}\big) \\ &\leq (2\tilde{C}_\Omega + 4\hat{C}\tilde{C}_T)\|\vec{f}\|^2_{L^2(Q;\mathbb{R}^2)},\end{aligned}$$

with $\tilde{C}_\Omega := \max\{1, C_\Omega^2\}$. Now, defining $C_Q^2 := 2\tilde{C}_\Omega + 6\hat{C}\tilde{C}_T$, we get

$$\begin{aligned}\|\vec{u}^*\|^2_U &= \|\partial_t u_2^*\|^2_{L^2(I;H^{-1}(\Omega))} + \|\partial_t u_1^*\|^2_{L^2(Q)} + \|u_1^*\|^2_{L^2(I;H_0^1(\Omega))} + \|u_2^*\|^2_{L^2(Q)} \\ &\leq C_Q \|\vec{f}\|^2_{L^2(Q;\mathbb{R}^2)}.\end{aligned}$$

□

Although, (20) admits a unique solution $\vec{u}^* \in U$ for all $\vec{f} \in H^1_{0,}(I; L^2(\Omega)) \times L^2(Q) \subset V'$ we will now show that this can not be extended to all $\vec{f} \in V'$. Thus, for the wave equation, unlike the previous examples, considering $\overline{U}$ and $\overline{B}$ instead of U and B, respectively, is in fact necessary for a well-posed formulation (instead of just the optimal stability) as $B : U \to V'$ isn't already an isomorphism.

Theorem 3.3. *The operator* B *in* (20) *does not define an isomorphism as there does not exists an inf-sup constant* $\beta > 0$ *such that* $\beta\|\vec{u}\|_U \leq \|B\vec{u}\|_{V'}$ *for all* $\vec{u} \in U$, *i.e. the inverse of* B *is not bounded.*

Proof. We construct a counterexample, following the ideas presented in [24, Theorem 4.2.24]. To this end, let $\phi_k \in H_0^1(\Omega)$ denote the normalized eigenfunctions and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$ the eigenvalues of the spatial Laplacian, given as $-\Delta_x \phi_k = \lambda_k \phi_k$ in Ω with $\|\phi_k\|_{L^2(\Omega)} = 1$, and consider the function

$$\vec{u}^k(t,x) := \begin{pmatrix} \phi_k(x) \int_0^t s \sin(\sqrt{\lambda_k} s)\, ds \\ \phi_k(x) t \sin(\sqrt{\lambda_k} t) \end{pmatrix},$$

which solves (20) for $\vec{f}^k := (0, f^k)$, $f^k(t,x) := 2\phi_k(x)\sin(\sqrt{\lambda_k} t)$. Elementary computations give $\|\vec{u}^k\|_U^2 = \lambda_k^{-3/2}\big(T\sqrt{\lambda_k} + \frac{2}{3}T^3\lambda_k^{3/2} - \frac{1}{2}T\,\sin(\sqrt{\lambda_k}T)\big)$ and $\|\vec{f}^k\|_{V'}^2 = \lambda_k^{-3/2}\big(2T\sqrt{\lambda_k} - \sin(2\sqrt{\lambda_k}T)\big)$. Therefore, taking the limit we get $\frac{\|\vec{f}^k\|_{V'}}{\|\vec{u}^k\|_U} \to 0$ for $k \to \infty$ which completes the proof. □

