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Abstract We consider a space–time variational formulation of the wave equation by including integration by parts also in the time variable. A standard finite element discretization by using lowest order piecewise linear continuous functions then requires a CFL condition to ensure stability. To overcome this restriction, and following the work of A. A. Zlotnik (1994), we consider, in the case of space–time tensor product discretizations, a stabilized variational problem which is unconditionally stable. We provide a stability and error analysis, and some numerical results which confirm the theoretical findings.

1 Introduction

While for the analysis of parabolic and hyperbolic partial differential equations a variety of approaches such as Fourier and Laplace methods, semigroup theory, or Galerkin methods, is available, see, for example, [9, 10, 11, 14, 21, 22], standard approaches for the numerical solution are in most cases based on semi–discretizations where the discretization in space and time is split accordingly, see, e.g., [19] for parabolic problems, and [5, 6, 15] for hyperbolic equations. More recently, there exist space–time approaches as for example in [1, 2, 12, 13, 16, 17, 20] for parabolic problems, and [3, 4, 7, 8, 23] for hyperbolic equations, see also [18] where the space–time discretization of the wave equation requires some CFL condition.

In this work we introduce a stabilized finite element method for a second order ordinary differential equation and we transfer this approach to the corresponding hyperbolic partial differential equation. As model problem we consider the Dirichlet problem for the wave equation,

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$$\left.\begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) = \partial_t u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$, $T > 0$ is a finite time and f is a given right–hand side. The variational formulation of (1) is to find $u \in H^{1,1}_{0;0,}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;L^2(\Omega)) \subset H^1(Q)$ such that

$$-\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)} = \langle f, w \rangle_Q \tag{2}$$

is satisfied for all $w \in H^{1,1}_{0;,0}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;L^2(\Omega)) \subset H^1(Q)$, where $f \in L^2(Q)$ is given. Here, we use the standard Sobolev and Bochner spaces with the subspaces

$$H^1_{0,}(0,T;L^2(\Omega)) := \left\{ v \in H^1(0,T;L^2(\Omega)) \colon v(\cdot,0) = 0 \right\}$$

and

$$H^1_{,0}(0,T;L^2(\Omega)) := \left\{ v \in H^1(0,T;L^2(\Omega)) \colon v(\cdot,T) = 0 \right\}.$$

Furthermore, $\langle \cdot,\cdot \rangle_Q$ denotes the duality pairing as extension of the inner product in $L^2(Q)$. Note that the initial condition $u(\cdot,0) = 0$ is considered in the strong sense, whereas the initial condition $\partial_t u(\cdot,0) = 0$ is incorporated in a weak sense. It is well known that for $f \in L^2(Q)$ there exists a unique solution $u \in H^{1,1}_{0;0,}(Q)$ of the variational formulation (2), see [9, Theorem 3.2 in Chapter IV], and [18].

Nevertheless, a conforming tensor–product space–time discretization of (2) by piecewise multilinear continuous functions requires the CFL condition [18]

$$h_t \le \frac{1}{\sqrt{d}}\, h_x, \tag{3}$$

where h_t and h_x are the uniform mesh sizes in time and space, see also Remark 1 in Section 3. To gain a deeper understanding of the CFL condition (3) a corresponding scalar ordinary differential equation

$$\partial_{tt}u(t) + \mu\, u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0, \tag{4}$$

where $\mu > 0$ and f are given, is analyzed and an unconditionally stable finite element method for (4) is introduced. Note that $\mu > 0$ is related to the eigenvalues of the Laplace operator for homogeneous Dirichlet conditions.

Instead of the variational formulation (2) we consider a stabilized formulation which generalizes the approach of [23]. This stabilization is first discussed for the scalar differential equation (4), and then transferred to the wave equation (1). The rest of this paper is organized as follows: In Section 2 we consider the second order ordinary differential equation (4), where we show unique solvability. In addition, we prove a discrete inf–sup condition to get an unconditionally stable numerical scheme and error estimates. We present some numerical examples to illustrate the theoretical results. In Section 3 we extend the ideas of Section 2 to the scalar wave equation, where we get an unconditionally stable finite element method for the wave equation. We present a numerical analysis of the discretization scheme, an error estimate with respect to $\|\cdot\|_{L^2(Q)}$ and we provide some numerical results for illustration.

2 Second order ordinary differential equations

As a model problem we consider the second order linear equation for $\mu > 0$,

$$\partial_{tt} u(t) + \mu\, u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0, \tag{5}$$

and the variational formulation to find $u \in H^1_{0,}(0,T)$ such that

$$a(u,w) = \langle f, w\rangle_{(0,T)} \tag{6}$$

is satisfied for all $w \in H^1_{,0}(0,T)$, where $T > 0$ and $f \in [H^1_{,0}(0,T)]'$ are given, and where the bilinear form is

$$a(u,w) := -\langle \partial_t u, \partial_t w\rangle_{L^2(0,T)} + \mu \langle u, w\rangle_{L^2(0,T)}.$$

Note that $\langle\cdot,\cdot\rangle_{(0,T)}$ denotes the duality pairing as extension of the inner product in $L^2(0,T)$, and the Sobolev spaces

$$H^1_{0,}(0,T) := \Big\{ v \in H^1(0,T) \colon v(0) = 0 \Big\},$$
$$H^1_{,0}(0,T) := \Big\{ v \in H^1(0,T) \colon v(T) = 0 \Big\}$$

are endowed with the inner products

$$\langle u, v\rangle_{H^1_{0,}(0,T)} := \langle u, v\rangle_{H^1_{,0}(0,T)} := \int_0^T \partial_t u(t) \partial_t v(t)\, dt,$$

and with the induced norm

$$|u|^2_{H^1(0,T)} := \|\partial_t u\|^2_{L^2(0,T)} = \int_0^T [\partial_t u(t)]^2\, dt.$$

The dual space $[H^1_{,0}(0,T)]'$ is characterized as completion of $L^2(0,T)$ with respect to the norm

$$\|f\|_{[H^1_{,0}(0,T)]'} := \sup_{0 \neq w \in H^1_{,0}(0,T)} \frac{\langle f, w\rangle_{(0,T)}}{|w|_{H^1(0,T)}}.$$

For $v \in H^1_{0,}(0,T)$ we define $w(t) = (\overline{\mathscr{H}}_T v)(t) := v(T) - v(t)$, i.e. $w \in H^1_{,0}(0,T)$. Then the variational formulation (6) is equivalent to the variational formulation to find $u \in H^1_{0,}(0,T)$ such that

$$-\langle \partial_t u, \partial_t \overline{\mathscr{H}}_T v \rangle_{L^2(0,T)} + \mu \langle u, \overline{\mathscr{H}}_T v \rangle_{L^2(0,T)} = \langle f, \overline{\mathscr{H}}_T v \rangle_{(0,T)} \tag{7}$$

is satisfied for all $v \in H^1_{0,}(0,T)$. Since the bilinear form

$$-\langle \partial_t u, \partial_t \overline{\mathscr{H}}_T v \rangle_{L^2(0,T)} = \langle \partial_t u, \partial_t v \rangle_{L^2(0,T)} \quad \text{for } u, v \in H^1_{0,}(0,T)$$

implies an elliptic operator $A : H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$, unique solvability of the variational formulation (7) follows by using some compact perturbation argument, and injectivity, see [18, Theorem 4.6].

Next, we consider a conforming finite element discretization for the variational formulation (7). For a time interval $(0,T)$ and a discretization parameter $N \in \mathbf{N}$ we define nodes

$$0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T,$$

finite elements $\tau_\ell = (t_{\ell-1}, t_\ell)$ of local mesh size $h_\ell = t_\ell - t_{\ell-1}$, $\ell = 1, \ldots, N$, the global mesh size $h = \max h_\ell$ and a related finite element space $S^1_h(0,T) = \operatorname{span}\{\varphi_k\}_{k=0}^N$ of piecewise linear continuous functions, where the basis functions φ_k are the usual hat functions. The Galerkin–Bubnov finite element discretization of the variational formulation (7) is to find $u_h \in V_h := S^1_h(0,T) \cap H^1_{0,}(0,T) = \operatorname{span}\{\varphi_k\}_{k=1}^N$ such that

$$-\langle \partial_t u_h, \partial_t \overline{\mathscr{H}}_T v_h \rangle_{L^2(0,T)} + \mu \langle u_h, \overline{\mathscr{H}}_T v_h \rangle_{L^2(0,T)} = \langle f, \overline{\mathscr{H}}_T v_h \rangle_{(0,T)} \tag{8}$$

is satisfied for all $v_h \in V_h$. It turns out that for a sufficiently small mesh size

$$h \le \frac{2\sqrt{3}}{(2 + \sqrt{\mu} T)\mu T} \tag{9}$$

there holds the discrete stability condition [18, Theorem 4.9]

$$c(\mu, T)\, |u_h|_{H^1(0,T)} \le \sup_{0 \ne v_h \in V_h} \frac{a(u_h, \overline{\mathscr{H}}_T v_h)}{|v_h|_{H^1(0,T)}} \quad \text{for all } u_h \in V_h,$$

implying the error estimate [18, Theorem 4.10], when assuming $u \in H^2(0,T)$,

$$|u - u_h|_{H^1(0,T)} \le c(\mu, T)\, h\, |u|_{H^2(0,T)} .$$

When considering the stability of the finite element scheme (8) in the case of a uniform mesh, i.e. when analyzing the root condition, instead of (9) we conclude the weaker mesh assumption [18]

$$h \le \sqrt{\frac{12}{\mu}} .$$

To overcome the mesh condition (9) we will stabilize the numerical scheme in (8) for which we need the following technical lemmata, where the trapezoidal rule is used analogously as in [23, Chapter 2]. In addition to $S_h^1(0,T)$ we also use the finite element space $S_h^0(0,T)$ of piecewise constant functions.

Lemma 1. *For all $f \in L^2(0,T)$ there holds*

$$\partial_t I_h \int_0^t f(s)ds = Q_h^0 f = \partial_t I_h \int_T^t f(s)ds, \tag{10}$$

where $I_h\colon C[0,T] \to S_h^1(0,T)$ is the piecewise linear nodal interpolation operator, and $Q_h^0\colon L^2(0,T) \to S_h^0(0,T)$ denotes the L^2 projection on the piecewise constant finite element space $S_h^0(0,T)$.

Proof. For $t \in \tau_\ell = (t_{\ell-1}, t_\ell)$, $\ell = 1, \ldots, N$, we have

$$\partial_t I_h \int_0^t f(s)ds = \frac{1}{h_\ell}\left[\int_0^{t_\ell} f(s)ds - \int_0^{t_{\ell-1}} f(s)ds\right] = \frac{1}{h_\ell}\int_{t_{\ell-1}}^{t_\ell} f(s)ds = Q_h^0 f,$$

and

$$\partial_t I_h \int_T^t f(s)ds = \frac{1}{h_\ell}\left[\int_T^{t_\ell} f(s)ds - \int_T^{t_{\ell-1}} f(s)ds\right] = \frac{1}{h_\ell}\int_{t_{\ell-1}}^{t_\ell} f(s)ds = Q_h^0 f.$$

□

Lemma 2. *For all $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ and $w_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$ there holds the representation*

$$\langle u_h, w_h \rangle_{L^2(0,T)} = \frac{1}{12}\sum_{\ell=1}^N h_\ell^2 \langle \partial_t u_h, \partial_t w_h \rangle_{L^2(\tau_\ell)} + \langle u_h, Q_h^0 w_h \rangle_{L^2(0,T)}, \tag{11}$$

where $Q_h^0\colon L^2(0,T) \to S_h^0(0,T)$ denotes the L^2 projection on the piecewise constant finite element space $S_h^0(0,T)$.

Proof. We consider the L^2 projection Q_h^0 on the finite element space $S_h^0(0,T)$, and with the error representation of the trapezoidal rule we obtain for each finite element τ_ℓ, $\ell = 1, \ldots, N$,

$$\begin{aligned} Q_h^0 \int_T^t w_h(s)\,ds &= \frac{1}{h_\ell}\int_{t_{\ell-1}}^{t_\ell}\int_T^t w_h(s)\,ds\,dt \\ &= \frac{1}{2}\left[\int_T^{t_{\ell-1}} w_h(s)\,ds + \int_T^{t_\ell} w_h(s)\,ds\right] - \frac{h_\ell^2}{12}\,\partial_t w_{h|\tau_\ell} \\ &= Q_h^0 I_h \int_T^t w_h(s)\,ds - \frac{h_\ell^2}{12}\,\partial_t w_{h|\tau_\ell}\,. \end{aligned}$$

Using integration by parts and (10) we further have

$$\begin{aligned} \int_0^T \partial_t u_h(t)\, I_h \int_T^t w_h(s)\,ds\,dt &= -\int_0^T u_h(t)\,\partial_t I_h \int_T^t w_h(s)\,ds\,dt \\ &= -\int_0^T u_h(t)\, Q_h^0 w_h(t)\,dt\,. \end{aligned}$$

With this we then conclude, by using integration by parts and the local definition of the L^2 projection Q_h^0,

$$\begin{aligned}
\langle u_h, w_h\rangle_{L^2(0,T)} &= \int_0^T u_h(t)\,\partial_t \int_T^t w_h(s)\,ds\,dt = -\int_0^T \partial_t u_h(t) \int_T^t w_h(s)\,ds\,dt \\
&= -\sum_{\ell=1}^N \int_{t_{\ell-1}}^{t_\ell} \partial_t u_h(t)\, Q_h^0 \int_T^t w_h(s) ds\,dt \\
&= \sum_{\ell=1}^N \frac{h_\ell^2}{12} \int_{t_{\ell-1}}^{t_\ell} \partial_t u_h(t)\,\partial_t w_h(t)\,dt - \sum_{\ell=1}^N \int_{t_{\ell-1}}^{t_\ell} \partial_t u_h(t)\, Q_h^0 I_h \int_T^t w_h(s)\,ds\,dt \\
&= \frac{1}{12}\sum_{\ell=1}^N h_\ell^2 \langle \partial_t u_h, \partial_t w_h\rangle_{L^2(\tau_\ell)} - \sum_{\ell=1}^N \int_{t_{\ell-1}}^{t_\ell} \partial_t u_h(t)\, I_h \int_T^t w_h(s)\,ds\,dt \\
&= \frac{1}{12}\sum_{\ell=1}^N h_\ell^2 \langle \partial_t u_h, \partial_t w_h\rangle_{L^2(\tau_\ell)} - \int_0^T \partial_t u_h(t)\, I_h \int_T^t w_h(s) ds\,dt \\
&= \frac{1}{12}\sum_{\ell=1}^N h_\ell^2 \langle \partial_t u_h, \partial_t w_h\rangle_{L^2(\tau_\ell)} + \int_0^T u_h(t)\,\partial_t I_h \int_T^t w_h(s) ds\,dt \\
&= \frac{1}{12}\sum_{\ell=1}^N h_\ell^2 \langle \partial_t u_h, \partial_t w_h\rangle_{L^2(\tau_\ell)} + \langle u_h, Q_h^0 w_h\rangle_{L^2(0,T)},
\end{aligned}$$

i.e. the representation (11). □

Now we are in a position to find an alternative representation of the bilinear form $a(\cdot,\cdot)$.

Corollary 1. *For $u_h \in S_h^1(0,T)\cap H_{0,}^1(0,T)$ and $w_h \in S_h^1(0,T)\cap H_{,0}^1(0,T)$ we have*

$$\begin{aligned}
a(u_h, w_h) &= -\langle \partial_t u_h, \partial_t w_h\rangle_{L^2(0,T)} + \mu\langle u_h, w_h\rangle_{L^2(0,T)} \\
&= -\langle \partial_t u_h, \partial_t w_h\rangle_{L^2(0,T)} + \sum_{\ell=1}^N \frac{\mu h_\ell^2}{12}\langle \partial_t u_h, \partial_t w_h\rangle_{L^2(\tau_\ell)} + \mu\langle u_h, Q_h^0 w_h\rangle_{L^2(0,T)} \\
&= \sum_{\ell=1}^N \left(\frac{\mu h_\ell^2}{12} - 1\right)\langle \partial_t u_h, \partial_t w_h\rangle_{L^2(\tau_\ell)} + \mu\langle u_h, Q_h^0 w_h\rangle_{L^2(0,T)}. \qquad (12)
\end{aligned}$$

Motivated by the representation (12) we now define the perturbed bilinear form

$$a_h(u_h, w_h) := -\langle \partial_t u_h, \partial_t w_h\rangle_{L^2(0,T)} + \mu\langle u_h, Q_h^0 w_h\rangle_{L^2(0,T)} \qquad (13)$$

for $u_h \in S_h^1(0,T)\cap H_{0,}^1(0,T)$ and $w_h \in S_h^1(0,T)\cap H_{,0}^1(0,T)$, and we consider the perturbed variational formulation to find $\widetilde{u}_h \in S_h^1(0,T)\cap H_{0,}^1(0,T)$ such that

$$a_h(\widetilde{u}_h, w_h) = \langle f, w_h\rangle_{(0,T)} \qquad (14)$$

is satisfied for all $w_h \in S_h^1(0,T)\cap H_{,0}^1(0,T)$.

Lemma 3. *The perturbed bilinear form* (13) *is bounded, i.e. we have*

$$\Big| a_h(u_h, w_h) \Big| \le \left(1 + \frac{1}{2}\mu T^2\right) |u_h|_{H^1(0,T)} |w_h|_{H^1(0,T)}$$

for all $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$, $w_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$.

Proof. With the Cauchy–Schwarz inequality, the L^2 stability of Q_h^0, and with the Poincaré inequality we have

$$\begin{aligned} \Big| a_h(u_h, w_h) \Big| &\le |u_h|_{H^1(0,T)} |w_h|_{H^1(0,T)} + \mu \, \|u_h\|_{L^2(0,T)} \|Q_h^0 w_h\|_{L^2(0,T)} \\ &\le \left(1 + \frac{1}{2}\mu T^2\right) |u_h|_{H^1(0,T)} |w_h|_{H^1(0,T)} \end{aligned}$$

for $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$, $w_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$, and so the assertion. □

To prove a discrete stability condition for the perturbed bilinear form (13) we need the following lemma which is analogous to [23, Theorem 2.1].