Thus, in contrast to elliptic and parabolic PDEs, in this case the norm induced by the operator is not equivalent to the norm on U. In order to determine $\|B\cdot\|_{V'}$, using $R_{H_0^1(\Omega)} \equiv -\Delta_x^{\mathrm{we}}$, it holds for all $\vec{u} \in U$, that

$$\begin{aligned}
\|B\vec{u}\|_{V'}^2 &\overset{(21)}{=} \|\partial_t u_1 - u_2\|_{L^2(Q)}^2 + \|\partial_t u_2 - \Delta_x^{\mathrm{we}} u_1\|_{L^2(I;H^{-1}(\Omega))}^2 \\
&= \|\partial_t u_1\|_{L^2(Q)}^2 + \|u_2\|_{L^2(Q)}^2 - 2(\partial_t u_1, u_2)_{L^2(Q)} \\
&\quad + \|\partial_t u_2\|_{L^2(I;H^{-1}(\Omega))}^2 + \|u_1\|_{L^2(I;H_0^1(\Omega))}^2 + 2\langle \partial_t u_2, u_1\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H_0^1(\Omega))} \\
&= \|\vec{u}\|_U^2 + 2\langle \partial_t \vec{u}, J\vec{u}\rangle_{V'\times V},
\end{aligned}$$

with $J := \left(\begin{smallmatrix} 0 & -Id \\ Id & 0 \end{smallmatrix}\right)$.

Remark 3.8. *Alternatively, using the Riesz operator* (1) *on* V*, given by* $R_V = \left(\begin{smallmatrix} Id & 0 \\ 0 & -\Delta_x^{\mathrm{we}} \end{smallmatrix}\right)$*, together with* (11)*, we get*

$$\begin{aligned}
\|B\vec{u}\|_{V'}^2 &= \langle B\vec{u}, R_V^{-1} B\vec{u}\rangle_{V'\times V} = \langle \partial_t \vec{u} + A\vec{u}, R_V^{-1}(\partial_t \vec{u} + A\vec{u})\rangle_{V'\times V} \\
&= \langle \partial_t \vec{u}, R_V^{-1}\partial_t \vec{u}\rangle_{V'\times V} + 2\langle \partial_t \vec{u}, R_V^{-1} A\vec{u}\rangle_{V'\times V} + \langle A^* R_V^{-1} A\vec{u}, \vec{u}\rangle_{\mathbb{F}V'\times \mathbb{F}V},
\end{aligned}$$

where $A^* = \left(\begin{smallmatrix} 0 & -\Delta_x^{\mathrm{we}} \\ -Id & 0 \end{smallmatrix}\right) : W \to \mathbb{F}W'$ *denotes the adjoint of* A*. To further simplify this expression, note that* $R_V = AJ^*$ *and* $J^{-1} = J^*$*, hence* $R_V^{-1}A = J$ *as well as* $A^* R_V^{-1} A = \left(\begin{smallmatrix} -\Delta_x^{\mathrm{we}} & 0 \\ 0 & Id \end{smallmatrix}\right) = R_{\mathbb{F}V}$*, with* $R_{\mathbb{F}V}$ *denoting the Riesz operator* (1) *on* $\mathbb{F}V$*. Thus, we get the same result, namely*

$$\begin{aligned}
\|B\vec{u}\|_{V'}^2 &= \langle \partial_t \vec{u}, R_V^{-1}\partial_t \vec{u}\rangle_{V'\times V} + 2\langle \partial_t \vec{u}, J\vec{u}\rangle_{V'\times V} + \langle R_{\mathbb{F}V}\vec{u}, \vec{u}\rangle_{\mathbb{F}V'\times\mathbb{F}V} \\
&= \|\partial_t \vec{u}\|_{V'}^2 + 2\langle \partial_t \vec{u}, J\vec{u}\rangle_{V'\times V} + \|\vec{u}\|_{\mathbb{F}V}^2 = \|\vec{u}\|_U^2 + 2\langle \partial_t \vec{u}, J\vec{u}\rangle_{V'\times V}.
\end{aligned} \tag{22}$$