Lemma 4. *For a given* $z_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$ *represented by*

$$z_h(t) = \sum_{i=0}^{N} z_i \varphi_i(t) \quad \text{with } z_N = 0$$

and a fixed index $j \in \{0, \dots, N-1\}$ *there exists a function* $\overline{v}_h^j \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ *with the following properties:*

i. For $t \in [0, t_j]$ *we have* $\overline{v}_h^j(t) = 0$.
ii. For $\ell = j+1, \dots, N$ *we have*

$$\langle \partial_t \overline{v}_h^j, \partial_t z_h \rangle_{L^2(\tau_\ell)} = \frac{1}{2}\Big(z_\ell^2 - z_{\ell-1}^2\Big)$$

as well as

$$\langle \overline{v}_h^j, Q_h^0 z_h \rangle_{L^2(\tau_\ell)} = \frac{1}{2}\left(\int_{t_j}^{t_\ell} z_h(s)ds\right)^2 - \frac{1}{2}\left(\int_{t_j}^{t_{\ell-1}} z_h(s)ds\right)^2.$$

iii. There holds the estimate

$$|\overline{v}_h^j|_{H^1(0,T)} \le \|z_h\|_{L^2(0,T)}.$$

Proof. For $z_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$ we consider the piecewise linear interpolation of the antiderivative, i.e. for $t \in [0,T]$ we define

$$\overline{v}_h(t) := \sum_{k=0}^{N} \left(\int_0^{t_k} z_h(s)ds\right) \varphi_k(t) = I_h \int_0^t z_h(s)ds\,, \quad \overline{v}_h \in S_h^1(0,T) \cap H^1_{0,}(0,T).$$

From (10) the relation $\partial_t \overline{v}_h = Q_h^0 z_h$ follows. For a fixed index $j \in \{0,\ldots,N-1\}$ we now define

$$z_h^j(t) = \sum_{i=0}^N z_i^j \varphi_i(t), \quad z_i^j = \begin{cases} (-1)^{j-i} z_j & \text{for } i = 0,\ldots,j, \\ z_i & \text{for } i = j+1,\ldots,N. \end{cases}$$

Note that $z_h^j \in S_h^1(0,T) \cap H_{,0}^1(0,T)$, and according to z_h^j we introduce $\overline{v}_h^j$ satisfying $\partial_t \overline{v}_h^j = Q_h^0 z_h^j$. In particular for $j > 0$ and $t \in \tau_\ell$ for $\ell = 1,\ldots,j$ we then have

$$\partial_t \overline{v}_h^j(t) = Q_h^0 z_h^j(t) = \frac{1}{h_\ell}\int_{t_{\ell-1}}^{t_\ell} z_h^j(s)\,ds = \frac{1}{2}\left(z_{\ell-1}^j + z_\ell^j\right) = 0,$$

and due to $\overline{v}_h^j(0) = 0$ we conclude $\overline{v}_h^j(t) = 0$ for $t \in [0,t_j]$, i.e. *i.*

To prove *ii.* we compute for $\ell = j+1,\ldots,N$

$$\begin{aligned} \langle \partial_t \overline{v}_h^j, \partial_t z_h \rangle_{L^2(\tau_\ell)} &= \langle Q_h^0 z_h^j, \partial_t z_h \rangle_{L^2(\tau_\ell)} \\ &= \frac{1}{2}\left(z_{\ell-1}^j + z_\ell^j\right)\left(z_\ell - z_{\ell-1}\right) \\ &= \frac{1}{2}\left(z_{\ell-1} + z_\ell\right)\left(z_\ell - z_{\ell-1}\right) = \frac{1}{2}\left(z_\ell^2 - z_{\ell-1}^2\right) \end{aligned}$$

as well as

$$\begin{aligned} \langle \overline{v}_h^j, Q_h^0 z_h \rangle_{L^2(\tau_\ell)} &= \int_{t_{\ell-1}}^{t_\ell} I_h \int_0^t z_h^j(s)ds\, Q_h^0 z_h(t)\,dt \\ &= Q_h^0 z_{h|\tau_\ell} \int_{t_{\ell-1}}^{t_\ell}\left[\int_0^{t_{\ell-1}} z_h^j(s)ds\,\varphi_{\ell-1}(t) + \int_0^{t_\ell} z_h^j(s)ds\,\varphi_\ell(t)\right]dt \\ &= \frac{1}{h_\ell}\int_{t_{\ell-1}}^{t_\ell} z_h(s)ds\;\frac{1}{2}h_\ell\left[\int_0^{t_{\ell-1}} z_h^j(s)ds + \int_0^{t_\ell} z_h^j(s)ds\right] \\ &= \frac{1}{2}\int_{t_{\ell-1}}^{t_\ell} z_h(s)ds\left[\int_{t_j}^{t_{\ell-1}} z_h(s)ds + \int_{t_j}^{t_\ell} z_h(s)ds\right] \\ &= \frac{1}{2}\left(\int_{t_{\ell-1}}^{t_\ell} z_h(s)\,ds\right)^2 + \int_{t_{\ell-1}}^{t_\ell} z_h(s)\,ds \int_{t_j}^{t_{\ell-1}} z_h(s)\,ds \\ &= \frac{1}{2}\left(\int_{t_{\ell-1}}^{t_\ell} z_h(s)\,ds + \int_{t_j}^{t_{\ell-1}} z_h(s)\,ds\right)^2 - \frac{1}{2}\left(\int_{t_j}^{t_{\ell-1}} z_h(s)\,ds\right)^2 \\ &= \frac{1}{2}\left(\int_{t_j}^{t_\ell} z_h(s)\,ds\right)^2 - \frac{1}{2}\left(\int_{t_j}^{t_{\ell-1}} z_h(s)ds\right)^2. \end{aligned}$$

From the L^2 stability of Q_h^0 we finally conclude the third assertion, i.e.

$$\begin{aligned} |\overline{v}_h^j|_{H^1(0,T)} = |\overline{v}_h^j|_{H^1(t_j,T)} &= \|Q_h^0 z_h^j\|_{L^2(t_j,T)} \\ &= \|Q_h^0 z_h\|_{L^2(t_j,T)} \leq \|Q_h^0 z_h\|_{L^2(0,T)} \leq \|z_h\|_{L^2(0,T)}. \qquad \square \end{aligned}$$

Lemma 5. *The variational formulation to find $z_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$ such that*

$$a_h(v_h, z_h) = \langle g, \partial_t v_h \rangle_{L^2(0,T)} \tag{15}$$

is satisfied for all $v_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ is uniquely solvable, where $g \in L^2(0,T)$ is given. Moreover, the stability estimate

$$\|z_h\|_{L^2(0,T)} \le 2T\, \|g\|_{L^2(0,T)} \tag{16}$$

holds for any mesh with maximal mesh size h.

Proof. The finite element stiffness matrix of the variational problem (15) is upper triangular with positive diagonal elements and hence, there exists a unique solution $z_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$ of (15).

For the stability estimate we consider for an index $j \in \{0, \dots, N-1\}$ the function $\overline{v}_h^j \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ as given in Lemma 4. Plugging $\overline{v}_h^j$ into (15) and by using the properties of Lemma 4 this gives

$$\begin{aligned}
&\langle g, \partial_t \overline{v}_h^j \rangle_{L^2(0,T)} = a_h(\overline{v}_h^j, z_h) \\
&\quad = -\langle \partial_t \overline{v}_h^j, \partial_t z_h \rangle_{L^2(0,T)} + \mu \langle \overline{v}_h^j, Q_h^0 z_h \rangle_{L^2(0,T)} \\
&\quad = -\sum_{\ell=j+1}^{N} \langle \partial_t \overline{v}_h^j, \partial_t z_h \rangle_{L^2(\tau_\ell)} + \mu \sum_{\ell=j+1}^{N} \langle \overline{v}_h^j, Q_h^0 z_h \rangle_{L^2(\tau_\ell)} \\
&\quad = -\frac{1}{2} \sum_{\ell=j+1}^{N} \left(z_\ell^2 - z_{\ell-1}^2 \right) + \frac{\mu}{2} \sum_{\ell=j+1}^{N} \left(\left(\int_{t_j}^{t_\ell} z_h(s)ds \right)^2 - \left(\int_{t_j}^{t_{\ell-1}} z_h(s)ds \right)^2 \right) \\
&\quad = \frac{1}{2} z_j^2 + \frac{\mu}{2} \left(\int_{t_j}^{T} z_h(s)ds \right)^2.
\end{aligned}$$

This result yields, with the Cauchy–Schwarz inequality, the Poincaré inequality, and the use of the properties of Lemma 4,

$$\begin{aligned}
&\|z_h\|^2_{L^2(0,T)} = \sum_{\ell=1}^{N} \|z_h\|^2_{L^2(\tau_\ell)} = \sum_{\ell=1}^{N} \frac{h_\ell}{3} \left(z_\ell^2 + z_\ell z_{\ell-1} + z_{\ell-1}^2 \right) \le \frac{1}{2} \sum_{\ell=1}^{N} h_\ell \left(z_\ell^2 + z_{\ell-1}^2 \right) \\
&\le \frac{1}{2} \sum_{j=1}^{N-1} h_j z_j^2 + \frac{1}{2} \sum_{j=0}^{N-1} h_{j+1} z_j^2 \\
&\le \sum_{j=1}^{N-1} h_j \, \langle g, \partial_t \overline{v}_h^j \rangle_{L^2(0,T)} + \sum_{j=0}^{N-1} h_{j+1} \, \langle g, \partial_t \overline{v}_h^j \rangle_{L^2(0,T)} \\
&\le \sum_{j=1}^{N-1} h_j \|g\|_{L^2(0,T)} |\overline{v}_h^j|_{H^1(0,T)} + \sum_{j=0}^{N-1} h_{j+1} \|g\|_{L^2(0,T)} |\overline{v}_h^j|_{H^1(0,T)} \\
&\le 2T\, \|g\|_{L^2(0,T)} \|z_h\|_{L^2(0,T)},
\end{aligned}$$

i.e. the assertion. □

Lemma 6. *For each $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ there holds the discrete inf–sup condition*

$$\frac{1}{1+\sqrt{2}\mu T^2}\,|u_h|_{H^1(0,T)} \leq \sup_{0\neq w_h \in S_h^1(0,T)\cap H^1_{,0}(0,T)} \frac{|a_h(u_h,w_h)|}{|w_h|_{H^1(0,T)}}.$$

Proof. For a fixed function $u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ let $w_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$ be the unique solution of (15) for $g := \partial_t u_h \in L^2(0,T)$, i.e. we have

$$a_h(v_h,w_h) = \langle \partial_t u_h, \partial_t v_h \rangle_{L^2(0,T)} \tag{17}$$

for all $v_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$. For the particular choice $v_h(t) = w_h(0) - w_h(t)$ with $v_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ we obtain

$$\langle \partial_t w_h, \partial_t w_h \rangle_{L^2(0,T)} - \mu \langle w_h - w_h(0), Q_h^0 w_h \rangle_{L^2(0,T)} = -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(0,T)},$$

and hence we conclude, by using the Cauchy–Schwarz and Poincaré inequalities, and the L^2 stability of the L^2 projection Q_h^0,

$$\begin{aligned}
|w_h|^2_{H^1(0,T)} &= -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(0,T)} + \mu \langle w_h - w_h(0), Q_h^0 w_h \rangle_{L^2(0,T)} \\
&\leq |u_h|_{H^1(0,T)} |w_h|_{H^1(0,T)} + \mu \, \|w_h - w_h(0)\|_{L^2(0,T)} \|Q_h^0 w_h\|_{L^2(0,T)} \\
&\leq |u_h|_{H^1(0,T)} |w_h|_{H^1(0,T)} + \frac{1}{\sqrt{2}} \mu T |w_h|_{H^1(0,T)} \|w_h\|_{L^2(0,T)} \\
&\leq \left(1+\sqrt{2}\mu T^2\right) |u_h|_{H^1(0,T)} |w_h|_{H^1(0,T)},
\end{aligned}$$

where in the last step we used the stability estimate (16).

The choice $v_h = u_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ in (17) and the estimate above yield

$$a_h(u_h,w_h) = |u_h|^2_{H^1(0,T)} \geq \frac{1}{1+\sqrt{2}\mu T^2} |u_h|_{H^1(0,T)} |w_h|_{H^1(0,T)}$$

and hence the discrete inf–sup condition follows. □

Theorem 1. *Let the unique solution u of (6) satisfy $u \in H^s(0,T)$ for some $s \in [1,2]$. There exists a unique solution $\widetilde{u}_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ of the Galerkin–Petrov finite element discretization (14) satisfying*

$$\begin{aligned}
|u - \widetilde{u}_h|_{H^1(0,T)} \leq & \left[1 + \left(1+\frac{1}{2}\mu T^2\right)\left(1+\sqrt{2}\mu T^2\right)\right] C_1(s)\, h^{s-1}\, |u|_{H^s(0,T)} \\
& + \frac{1}{12}\mu \left(1+\sqrt{2}\mu T^2\right) h^2 \, C_2 \, |u|_{H^1(0,T)},
\end{aligned}$$

where the constants $C_1, C_2 > 0$ are coming from standard interpolation error and stability estimates.

Proof. For any $v_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ we first have

$$|u-\widetilde{u}_h|_{H^1(0,T)} \leq |u-v_h|_{H^1(0,T)} + |\widetilde{u}_h - v_h|_{H^1(0,T)}$$

and it remains to bound the second term. With the discrete inf–sup condition in Lemma 6 and using the Galerkin orthogonality for the variational formulations (6) and (14), we first have

$$\begin{aligned}
\frac{1}{1+\sqrt{2}\mu T^2}|\widetilde{u}_h - v_h|_{H^1(0,T)} &\leq \sup_{0\neq w_h\in S_h^1(0,T)\cap H^1_{,0}(0,T)} \frac{|a_h(\widetilde{u}_h - v_h, w_h)|}{|w_h|_{H^1(0,T)}} \\
&= \sup_{0\neq w_h\in S_h^1(0,T)\cap H^1_{,0}(0,T)} \frac{|a_h(\widetilde{u}_h, w_h) - a_h(v_h,w_h)|}{|w_h|_{H^1(0,T)}} \\
&= \sup_{0\neq w_h\in S_h^1(0,T)\cap H^1_{,0}(0,T)} \frac{|a(u, w_h) - a_h(v_h,w_h)|}{|w_h|_{H^1(0,T)}} \\
&= \sup_{0\neq w_h\in S_h^1(0,T)\cap H^1_{,0}(0,T)} \frac{|a(u-v_h, w_h) + a(v_h,w_h) - a_h(v_h,w_h)|}{|w_h|_{H^1(0,T)}}.
\end{aligned}$$

Now, with the boundedness of the bilinear form $a(\cdot,\cdot)$ and the Poincaré inequality we further conclude

$$\begin{aligned}
a(u-v_h,w_h) &= -\langle \partial_t(u-v_h), \partial_t w_h\rangle_{L^2(0,T)} + \mu\langle u-v_h, w_h\rangle_{L^2(0,T)} \\
&\leq |u-v_h|_{H^1(0,T)}|w_h|_{H^1(0,T)} + \mu\|u-v_h\|_{L^2(0,T)}\|w_h\|_{L^2(0,T)} \\
&\leq \left(1+\frac{1}{2}\mu T^2\right)|u-v_h|_{H^1(0,T)}|w_h|_{H^1(0,T)}.
\end{aligned}$$

Moreover, using the representation (12) we can estimate the consistency error by

$$\begin{aligned}
|a(v_h,w_h) - a_h(v_h,w_h)| &= \frac{1}{12}\mu\left|\sum_{\ell=1}^N h_\ell^2 \langle \partial_t v_h, \partial_t w_h\rangle_{L^2(\tau_\ell)}\right| \\
&\leq \frac{1}{12}\mu\, h^2\, |v_h|_{H^1(0,T)}|w_h|_{H^1(0,T)}.
\end{aligned}$$

Hence we have

$$\frac{1}{1+\sqrt{2}\mu T^2}|\widetilde{u}_h - v_h|_{H^1(0,T)} \leq \left(1+\frac{1}{2}\mu T^2\right)|u-v_h|_{H^1(0,T)} + \frac{1}{12}\mu\, h^2\, |v_h|_{H^1(0,T)},$$

and therefore

$$\begin{aligned}
|u-\widetilde{u}_h|_{H^1(0,T)} \leq &\left[1+\left(1+\frac{1}{2}\mu T^2\right)\left(1+\sqrt{2}\mu T^2\right)\right]|u-v_h|_{H^1(0,T)} \\
&+\frac{1}{12}\mu\left(1+\sqrt{2}\mu T^2\right)h^2\,|v_h|_{H^1(0,T)}
\end{aligned}$$

follows. In particular for the piecewise linear nodal interpolation $v_h = I_h u$ we have

$$\|u - I_h u\|_{H^1(0,T)} \leq C_1(s)\, h^{s-1}\, |u|_{H^s(0,T)}, \quad \|I_h u\|_{H^1(0,T)} \leq C_2\, |u|_{H^1(0,T)}. \qquad \Box$$

As numerical example for the Galerkin finite element methods (8) and (14) we consider a uniform discretization of the time interval $(0,T)$ with $T = 10$ and a mesh size $h = T/N$. For $\mu = 1000$ we consider the solution $u(t) = \sin^2\left(\frac{5}{4}\pi t\right)$ and we compute the appearing integrals for the related right–hand side in (8) and (14) by the usage of high order integration rules.

In Table 1 we present the results for the stabilized variational formulation (14) which is unconditionally stable, and where the error estimate in the energy norm of Theorem 1 is confirmed. In addition we also present the error in $L^2(0,T)$ where we observe a second order convergence, as expected. But at this point we do not include any further discussion of error estimates in $L^2(0,T)$ since this is behind the scope of this contribution.

N	h	$\|u-\widetilde{u}_h\|_{L^2(0,10)}$	eoc	$\|u-\widetilde{u}_h\|_{H^1(0,10)}$	eoc
4	2.5000000	1.7722e+00	0.00	9.0867e+00	0.00
8	1.2500000	6.0704e+00	-1.78	2.0130e+01	-1.15
16	0.6250000	1.2687e+00	2.26	9.4204e+00	1.10
32	0.3125000	5.7861e+00	-2.19	6.0121e+01	-2.67
64	0.1562500	3.3966e-01	4.09	6.1941e+00	3.28
128	0.0781250	7.6647e-02	2.15	2.2955e+00	1.43
256	0.0390625	2.0315e-02	1.92	9.4091e-01	1.29
512	0.0195312	5.2649e-03	1.95	4.1539e-01	1.18
1024	0.0097656	1.3365e-03	1.98	1.9803e-01	1.07
2048	0.0048828	3.3682e-04	1.99	9.7671e-02	1.02
4096	0.0024414	8.4229e-05	2.00	4.8663e-02	1.01
8192	0.0012207	2.1057e-05	2.00	2.4310e-02	1.00
16384	0.0006104	5.2644e-06	2.00	1.2152e-02	1.00
32768	0.0003052	1.3161e-06	2.00	6.0758e-03	1.00

Table 1 Numerical results for the stabilized variational formulation (14), $\mu = 1000$, $T = 10$.

In Table 2 we present the related results for the variational formulation (8) without stabilization. We observe that we have convergence for a sufficiently small mesh size only. Note that $\sqrt{12/\mu} \approx 0.1095$.

3 Wave equation

Instead of the ordinary differential equation (5) we now consider the wave equation (1) and the related variational formulation (2), and we aim to extend the results of Section 2. For $u \in H^{1,1}_{0;0,}(Q)$ and $w \in H^{1,1}_{0;,0}(Q)$ we define the bilinear form

$$a(u,w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)}.$$

N	h	$\|u-u_h\|_{L^2(0,10)}$	eoc	$\|u-u_h\|_{H^1(0,10)}$	eoc
4	2.5000000	7.0573e+01	0.00	9.8785e+01	0.00
8	1.2500000	1.6871e+03	-4.58	3.7166e+03	-5.23
16	0.6250000	9.1421e+07	-15.73	3.7247e+08	-16.61
32	0.3125000	2.3915e+15	-24.64	1.9496e+16	-25.64
64	0.1562500	1.6337e+22	-22.70	2.9536e+23	-23.85
128	0.0781250	3.1417e-02	78.78	1.7859e+00	77.13
256	0.0390625	9.2885e-03	1.76	8.2361e-01	1.12
512	0.0195312	2.4767e-03	1.91	3.9567e-01	1.06
1024	0.0097656	6.3105e-04	1.97	1.9532e-01	1.02
2048	0.0048828	1.5839e-04	1.99	9.7325e-02	1.00
4096	0.0024414	3.9633e-05	2.00	4.8620e-02	1.00
8192	0.0012207	9.9106e-06	2.00	2.4304e-02	1.00
16384	0.0006104	2.4778e-06	2.00	1.2152e-02	1.00
32768	0.0003052	6.1946e-07	2.00	6.0757e-03	1.00

Table 2 Numerical results for the variational formulation (8), $\mu = 1000$, $T = 10$.