As we can see, the representation of the norm $\|B\cdot\|_{V'}$ is quite similar to the one derived for the heat equation in (19). However, the additional term

$$2\langle \partial_t \vec{u}, J\vec{u}\rangle_{V'\times V} = 2\langle \partial_t u_2, u_1\rangle_{L^2(I;H^{-1}(\Omega))\times L^2(I;H_0^1(\Omega))} - 2(\partial_t u_1, u_2)_{L^2(Q)}$$

can not be bounded from below by zero. In fact, for $\vec{u}^k$ as defined in the proof of theorem 3.3, we compute $\langle \partial_t \vec{u}^k, J\vec{u}^k\rangle_{V'\times V} = -\frac{T^3}{3} - \frac{\sin(2\sqrt{\lambda_k}T)}{4\lambda_k^{3/2}} + \frac{T}{2\lambda_k}$, which goes to $-\frac{T^3}{3} < 0$

as $k \to \infty$. Moreover, as the function $\vec{u}^k$ solves (20) for the right-hand side $\vec{f}^k$, it holds $B\vec{u}^k = \vec{f}^k$ and we get

$$\frac{\|\vec{f}^k\|_{V'}}{\|B\vec{u}^k\|_{V'}} = \frac{\|\vec{f}^k\|_{V'}}{\|\vec{f}^k\|_{V'}} = 1 \qquad \text{but} \qquad \frac{\|\vec{f}^k\|_{V'}}{\|\vec{u}^k\|_U} \to 0, \quad k \to \infty.$$

Hence, the norms $\|B\cdot\|_{V'}$ and $\|\cdot\|_U$ are not equivalent. By (8) and (22), we have a characterization of $\|\cdot\|_{\overline{U}}$ on $U \subseteq_d \overline{U}$. Now, with the norm $\|\cdot\|_{\overline{U}}$ characterized (at least on U), let us consider the space $\overline{U}$. Unlike to the elliptic and parabolic equations, we cannot give $\overline{U}$ explicitly, but we can characterize it to some extend by theorem 2.4.

Corollary 3.4. *The adjoint problem of* (20)*, namely to find* $\vec{v}^* \in V$ *such that*

$$\langle B^*\vec{v}^*, \vec{u}\rangle_{U'\times U} = \langle \vec{g}, \vec{u}\rangle_{U'\times U} \quad \forall \vec{u} \in U,$$

possess a solution for each $\vec{g} \in X := L^2(Q) \times H^1_{,0}(I; L^2(\Omega)) \subset_d G$*. Further, there exists* $C_Q^* < \infty$ *independent of* $\vec{v}^*$ *and* $\vec{g}$ *such that* $\|\vec{v}^*\|_V \le C_Q^*\|\vec{g}\|_{L^2(Q;\mathbb{R}^2)}$*. In particular theorem 2.4 holds by remark 2.6.*

Proof. In addition to the flip operator $\mathbb{F}$, we define the time reversal operator by $\mathbb{T}f(t) := f(T-t)$ for almost all $t \in I$. It is easy to see, that $\mathbb{F}^2 = Id$, $\mathbb{T}^2 = Id$, $\mathbb{F}^* = \mathbb{F}$, $\mathbb{T}^* = \mathbb{T}$, $\mathbb{TF} = \mathbb{FT}$, $\partial_t\mathbb{TF} = -\mathbb{TF}\partial_t$ and $\mathbb{TF}A = A^*\mathbb{TF}$. Further, it holds $\mathbb{TF}\vec{u} \in V$, $(\mathbb{TF}\vec{u})(T) = 0$ and $\mathbb{TF}A\vec{u} \in \mathbb{F}V'$ for all $\vec{u} \in U$. Now, let $\vec{g} \in X$ be arbitrary but fixed, define $\vec{f} := \mathbb{TF}\vec{g} \in Y$ and denote by $\vec{u}^* \in U$ the solution of $B\vec{u}^* = \vec{f}$ in V' given by corollary 3.2. Defining $\vec{v}^* := \mathbb{TF}\vec{u}^* \in V$, it holds for all $\vec{u} \in U$, that