The Hilbert spaces $H^{1,1}_{0;0,}(Q)$ and $H^{1,1}_{0;,0}(Q)$, defined in Section 1, are endowed with the inner product

$$\langle u,v\rangle_{H^{1,1}_{0;0,}(Q)} = \langle u,v\rangle_{H^{1,1}_{0;,0}(Q)} := \int_0^T \int_\Omega \Big[\partial_t u(x,t)\partial_t v(x,t) + \nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big]\,dx\,dt,$$

where the induced norm

$$|u|^2_{H^1(Q)} := \int_0^T \int_\Omega \Big[|\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2\Big]\,dx\,dt$$

is well–defined due to Poincaré inequalities with respect to space and time. As in [9] we have unique solvability of (2) when assuming $f \in L^2(Q)$, in particular we have, see [18],

$$|u|_{H^1(Q)} \le \frac{1}{\sqrt{2}}\,T\,\|f\|_{L^2(Q)}.$$

Next, we examine a conforming finite element discretization for the variational formulation (2) in the case where $\Omega = (0,L)$ is an interval for $d = 1$, or Ω is polygonal for $d = 2$, or Ω is polyhedral for $d = 3$. For a tensor–product ansatz we consider a sequence $(\mathscr{T}_N)_{N\in\mathbb{N}}$ of admissible decompositions

$$\overline{Q} = \overline{\mathscr{T}_N} = \overline{\Omega}\times[0,T] = \bigcup_{i=1}^{N_x}\overline{\omega}_i \times \bigcup_{\ell=1}^{N_t}\overline{\tau}_\ell$$

with $N := N_x \cdot N_t$ space–time elements, where the time intervals $\tau_\ell = (t_{\ell-1}, t_\ell)$ with mesh size $h_{t,\ell}$ are defined via the decomposition

$$0 = t_0 < t_1 < t_2 < \ldots < t_{N_t-1} < t_{N_t} = T$$

of the time interval $(0,T)$. For the spatial domain Ω we consider an admissible and globally quasi–uniform decomposition into finite elements ω_i with mesh size $h_{x,i}$ which can be represented by using standard maps with respect to related reference elements. Here, the spatial elements ω_i are intervals for $d=1$, triangles or quadrilaterals for $d=2$, and tetrahedra or hexahedra for $d=3$. Next, we consider the finite element space

$$Q_h^1(Q) := V_{h_x}(\Omega) \otimes S_{h_t}^1(0,T)$$

of piecewise multilinear continuous functions, i.e.

$$V_{h_x}(\Omega) = \operatorname{span}\{\psi_j\}_{j=1}^{M_x} \subset H_0^1(\Omega), \quad S_{h_t}^1(0,T) = \operatorname{span}\{\varphi_\ell\}_{\ell=0}^{N_t} \subset H^1(0,T).$$

In fact, $V_{h_x}(\Omega)$ is either the space $S_{h_x}^1(\Omega)$ of piecewise linear continuous functions on intervals ($d=1$), triangles ($d=2$), and tetrahedra ($d=3$), or $V_{h_x}(\Omega)$ is the space $Q_{h_x}^1(\Omega)$ of piecewise linear/bilinear/trilinear continuous functions on intervals ($d=1$), quadrilaterals ($d=2$), and hexahedra ($d=3$). A finite element function $u_h \in Q_h^1(Q)$ admits the representation

$$u_h(x,t) = \sum_{\ell=0}^{N_t} \sum_{j=1}^{M_x} u_j^\ell \psi_j(x) \varphi_\ell(t) \quad \text{for } (x,t) \in Q. \tag{18}$$

The Galerkin–Petrov finite element discretization of the variational formulation (2) is to find $u_h \in Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q)$ such that

$$a(u_h,w_h) = -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x w_h \rangle_{L^2(Q)} = \langle f, w_h \rangle_Q \tag{19}$$

is satisfied for all $w_h \in Q_h^1(Q) \cap H_{0;,0}^{1,1}(Q)$. After an appropriate ordering of the degrees of freedom, the discrete variational formulation (19) is equivalent to the linear system $K_h \underline{u} = \underline{f}$ with the system matrix

$$K_h := -M_{h_x} \otimes A_{h_t} + A_{h_x} \otimes M_{h_t} \in \mathbb{R}^{M_x \cdot N_t \times M_x \cdot N_t}, \tag{20}$$

where $M_{h_x}, A_{h_x} \in \mathbb{R}^{M_x}$ are the mass and stiffness matrix with respect to space, and $M_{h_t}, A_{h_t} \in \mathbb{R}^{N_t}$ are the mass and stiffness matrix with respect to time, respectively.

Remark 1. With the help of a von Neumann type stability analysis [5] for the matrix (20) of the Galerkin–Petrov finite element method (19) for a uniform discretization in time with mesh size h_t and a uniform discretization in space with mesh size h_x for piecewise linear/bilinear/trilinear continuous functions on intervals ($d=1$), squares ($d=2$), or cubes ($d=3$) we can show stability of the corresponding numerical scheme, if the condition

$$h_t \le \frac{1}{\sqrt{d}} h_x$$

is satisfied, see [18].

From Remark 1 we conclude that we have only conditional stability of (19). To stabilize the numerical scheme in (19) we use as in (12) again Zlotnik's idea [23] and we prove the following representation.

Lemma 7. *For all* $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ *and* $w_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$ *the bilinear form in (19) has the representation*

$$a(u_h, w_h) = -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \langle \partial_{x_m} u_h, Q^0_{h_t} \partial_{x_m} w_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \partial_{x_m} u_h, \partial_t \partial_{x_m} w_h \rangle_{L^2(\Omega \times \tau_\ell)}, \tag{21}$$

where $Q^0_{h_t} \colon L^2(0,T) \to S^0_{h_t}(0,T)$ *denotes the* L^2 *projection with respect to time on the space* $S^0_{h_t}(0,T)$ *of piecewise constant functions.*

Proof. Let $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ and $w_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$ be given. With the representation (18) we have for $(x,t) \in Q$

$$u_h(x,t) = \sum_{\ell=1}^{N_t} \sum_{j=1}^{M_x} u_j^\ell \psi_j(x) \varphi_\ell(t) = \sum_{j=1}^{M_x} U_{j,h}(t) \psi_j(x), \quad U_{j,h}(t) = \sum_{\ell=1}^{N_t} u_j^\ell \varphi_\ell(t),$$

as well as

$$w_h(x,t) = \sum_{\ell=0}^{N_t-1} \sum_{j=1}^{M_x} w_j^\ell \psi_j(x) \varphi_\ell(t) = \sum_{j=1}^{M_x} W_{j,h}(t) \psi_j(x), \quad W_{j,h}(t) = \sum_{\ell=0}^{N_t-1} w_j^\ell \varphi_\ell(t).$$

Hence we have, for $m = 1, \ldots, d$, and by using (11),

$$\begin{aligned}
\langle \partial_{x_m} u_h, \partial_{x_m} w_h \rangle_{L^2(Q)} &= \sum_{i=1}^{M_x} \sum_{j=1}^{M_x} \int_0^T U_{i,h}(t) W_{j,h}(t) dt \int_\Omega \partial_{x_m} \psi_i(x) \partial_{x_m} \psi_j(x)\, dx \\
&= \sum_{i=1}^{M_x} \sum_{j=1}^{M_x} \left[\frac{1}{12} \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t U_{i,h}, \partial_t W_{j,h} \rangle_{L^2(\tau_\ell)} + \langle U_{i,h}, Q^0_{h_t} W_{j,h} \rangle_{L^2(0,T)} \right] \\
&\qquad \cdot \int_\Omega \partial_{x_m} \psi_i(x) \partial_{x_m} \psi_j(x)\, dx \\
&= \langle \partial_{x_m} u_h, Q^0_{h_t} \partial_{x_m} w_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \partial_{x_m} u_h, \partial_t \partial_{x_m} w_h \rangle_{L^2(\Omega \times \tau_\ell)}.
\end{aligned}$$

□

Due to the representation (21) we define for $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ and $w_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$ the perturbed bilinear form

$$a_h(u_h, w_h) := -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \langle \partial_{x_m} u_h, Q^0_{h_t} \partial_{x_m} w_h \rangle_{L^2(Q)} \tag{22}$$

$$= -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \langle Q^0_{h_t} \partial_{x_m} u_h, \partial_{x_m} w_h \rangle_{L^2(Q)}, \qquad (23)$$

and we consider the perturbed variational problem to find $\widetilde{u}_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ such that

$$a_h(\widetilde{u}_h, w_h) = \langle f, w_h \rangle_Q \qquad (24)$$

is satisfied for all $w_h \in Q^1_h(Q) \cap H^{1,1}_{0;,0}(Q)$.

To prove the existence and uniqueness of a solution $\widetilde{u}_h$ of (24) we show the following lemma, which is analogous to Lemma 4.

Lemma 8. *For a given $v_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ represented by*

$$v_h(x,t) = \sum_{\ell=0}^{N_t} V_{\ell,h}(x)\varphi_\ell(t), \quad V_{\ell,h}(x) = \sum_{j=1}^{M_x} v_j^\ell \psi_j(x) \quad \textit{for } (x,t) \in Q$$

with $V_{0,h}(x) = 0$ for $x \in \Omega$ and $V_{\ell,h} \in V_{h_x}(\Omega)$, and for a fixed index $j \in \{1,\ldots,N_t\}$ there exists a function $\overline{z}^j_h \in Q^1_h(Q) \cap H^{1,1}_{0;,0}(Q)$ with the following properties:

i. For $(x,t) \in \overline{\Omega} \times [t_j, T]$ we have $\overline{z}^j_h(x,t) = 0$.

ii. For $\ell = 1,\ldots,j$ and for $x \in \overline{\Omega}$ we have

$$\langle \partial_t \overline{z}^j_h(x,\cdot), \partial_t v_h(x,\cdot) \rangle_{L^2(\tau_\ell)} = \frac{1}{2}\Big([V_{\ell-1,h}(x)]^2 - [V_{\ell,h}(x)]^2 \Big),$$

and for $m = 1,\ldots,d$

$$\langle \partial_{x_m} \overline{z}^j_h(x,\cdot), Q^0_{h_t} \partial_{x_m} v_h(x,\cdot) \rangle_{L^2(\tau_\ell)} = \\ = \frac{1}{2}\left(\int_{t_{\ell-1}}^{t_j} \partial_{x_m} v_h(x,s) ds \right)^2 - \frac{1}{2}\left(\int_{t_\ell}^{t_j} \partial_{x_m} v_h(x,s) ds \right)^2.$$

iii. For $x \in \overline{\Omega}$ there holds the estimate

$$\|\partial_t \overline{z}^j_h(x,\cdot)\|_{L^2(0,T)} \le \|v_h(x,\cdot)\|_{L^2(0,T)}.$$

Proof. For $v_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ we define

$$u^j_h(x,t) = \sum_{i=0}^{N_t} U^j_{i,h}(x)\varphi_i(t), \quad U^j_{i,h}(x) := \begin{cases} V_{i,h}(x) & \text{for } i = 0,\ldots,j, \\ (-1)^{j-i} V_{j,h}(x) & \text{for } i = j+1,\ldots,N_t, \end{cases}$$

and for $(x,t) \in Q$ we set

$$\overline{z}^j_h(x,t) := -\sum_{k=0}^{N_t} \int_T^{t_k} u^j_h(x,s)\, ds \varphi_k(t) = -I_{h_t} \int_T^t u^j_h(x,s)\, ds,$$

where $I_{h_t}\colon C[0,T] \to S_h^1(0,T)$ is the interpolation operator with respect to time. Note that $\overline{z}_h^j \in Q_h^1(Q) \cap H_{0;,0}^{1,1}(Q)$. For $x \in \Omega$ it follows from relation (10) that

$$\partial_t \overline{z}_h^j(x,\cdot) = -Q_{h_t}^0 u_h^j(x,\cdot).$$

In particular for $j < N_t$, $x \in \overline{\Omega}$, and for $t \in \tau_\ell$ for $\ell = j+1,\ldots,N_t$ we then have

$$-\partial_t \overline{z}_h^j(x,t) = Q_{h_t}^0 u_h^j(x,t) = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} u_h^j(x,s)\,ds = \frac{1}{2}\left(U_{\ell-1,h}^j(x) + U_{\ell,h}^j(x)\right) = 0,$$

and due to $\overline{z}_h^j(x,T) = 0$ we conclude $\overline{z}_h^j(x,t) = 0$ for $t \in [t_j,T]$, i.e. *i.*

To prove *ii.* we first compute for $x \in \overline{\Omega}$ and for $\ell = 1,\ldots,j$

$$\begin{aligned}
\langle \partial_t \overline{z}_h^j(x,\cdot), \partial_t v_h(x,\cdot)\rangle_{L^2(\tau_\ell)} &= \int_{t_{\ell-1}}^{t_\ell} \partial_t \overline{z}_h^j(x,t)\, \partial_t v_h(x,t)\,dt \\
&= -\frac{1}{2}\left(U_{\ell-1,h}^j(x) + U_{\ell,h}^j(x)\right) \int_{t_{\ell-1}}^{t_\ell} \partial_t v_h(x,t)\,dt \\
&= -\frac{1}{2}\left(U_{\ell-1,h}^j(x) + U_{\ell,h}^j(x)\right)\left(V_{\ell,h}(x) - V_{\ell-1,h}(x)\right) \\
&= \frac{1}{2}\left(V_{\ell-1,h}(x) + V_{\ell,h}(x)\right)\left(V_{\ell-1,h}(x) - V_{\ell,h}(x)\right) \\
&= \frac{1}{2}\left([V_{\ell-1,h}(x)]^2 - [V_{\ell,h}(x)]^2\right).
\end{aligned}$$

Moreover, for $m = 1,\ldots,d$ we have for $x \in \Omega$ and for $\ell = 1,\ldots,j$

$$\begin{aligned}
&\langle \partial_{x_m} \overline{z}_h^j(x,\cdot), Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot)\rangle_{L^2(\tau_\ell)} = Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot)_{|\tau_\ell} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} \overline{z}_h^j(x,t)\,dt \\
&= -Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot)_{|\tau_\ell} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m}\left[\int_T^{t_{\ell-1}} u_h^j(x,s)ds\,\varphi_{\ell-1}(t) + \int_T^{t_\ell} u_h^j(x,s)ds\,\varphi_\ell(t)\right] dt \\
&= -\frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} v_h(x,t)dt\, \frac{1}{2} h_{t,\ell} \left[\int_T^{t_{\ell-1}} \partial_{x_m} u_h^j(x,s)ds + \int_T^{t_\ell} \partial_{x_m} u_h^j(x,s)ds\right] \\
&= -\frac{1}{2}\left[\int_{t_j}^{t_\ell} \partial_{x_m} v_h(x,t)dt - \int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h(x,t)dt\right] \\
&\qquad\qquad \cdot \left[\int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h^j(x,s)ds + \int_{t_j}^{t_\ell} \partial_{x_m} v_h^j(x,s)ds\right] \\
&= \frac{1}{2}\left(\int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h^j(x,s)ds\right)^2 - \frac{1}{2}\left(\int_{t_j}^{t_\ell} \partial_{x_m} v_h^j(x,s)ds\right)^2.
\end{aligned}$$

From the L^2 stability of $Q_{h_t}^0$ and by using

$$\|\partial_t \overline{z}_h^j(x,\cdot)\|_{L^2(0,T)} = \|\partial_t \overline{z}_h^j(x,\cdot)\|_{L^2(0,t_j)} = \|Q_{h_t}^0 u_h^j(x,\cdot)\|_{L^2(0,t_j)}$$

$$= \|Q^0_{h_t} v_h(x,\cdot)\|_{L^2(0,t_j)} \leq \|Q^0_{h_t} v_h(x,\cdot)\|_{L^2(0,T)} \leq \|v_h(x,\cdot)\|_{L^2(0,T)}$$

for $x \in \overline{\Omega}$ we finally conclude the third property. □

With the last lemma we are now able to prove existence, uniqueness, and stability of a solution $\widetilde{u}_h$ of (24).

Lemma 9. *For $f_0 \in [H^1_{,0}(0,T;L^2(\Omega))]'$ and $f_1, f_2 \in L^2(Q)$ the variational formulation to find $w_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ such that*

$$a_h(w_h, v_h) = \langle f_0, v_h \rangle_Q + \langle f_1, \partial_t v_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell} \langle f_2, \partial_t v_h \rangle_{L^2(\Omega\times\tau_\ell)} \tag{25}$$

is satisfied for all $v_h \in Q^1_h(Q) \cap H^{1,1}_{0;,0}(Q)$ is uniquely solvable, and there holds the stability estimate

$$\|w_h\|_{L^2(Q)} \leq 2T \left\{ \|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h^2_t \, \|f_2\|_{L^2(Q)} \right\}. \tag{26}$$

Proof. Let $w^0_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ be any solution of the homogeneous variational formulation (25) with $f_i \equiv 0$, and with the representation (18), i.e.

$$w^0_h(x,t) = \sum_{\ell=0}^{N_t} W^0_{\ell,h}(x)\varphi_\ell(t) \quad \text{for } (x,t) \in Q, \quad W^0_{\ell,h} \in V_{h_x}(\Omega), \quad W^0_{0,h}(x) = 0.$$

For an index $j \in \{1,\ldots,N_t\}$ we now consider an element $\overline{z}^j_h \in Q^1_h(Q) \cap H^{1,1}_{0;,0}(Q)$ as given in Lemma 8. Plugging $\overline{z}^j_h$ into (25) and by using the properties of Lemma 8 this gives

$$\begin{aligned}
0 &= a_h(w^0_h, \overline{z}^j_h) \\
&= -\langle \partial_t w^0_h, \partial_t \overline{z}^j_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \langle Q^0_{h_t} \partial_{x_m} w^0_h, \partial_{x_m} \overline{z}^j_h \rangle_{L^2(Q)} \\
&= -\sum_{\ell=1}^{j} \langle \partial_t w^0_h, \partial_t \overline{z}^j_h \rangle_{L^2(\Omega\times\tau_\ell)} + \sum_{m=1}^{d}\sum_{\ell=1}^{j} \langle Q^0_{h_t} \partial_{x_m} w^0_h, \partial_{x_m} \overline{z}^j_h \rangle_{L^2(\Omega\times\tau_\ell)} \\
&= -\int_\Omega \sum_{\ell=1}^{j} \left(\frac{1}{2}[W^0_{\ell-1,h}(x)]^2 - \frac{1}{2}[W^0_{\ell,h}(x)]^2 \right) dx \\
&\quad + \sum_{m=1}^{d} \int_\Omega \sum_{\ell=1}^{j} \left(\frac{1}{2} \left(\int_{t_{\ell-1}}^{t_j} \partial_{x_m} w^0_h(x,s)ds \right)^2 - \frac{1}{2} \left(\int_{t_\ell}^{t_j} \partial_{x_m} w^0_h(x,s)ds \right)^2 \right) dx \\
&= \frac{1}{2} \int_\Omega [W^0_{j,h_x}(x)]^2 \, dx + \frac{1}{2} \sum_{m=1}^{d} \int_\Omega \left(\int_0^{t_j} \partial_{x_m} w^0_h(x,s)ds \right)^2 dx.
\end{aligned}$$