$$\begin{aligned}
\langle B^*\vec{v}^*, \vec{u}\rangle_{U'\times U} &= \langle B\vec{u}, \vec{v}^*\rangle_{V'\times V} = \langle \partial_t\vec{u} + A\vec{u}, \vec{v}^*\rangle_{V'\times V} = \langle \partial_t\vec{u} + A\vec{u}, \mathbb{TF}\vec{u}^*\rangle_{V'\times V} \\
&= \langle \mathbb{TF}\partial_t\vec{u} + \mathbb{TF}A\vec{u}, \vec{u}^*\rangle_{\mathbb{F}V'\times\mathbb{F}V} = \langle -\partial_t\mathbb{TF}\vec{u} + A^*\mathbb{TF}\vec{u}, \vec{u}^*\rangle_{\mathbb{F}V'\times\mathbb{F}V} \\
&= \langle \partial_t\vec{u}^* + A\vec{u}^*, \mathbb{TF}\vec{u}\rangle_{V'\times V} + (\vec{u}^*(T), \underbrace{(\mathbb{TF}\vec{u})(T)}_{=0})_{L^2(\Omega)} - (\underbrace{\vec{u}^*(0)}_{=0}, (\mathbb{TF}\vec{u})(0))_{L^2(\Omega)} \\
&= \langle B\vec{u}^*, \mathbb{TF}\vec{u}\rangle_{V'\times V} = \langle \vec{f}, \mathbb{TF}\vec{u}\rangle_{V'\times V} = \langle \mathbb{TF}\vec{g}, \mathbb{TF}\vec{u}\rangle_{V'\times V} = \langle \vec{g}, \vec{u}\rangle_{U'\times U},
\end{aligned}$$

i.e., $\vec{v}^* \in V$ is a solution of the adjoint problem. Thereby we used in the last step, that $\vec{g}, \vec{u}, \mathbb{TF}\vec{g}, \mathbb{TF}\vec{u} \in L^2(Q;\mathbb{R}^2)$ and thus $\langle \mathbb{TF}\vec{g}, \mathbb{TF}\vec{u}\rangle_{V'\times V} = (\mathbb{TF}\vec{g}, \mathbb{TF}\vec{u})_{L^2(Q;\mathbb{R}^2)} = (\vec{g}, \vec{u})_{L^2(Q;\mathbb{R}^2)} = \langle \vec{g}, \vec{u}\rangle_{U'\times U}$ by (3) and the Gelfand triples (V, H, V') and (U, G, U'). Finally, using the stability estimate for $\vec{u}^*$ provided by corollary 3.2, it holds $\|\vec{v}^*\|_V = \|\vec{u}^*\|_{\mathbb{F}V} \le \|\vec{u}^*\|_U \le C_Q\|\vec{f}\|_{L^2(Q;\mathbb{R}^2)}$. □

Let us collect our results.

Corollary 3.5. *For the well-posed extension* (9) *of example 2.5 holds*

(i) $U \hookrightarrow_d \overline{U} \hookrightarrow_d L^2(Q;\mathbb{R}^2)$*, and the embedding constants read* $\|u\|_{L^2(Q;\mathbb{R}^2)} \le C_Q\|u\|_{\overline{U}}$*,* $u \in \overline{U}$ *and* $\|u\|_{\overline{U}} \le \sqrt{2}\|u\|_U$*,* $u \in U$*, with the constant* $C_Q = \sqrt{\max\{1, C_\Omega^2\} + \max\{3T^2, \frac{3}{2}T^4\}}$*;*

(ii) *the other directions of the norm inequalities do not holds, i.e.* $\|\cdot\|_{\overline{U}}$ *is neither equivalent to* $\|\cdot\|_{L^2(Q;\mathbb{R}^2)}$ *on* $\overline{U}$*, nor equivalent to* $\|\cdot\|_U$ *on* U*;*

(iii) $U \subsetneq \overline{U} \subsetneq L^2(Q;\mathbb{R}^2)$*;*

(iv) *for* $\vec{u} \in U$*, we have* $\|\vec{u}\|_{\overline{U}} = \sqrt{\|\vec{u}\|_U^2 + 2\langle \partial_t \vec{u}, J\vec{u}\rangle_{V'\times V}}$*, where* $J := \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$.

Proof. The first statement is given by corollary 3.4 and remark 2.7 implying $C_Q^* \equiv C_Q$, together with the definition of C_Q and $\hat{C}$ in corollary 3.2 and remark 3.7, respectively, while $\|B\|_{\mathcal{L}(U,V')} \leq \sqrt{2}$ holds by remark 3.6. The last statement was shown in (22). Further, that $\|\cdot\|_{\overline{U}}$ is not equivalent to $\|\cdot\|_U$ follows by theorem 3.3 and (8). Next, if $\|\cdot\|_{\overline{U}}$ would be equivalent to $\|\cdot\|_{L^2(Q;\mathbb{R}^2)}$, we would get $\overline{U} = L^2(Q;\mathbb{R}^2)$ since $\overline{U}$ is dense in $L^2(Q;\mathbb{R}^2)$ by (i) but also complete w.r.t. $\|\cdot\|_{\overline{U}}$. Since $\overline{B} : \overline{U} \to V'$ is an isomorphism, we would end up with $L^2(Q;\mathbb{R}^2)$ being isomorph to $V' = L^2(I; L^2(\Omega) \times H^{-1}(\Omega))$, which is not the case as $L^2(\Omega)$ is not isomorph to $H^{-1}(\Omega)$. Thus, we have also shown, that $\overline{U} \neq L^2(Q;\mathbb{R}^2)$ has to hold. Finally, by the construction of $\overline{U}$, it would follow from $U = \overline{U}$, that $R(B) = V'$, i.e. $B \in \mathcal{L}_{\text{is}}(U, V')$ by the bounded inverse theorem/open mapping theorem, (B1) and remark 3.6. This is a contradiction to theorem 3.3, and we conclude $U \neq \overline{U}$. □

Remark 3.9. *By remark 3.2, we can replace* $-\Delta_x^{\text{we}}$ *by any bounded self-adjoint coercive operator* $A_x : H_0^1(\Omega) \to H^{-1}(\Omega)$ *provided that* A_x *satisfies the assumptions of [16, Ch. 3 Thm. 8.1] (i.e. theorem 3.1 holds true), by replacing the norm on* $H_0^1(\Omega)$ *by* $\|A_x \cdot\|_{H^{-1}(\Omega)}$*.*

3.3.2 Weak in time

We consider the weak in time variational formulation of the wave equation as given in example 2.6.

Remark 3.10. *The operator* B *in* (7) *is bounded, i.e.* $B \in \mathcal{L}(U, V')$ *as it holds for* $u \in U$ *and* $v \in V$ *by Young's inequality*[7]*, that* $|\langle Bu, v\rangle_{V'\times V}| \leq \|\partial_t u\|_{L^2(Q)}\|\partial_t v\|_{L^2(Q)} + \|\nabla_x u\|_{L^2(Q)}\|\nabla_x v\|_{L^2(Q)} = \|u\|_U\|v\|_V$*.*

Theorem 3.6. *The variational problem* (7) *possess a unique solution* $u^* \in U$ *for all* $f \in L^2(Q)$ *and the solution satisfies* $\|u^*\|_{H_0^1(\Omega)} \leq \frac{T}{\sqrt{2}}\|f\|_{L^2(Q)}$*.*

Proof. This is a well known result, see e.g. [19, Theorem 5.1] for this exact statement or [15, ch. IV Theorems 3.1 & 3.2, eq. (3.17)] for a more general setting. □

Remark 3.11. *By the above existence result, theorem 2.2 holds by lemma 2.1 and remark 2.3, while theorem 2.4 holds as* B *is symmetric and thus self-adjoint (except for a switch of the initial/terminal conditions encoded in* U *and* V*, but* U *and* V *are isometric isomorphic and thus interchangeable).*

As for the strong in time formulation, the above existence result cannot be generalized to all right-hand sides $f \in V'$ as the operator is not inf-sup stable.