This result yields, with the Cauchy–Schwarz inequality and the use of the properties of Lemma 8,

$$\begin{aligned}
\|w_h^0\|^2_{L^2(Q)} &= \sum_{\ell=1}^{N_t} \|w_h^0\|^2_{L^2(\Omega\times\tau_\ell)} \\
&= \int_\Omega \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}}{3}\Big([W^0_{\ell,h}(x)]^2 + W^0_{\ell,h}(x)W^0_{\ell-1,h}(x) + [W^0_{\ell-1,h}(x)]^2\Big)\,dx \\
&\le \int_\Omega \sum_{j=1}^{N_t} \frac{h_{t,j}}{2}[W^0_{j,h}(x)]^2\,dx + \int_\Omega \sum_{j=1}^{N_t-1} \frac{h_{t,j+1}}{2}[W^0_{j,h}(x)]^2\,dx \le 0,
\end{aligned}$$

which implies $w_h^0 \equiv 0$. By using

$$\dim Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q) = \dim Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$$

we therefore conclude unique solvability of the variational formulation (25) for any right–hand side $f_0 \in [H^1_{,0}(0,T;L^2(\Omega))]'$ and $f_1, f_2 \in L^2(Q)$. Following the approach as above we then obtain

$$\begin{aligned}
\langle f_0, \overline{z}_h^j\rangle_Q + \langle f_1, \partial_t \overline{z}_h^j\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle f_2, \partial_t \overline{z}_h^j\rangle_{L^2(\Omega\times\tau_\ell)} \\
= a_h(w_h, \overline{z}_h^j) \ge \frac{1}{2}\int_\Omega [W_{j,h_x}(x)]^2\,dx
\end{aligned}$$

and

$$\begin{aligned}
\|w_h\|^2_{L^2(Q)} &\le \int_\Omega \sum_{j=1}^{N_t} \frac{h_{t,j}}{2}[W_{j,h}(x)]^2\,dx + \int_\Omega \sum_{j=1}^{N_t-1} \frac{h_{t,j+1}}{2}[W_{j,h}(x)]^2\,dx \\
&\le \sum_{j=1}^{N_t} h_{t,j}\left\{\langle f_0, \overline{z}_h^j\rangle_Q + \langle f_1, \partial_t \overline{z}_h^j\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle f_2, \partial_t \overline{z}_h^j\rangle_{L^2(\Omega\times\tau_\ell)}\right\} \\
&\quad + \sum_{j=1}^{N_t-1} h_{t,j+1}\left\{\langle f_0, \overline{z}_h^j\rangle_Q + \langle f_1, \partial_t \overline{z}_h^j\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle f_2, \partial_t \overline{z}_h^j\rangle_{L^2(\Omega\times\tau_\ell)}\right\} \\
&\le \sum_{j=1}^{N_t} h_{t,j}\Big\{\|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2\,\|f_2\|_{L^2(Q)}\Big\}\|\partial_t \overline{z}_h^j\|_{L^2(Q)} \\
&\quad + \sum_{j=1}^{N_t-1} h_{t,j+1}\Big\{\|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2\,\|f_2\|_{L^2(Q)}\Big\}\|\partial_t \overline{z}_h^j\|_{L^2(Q)} \\
&\le 2T\left\{\|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2\,\|f_2\|_{L^2(Q)}\right\}\|w_h\|_{L^2(Q)},
\end{aligned}$$

and hence the stability estimate is proven. □

As a consequence of Lemma 9 we obtain unique solvability of the variational formulation (24), and the stability estimate

$$\|\widetilde{u}_h\|_{L^2(Q)} \leq 2\,T\,\|f\|_{[H^1_{,0}(0,T;L^2(\Omega))]'}.$$

To derive an estimate for the L^2 error $\|u-\widetilde{u}_h\|_{L^2(Q)}$ we introduce, as in [2, Section 2], a space–time projection $Q^1_{h_t}Q^1_{h_x}v \in Q^1_h(Q)\cap H^{1,1}_{0;0,}(Q)$ when $v \in H^{1,1}_{0;0,}(Q)$ is given. First, the H^1_0 projection $Q^1_{h_x} : L^2(0,T;H^1_0(\Omega)) \to V_{h_x}(\Omega)\otimes L^2(0,T)$ is defined by

$$\int_0^T \int_\Omega \nabla_x Q^1_{h_x} v(x,t)\cdot \nabla_x v_{h_x}(x,t)\,dx\,dt \,=\, \int_0^T \int_\Omega \nabla_x v(x,t)\cdot \nabla_x v_{h_x}(x,t)\,dx\,dt \qquad (27)$$

for all $v_{h_x} \in V_{h_x}(\Omega)\otimes L^2(0,T)$. Note that we have the stability estimate

$$\|\nabla_x Q^1_{h_x} v\|_{L^2(Q)} \leq \|\nabla_x v\|_{L^2(Q)},$$

and, when assuming $v \in L^2(0,T;H^{1+s}(\Omega))$ for some $s\in[0,1]$, the standard error estimate

$$\|v-Q^1_{h_x}v\|_{L^2(\mathscr{T}_N)} \leq c\,h_x^{1+s}\,\|v\|_{L^2(0,T;H^{1+s}(\Omega))}, \qquad (28)$$

if Ω is sufficiently regular. Second, we define the $H^1_{0,}$ projection $Q^1_{h_t} : H^1_{0,}(0,T;L^2(\Omega)) \to L^2(\Omega)\otimes S^1_{h_t}(0,T)\cap H^1_{0,}(0,T)$ by

$$\int_0^T \int_\Omega \partial_t Q^1_{h_t} v(x,t)\,\partial_t v_{h_t}(x,t)\,dx\,dt \,=\, \int_0^T \int_\Omega \partial_t v(x,t)\,\partial_t v_{h_t}(x,t)\,dx\,dt \qquad (29)$$

for all $v_{h_t} \in L^2(\Omega)\otimes S^1_{h_t}(0,T)\cap H^1_{0,}(0,T)$. Again we have the stability estimate

$$\|\partial_t Q^1_{h_t} v\|_{L^2(Q)} \leq \|\partial_t v\|_{L^2(Q)},$$

and, when assuming $v \in H^{1+s}(0,T;L^2(\Omega))$ for some $s\in[0,1]$, the standard error estimate

$$\|v-Q^1_{h_t}v\|_{L^2(Q)} \leq c\,h_t^{1+s}\,\|v\|_{H^{1+s}(0,T;L^2(\Omega))}. \qquad (30)$$

The next lemma shows that $Q^1_{h_t}Q^1_{h_x}v \in Q^1_h(Q)\cap H^{1,1}_{0;0,}(Q)$ is well–defined under some regularity assumptions on v, and that the projection operators in space and time commute, see also [2, Lemma 2.1].

Lemma 10. *For a given function $v \in H^{1,1}_{0;0,}(Q)$ satisfying $\partial_t v \in L^2(0,T;H^1_0(\Omega))$ and $\partial_{x_m} v \in H^1_{0,}(0,T;L^2(\Omega))$, $m=1,\ldots,d$, there hold the following relations:*

i. $\partial_t Q^1_{h_x} v = Q^1_{h_x}\partial_t v \in V_{h_x}(\Omega)\otimes L^2(0,T)$,
ii. $\partial_{x_m} Q^1_{h_t} v = Q^1_{h_t}\partial_{x_m} v \in L^2(\Omega)\otimes S^1_{h_t}(0,T)\cap H^1_{0,}(0,T)$, $m=1,\ldots,d$,
iii. $Q^1_{h_t}Q^1_{h_x}v = Q^1_{h_x}Q^1_{h_t}v \in Q^1_h(Q)\cap H^{1,1}_{0;0,}(Q)$. *In particular, the space–time projections* $Q^1_{h_t}Q^1_{h_x}v$ *and* $Q^1_{h_x}Q^1_{h_t}v$ *are well–defined.*

Moreover, there holds the error estimate

$$\|v - Q^1_{h_t} Q^1_{h_x} v\|_{L^2(Q)} \leq \|v - Q^1_{h_t} v\|_{L^2(Q)} + \|v - Q^1_{h_x} v\|_{L^2(Q)} + c\, h_x\, h_t\, \|\partial_t \nabla_x v\|_{L^2(Q)}.$$

Proof. For $\partial_t v \in L^2(0,T; H^1_0(\Omega))$ we consider $Q^1_{h_x} \partial_t v \in V_{h_x}(\Omega) \otimes L^2(0,T)$ as the unique solution of the variational formulation

$$\int_0^T \int_\Omega \nabla_x Q^1_{h_x} \partial_t v(x,t) \cdot \nabla_x v_{h_x}(x,t)\, dx\, dt = \int_0^T \int_\Omega \nabla_x \partial_t v(x,t) \cdot \nabla_x v_{h_x}(x,t)\, dx\, dt$$

for all $v_{h_x} \in V_{h_x}(\Omega) \otimes C^\infty_0(0,T)$. By using integration by parts in time twice this gives

$$\begin{aligned} \int_0^T \int_\Omega \nabla_x Q^1_{h_x} \partial_t v(x,t) \cdot \nabla_x v_{h_x}(x,t)\, dx\, dt &= -\int_0^T \int_\Omega \nabla_x v(x,t) \cdot \nabla_x \partial_t v_{h_x}(x,t)\, dx\, dt \\ &= -\int_0^T \int_\Omega \nabla_x Q^1_{h_x} v(x,t) \cdot \nabla_x \partial_t v_{h_x}(x,t)\, dx\, dt \\ &= \int_0^T \int_\Omega \nabla_x \partial_t Q^1_{h_x} v(x,t) \cdot \nabla_x v_{h_x}(x,t)\, dx\, dt \end{aligned}$$

for all $v_{h_x} \in V_{h_x}(\Omega) \otimes C^\infty_0(0,T)$. Since $C^\infty_0(0,T)$ is dense in $L^2(0,T)$ this holds true for all $v_{h_x} \in V_{h_x}(\Omega) \otimes L^2(0,T)$, i.e. *i.* The proof of *ii.* follows in the same manner.