[7] $ab + cd = \sqrt{(ab+cd)^2} \leq \sqrt{a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2} = \sqrt{a^2+c^2}\sqrt{b^2+d^2}$ for all $a, b, c, d \geq 0$.

Theorem 3.7 ([20, Theorem 1.1])**.** *The operator B in* (7) *does not define an isomorphism. In particular, there does not exist an inf-sup constant $\beta > 0$ such that $\beta\|u\|_U \leq \|Bu\|_{V'}$ for all $u \in U$, i.e., the inverse of B is not bounded.*

Now, denoting the well-posed and optimally stable extension of U and B by $\overline{U}$ and $\overline{B}$, respectively, we get the following result, which is proven analogously to corollary 3.5.

Corollary 3.8. *For the well-posed extension* (9) *of* (7) *holds*

(i) $U = H^1_{0,}(I; H^1_0(\Omega)) \hookrightarrow_d \overline{U} \hookrightarrow_d L^2(Q)$ *and the embedding constants read* $\|u\|_{L^2(Q)} \leq \frac{T}{\sqrt{2}}\|u\|_{\overline{U}}$, $u \in \overline{U}$ *and* $\|u\|_{\overline{U}} \leq \|u\|_{H^1_0(Q)}$, $u \in U$;

(ii) the other directions of the norm inequalities do not holds, i.e. $\|\cdot\|_{\overline{U}}$ *is neither equivalent to* $\|\cdot\|_{L^2(Q)}$ *on* $\overline{U}$, *nor equivalent to* $\|\cdot\|_{H^1_0(Q)}$ *on* U;

(iii) $U \subsetneq \overline{U} \subsetneq L^2(Q)$, *i.e.* $\overline{U}$ *is a strict subset of* $L^2(Q)$ *and a strict superset of* U.

Remark 3.12. *Although constructed by completely different means, the well-posed and optimally stable extension* (9) *of the weak wave equation in example 2.6 corresponds (up to isometric isomorphism) exactly with the well-posed and optimally stable formulation of the wave equation introduced in [20], with* $\overline{U} \equiv \mathcal{H}_{0;0,}$ *and* $\overline{B} \equiv \mathcal{E}^*\widetilde{\Box(\cdot)}$. *This becomes clear by the uniqueness of the completion and continuous extension as stated in remark 2.4. Moreover, our framework also covers other formulations, e.g., [9], where the wave equation is reformulated as a first order system in the velocity and the flux variable.*

Remark 3.13. *By remark 3.2, we can replace* $-\Delta_x^{\mathrm{we}}$ *by any bounded self-adjoint coercive operator* $A_x : H^1_0(\Omega) \to H^{-1}(\Omega)$ *provided that* A_x *satisfies the assumptions of [15, ch. IV Theorems 3.1 & 3.2] (i.e. theorem 3.6 holds true), by replacing the norm on* $H^1_0(\Omega)$ *by* $\|A_x\cdot\|_{H^{-1}(\Omega)}$.

3.3.3 Ultra-weak in time

We consider the ultra-weak in time variational formulation of the wave equation as given in example 2.7. As this setting can be reduced to the strong in time formulation given in example 2.5, we denote by B, U and V the strong in time operator, trial space and test space, respectively, as given in example 2.5 and denote the ultra-weak in time operator, trial space and test space as given in example 2.7 by B^{uw}, U^{uw} and V^{uw}, respectively. Using the flip operator $\mathbb{F}\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \coloneqq \left(\begin{smallmatrix} x_2 \\ x_1 \end{smallmatrix}\right)$ and the time reversal operator $\mathbb{T}f(t) \coloneqq f(T-t)$ as introduced in §3.3.1, we get

$$U^{\mathrm{uw}} = \mathbb{T}\mathbb{F}V = \mathbb{F}V, \quad V^{\mathrm{uw}} \coloneqq \mathbb{T}\mathbb{F}U, \quad \langle B^{\mathrm{uw}}\vec{u}, \vec{v}\rangle_{[V^{\mathrm{uw}}]' \times V} = \langle \vec{u}, -\partial_t\vec{v} + A^*\vec{v}\rangle_{\mathbb{F}V \times \mathbb{F}V'},$$

for all $\vec{u} \in U^{\mathrm{uw}}$ and all $\vec{v} \in V^{\mathrm{uw}}$. Using $-\mathbb{T}\mathbb{F}\partial_t = \partial_t\mathbb{T}\mathbb{F}$, $\mathbb{T}\mathbb{F}A^* = A\mathbb{T}\mathbb{F}$ and $(\mathbb{T}\mathbb{F})^* = \mathbb{T}\mathbb{F}$ as stated in the proof of corollary 3.4, it holds for all $\vec{u} \in U^{\mathrm{uw}}$ and all $\vec{v} \in V^{\mathrm{uw}}$, that

$$\begin{aligned}
\langle B^{\mathrm{uw}}\vec{u}, \vec{v}\rangle_{[V^{\mathrm{uw}}]' \times V^{\mathrm{uw}}} &= \langle -\partial_t\vec{v} + A^*\vec{v}, \vec{u}\rangle_{\mathbb{T}\mathbb{F}V' \times \mathbb{T}\mathbb{F}V} = \langle -\mathbb{T}\mathbb{F}\partial_t\vec{v} + \mathbb{T}\mathbb{F}A^*\vec{v}, \mathbb{T}\mathbb{F}\vec{u}\rangle_{V' \times V} \\
&= \langle \partial_t\mathbb{T}\mathbb{F}\vec{v} + A\mathbb{T}\mathbb{F}\vec{v}, \mathbb{T}\mathbb{F}\vec{u}\rangle_{V' \times V} = \langle B\mathbb{T}\mathbb{F}\vec{v}, \mathbb{T}\mathbb{F}\vec{u}\rangle_{V' \times V} = \langle B^*\mathbb{T}\mathbb{F}\vec{u}, \mathbb{T}\mathbb{F}\vec{v}\rangle_{U' \times U} \\
&= \langle \mathbb{T}\mathbb{F}B^*\mathbb{T}\mathbb{F}\vec{u}, \vec{v}\rangle_{[V^{\mathrm{uw}}]' \times V^{\mathrm{uw}}}.
\end{aligned}$$

Thus, it holds $B^{\mathrm{uw}} = \mathbb{TF}B^*\mathbb{TF}$ and by a similar calculation $(B^{\mathrm{uw}})^* = \mathbb{TF}B\mathbb{TF}$. With that, we can use remark 3.6, corollaries 3.2 and 3.4, and theorem 3.3 to state similar results regarding B^{uw} and $(B^{\mathrm{uw}})^*$, noting, that B^{uw} is bounded / isomorphic if and only if $(B^{\mathrm{uw}})^*$ is bounded / isomorphic. The only result in §3.3.1, that cannot be reproduced for B^{uw} is the norm representation (22) as we do not have an explicit formula for the dual norm or for the inverse of the Riesz operator on U (and hence on $V^{\mathrm{uw}} = \mathbb{TF}U$) as it was the case in (21). Thus, we are not able to derive a representation formula for $\|B^{\mathrm{uw}}\cdot\|_{[V^{\mathrm{uw}}]'}$, neither by a direct calculation nor by using (11). Now, denoting the well-posed and optimally stable extension of U^{uw} and B^{uw} by $\overline{U}^{\mathrm{uw}}$ and $\overline{B}^{\mathrm{uw}}$, respectively, corollary 3.5 holds true after replacing every U and $\overline{U}$ by U^{uw} and $\overline{U}^{\mathrm{uw}}$, respectively, except for the norm representation (iv).

4 Conclusions and Outlook

In this paper, we presented a general abstract framework towards well-posed and (optimally) stable formulations of linear operator equations. The starting point is always a classical formulation, which does not need to be well-posed. This requires a Gelfand triple structure for the test space. The second step is to restrict the space for the right-hand sides in such a way that the operator equation admits a solution for such right-hand side data (think of smooth functions). Finally, we form a completion for the trial space and a unique continuous extension of the operator. This procedure can be made more explicit if also the trial space allows for a Gelfand triple structure.

This general framework is applied to the Poisson, heat and wave equation, the latter two in a variational space-time setting. Our findings are summarized in Table 1. We reproduce well-known results concerning strong, weak and ultra-weak formulations of elliptic and parabolic problems. However, the presented setting also applies for the hyperbolic wave equation, where we derive well-posed formulations, which are (to the very best of our knowledge) new. We can characterize trial spaces and induced norms in such a way that the formulation of the wave equation in these spaces is well-posed and optimally stable. It turns out that these are non-standard Sobolev-type spaces.

As already indicated above, this paper is meant to lay the theoretical foundation for a subsequent numerical discretization (in terms of Galerkin and Petrov-Galerkin schemes) and also for model reduction of parameterized linear operator equations. This will be the topic of subsequent parts of this work. It is clear that the numerical realization of the involved norms will be a challenge. Concerning model reduction, we will investigate to which extend the combination of parameter-dependent trial spaces (i.e., leaving the realm of linear model reduction and the known barrier of the Kolmogorov n-width for transport- and wave-type problems) and parameter-independent test spaces (allowing for an efficient computation of the norm of the residual as an a posteriori error estimator) might be beneficial.

Moreover, we are aiming to apply the presented framework also to first order transport problems, singular integral and Schrödinger-type operators.

Eq.	Form	U	V	$\bar{U}$	Ref.
Poisson	strong	$H^{\Delta} \cap H_0^1$	L^2	U	§3.1.1
	weak	H_0^1	H_0^1	U	§3.1.2
	ultra-weak	L^2	$H^{\Delta} \cap H_0^1$	U	§3.1.3
heat	strong in t	$L^2(I; H_0^1) \cap H_{0,}^1(I; H^{-1})$	$L^2(I; H_0^1)$	U	§3.2.1
	weak in t	$L^2(I; H_0^1)$	$L^2(I; H_0^1) \cap H_{,0}^1(I; H^{-1})$	U	§3.2.2
wave	strong in t (1^{st} or.)	$H_{0,}^1(I; L^2 \times H^{-1}) \cap L^2(I; H_0^1 \times L^2)$	$L^2(I; L^2 \times H_0^1)$	$U \hookrightarrow_d \overline{U} \hookrightarrow_d L^2(Q)^2$ $U \subsetneq \overline{U} \subsetneq L^2(Q)^2$	§3.3.1
	weak in t	$H_{0,}^1(I; H_0^1)$	$H_{,0}^1(I; H_0^1)$	$U \hookrightarrow_d \overline{U} \hookrightarrow_d L^2(Q)$ $U \subsetneq \overline{U} \subsetneq L^2(Q)$	§3.3.2
	ultra-weak in t (1^{st} or.)	$L^2(I; H_0^1 \times L^2)$	$H_{,0}^1(I; H^{-1} \times L^2) \cap L^2(I; L^2 \times H_0^1)$	$U \hookrightarrow_d \overline{U} \hookrightarrow_d L^2(Q)^2$ $U \subsetneq \overline{U} \subsetneq L^2(Q)^2$	§3.3.3

Table 1: Summary of the application of the general framework to the described examples. We omit the dependency on Ω for brevity.

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