To prove *iii.* we first note that $Q^1_{h_x} v \in V_{h_x}(\Omega) \otimes H^1_{0,}(0,T) \subset H^1_{0,}(0,T; L^2(\Omega))$, and so $Q^1_{h_t} Q^1_{h_x} v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ is well–defined. Analogously we have that $Q^1_{h_x} Q^1_{h_t} v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ is well–defined. Now, with *i.*, *ii.*, and the definitions (27), (29) we have that

$$\begin{aligned} \langle \partial_t \nabla_x Q^1_{h_t} Q^1_{h_x} v, \partial_t \nabla_x v_h \rangle_{L^2(Q)} &= \langle \partial_t Q^1_{h_t} \nabla_x Q^1_{h_x} v, \partial_t \nabla_x v_h \rangle_{L^2(Q)} \\ &= \langle \partial_t \nabla_x Q^1_{h_x} v, \partial_t \nabla_x v_h \rangle_{L^2(Q)} \\ &= \langle \partial_t \nabla_x v, \partial_t \nabla_x v_h \rangle_{L^2(Q)} \end{aligned}$$

as well as

$$\langle \partial_t \nabla_x Q^1_{h_x} Q^1_{h_t} v, \partial_t \nabla_x v_h \rangle_{L^2(Q)} = \langle \partial_t \nabla_x v, \partial_t \nabla_x v_h \rangle_{L^2(Q)}$$

for all $v_h \in Q^1_h(\mathscr{T}_N) \cap H^{1,1}_{0;0,}(Q)$, i.e. *iii.*

The error estimate follows from the triangle inequality

$$\begin{aligned} &\|v - Q^1_{h_t} Q^1_{h_x} v\|_{L^2(Q)} \leq \|v - Q^1_{h_t} v\|_{L^2(Q)} + \|Q^1_{h_t}(v - Q^1_{h_x} v)\|_{L^2(Q)} \\ &\quad \leq \|v - Q^1_{h_t} v\|_{L^2(Q)} + \|v - Q_{h_x} v\|_{L^2(Q)} + \|(I - Q^1_{h_t})(I - Q^1_{h_x}) v\|_{L^2(Q)} \\ &\quad \leq \|v - Q^1_{h_t} v\|_{L^2(Q)} + \|v - Q_{h_x} v\|_{L^2(Q)} + c\, h_t\, \|\partial_t (I - Q^1_{h_x}) v\|_{L^2(Q)} \\ &\quad \leq \|v - Q^1_{h_t} v\|_{L^2(Q)} + \|v - Q_{h_x} v\|_{L^2(Q)} + c\, h_t\, h_x\, \|\partial_t \nabla_x v\|_{L^2(Q)}. \end{aligned}$$

□

Now we are in a position to prove an error estimate for the approximate solution $\widetilde{u}_h$.

Theorem 2. *Let $u \in H^{1,1}_{0;0,}(Q)$ be the unique solution of the variational formulation (2) satisfying $\partial_t u \in L^2(0,T;H^1_0(\Omega))$ and $\partial_{x_m} u \in H^1_{0,}(0,T;L^2(\Omega))$, $m=1,\ldots,d$, and $\Delta_x u \in H^1_{0,}(0,T;L^2(\Omega))$. Then, the unique solution $\widetilde{u}_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ of the Galerkin–Petrov finite element discretization (24) satisfies the error estimate*

$$\begin{aligned}
&\|u-\widetilde{u}_h\|_{L^2(Q)} \leq \\
&\quad \leq \|u-Q^1_{h_t}u\|_{L^2(Q)} + \|u-Q^1_{h_x}u\|_{L^2(Q)} + c\,h_x h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\
&\quad\quad +2T\left\{ \|\Delta_x(u-Q^1_{h_t}u)\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|\partial_t(Q^1_{h_x}u-u)\|_{L^2(Q)} + \frac{h_t^2}{12}\|\partial_t \Delta_x u\|_{L^2(Q)} \right\}.
\end{aligned}$$

Proof. Since the solution u fulfills the assumptions of Lemma 10, the space–time projection $Q^1_{h_t}Q^1_{h_x}u \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ is well–defined. When using the representation (21), the properties of $Q^1_{h_t}Q^1_{h_x}$ as given in Lemma 10, and applying integration by parts, we conclude for all $w_h \in Q^1_h(Q) \cap H^{1,1}_{0;,0}(Q)$ that

$$\begin{aligned}
a_h(\widetilde{u}_h - Q^1_{h_t}Q^1_{h_x}u, w_h) &= a_h(\widetilde{u}_h, w_h) - a_h(Q^1_{h_t}Q^1_{h_x}u, w_h) \\
&= a(u,w_h) - a_h(Q^1_{h_t}Q^1_{h_x}u, w_h) \\
&= a(u,w_h) - a(Q^1_{h_t}Q^1_{h_x}u, w_h) + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q^1_{h_t}Q^1_{h_x}u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega\times\tau_\ell)} \\
&= -\langle \partial_t u, \partial_t w_h\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w_h\rangle_{L^2(Q)} + \langle \partial_t Q^1_{h_t}Q^1_{h_x}u, \partial_t w_h\rangle_{L^2(Q)} \\
&\qquad - \langle \nabla_x Q^1_{h_t}Q^1_{h_x}u, \nabla_x w_h\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q^1_{h_t}Q^1_{h_x}u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega\times\tau_\ell)} \\
&= -\langle \partial_t u, \partial_t w_h\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w_h\rangle_{L^2(Q)} + \langle \partial_t Q^1_{h_x}u, \partial_t w_h\rangle_{L^2(Q)} \\
&\qquad - \langle \nabla_x Q^1_{h_t}u, \nabla_x w_h\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q^1_{h_t}u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega\times\tau_\ell)} \\
&= \langle \partial_t (Q^1_{h_x}u-u), \partial_t w_h\rangle_{L^2(Q)} + \langle \nabla_x (u-Q^1_{h_t}u), \nabla_x w_h\rangle_{L^2(Q)} \\
&\qquad\qquad + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q^1_{h_t}u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega\times\tau_\ell)} \\
&= \langle \partial_t (Q^1_{h_x}u-u), \partial_t w_h\rangle_{L^2(Q)} + \langle -\Delta_x (u-Q^1_{h_t}u), w_h\rangle_{L^2(Q)} \\
&\qquad\qquad - \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \Delta_x Q^1_{h_t}u, \partial_t w_h \rangle_{L^2(\Omega\times\tau_\ell)}.
\end{aligned}$$

In particular we observe that $\widetilde{u}_h - Q^1_{h_t}Q^1_{h_x}u$ is the unique solution of (25) in the case

$$f_0 = -\Delta_x(u-Q^1_{h_t}u), \quad f_1 = \partial_t(Q^1_{h_x}u-u), \quad f_2 = -\frac{1}{12}\partial_t \Delta_x Q^1_{h_t}u.$$

Therefore, the stability estimate (26) and the stability of $Q^1_{h_t}$ in $H^1_{0,}(0,T)$ give

$$\begin{aligned}\|\widetilde{u}_h - Q^1_{h_t} Q^1_{h_x} u\|_{L^2(Q)} &\leq 2T\Big\{\|\Delta_x(u - Q^1_{h_t})u\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} \\ &\qquad + \|\partial_t(Q^1_{h_x}u - u)\|_{L^2(Q)} + \frac{h_t^2}{12}\|\partial_t \Delta_x Q^1_{h_t} u\|_{L^2(Q)}\Big\} \\ &\leq 2T\Big\{\|\Delta_x(u - Q^1_{h_t})u\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} \\ &\qquad + \|\partial_t(Q^1_{h_x}u - u)\|_{L^2(Q)} + \frac{h_t^2}{12}\|\partial_t \Delta_x u\|_{L^2(Q)}\Big\}.\end{aligned}$$

With the last estimate, the triangle inequality, and the error estimate of Lemma 10 we finally obtain

$$\begin{aligned}\|u - \widetilde{u}_h\|_{L^2(Q)} &\leq \|u - Q^1_{h_t} Q^1_{h_x} u\|_{L^2(Q)} + \|\widetilde{u}_h - Q^1_{h_t} Q^1_{h_x} u\|_{L^2(Q)} \\ &\leq \|u - Q^1_{h_t} u\|_{L^2(Q)} + \|u - Q^1_{h_x} u\|_{L^2(Q)} + c\, h_x\, h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\ &\quad + 2T\Big\{\|\Delta_x(u - Q^1_{h_t} u)\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|\partial_t(Q^1_{h_x}u - u)\|_{L^2(Q)} + \frac{h_t^2}{12}\|\partial_t \Delta_x u\|_{L^2(Q)}\Big\}.\end{aligned}$$

□

By using the error estimates (30) for the $H^1_{0,}$ projection $Q^1_{h_t}$ and (28) for the H^1_0 projection $Q^1_{h_x}$ we now conclude from Theorem 2 the following error estimate.

Corollary 2. *Let the assumptions of Theorem 2 be satisfied. If in addition the unique solution u of (2) is sufficiently smooth and Ω is sufficiently regular, we obtain the error estimate*

$$\begin{aligned}\|u - \widetilde{u}_h\|_{L^2(Q)} &\leq c\, h_x^2 \left(\|u\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^2(\Omega))}\right) \\ &\quad + c\, h_x\, h_t \|\partial_t \nabla_x u\|_{L^2(Q)} + c\, h_t^2 \left(\|\partial_{tt} u\|_{L^2(Q)} + \|\partial_{tt}\Delta_x u\|_{L^2(Q)} + \|\partial_t \Delta_x u\|_{L^2(Q)}\right).\end{aligned} \tag{31}$$

As a numerical example for the Galerkin–Petrov finite element method (24) we consider the one–dimensional spatial domain $\Omega = (0,1)$, i.e. we have the rectangular space–time domain $Q := \Omega \times (0,T) := (0,1) \times (0,10)$. The discretization is done with respect to nonuniform meshes as shown in Fig. 1 where we apply a uniform refinement strategy. Note that these meshes do not fulfill the CFL condition (3). As exact solutions we choose for $(x,t) \in Q$

$$u_1(x,t) = \sin(\pi x)\sin^2\left(\frac{5}{4}\pi t\right), \quad u_2(x,t) = \sin(\pi x) t^2 (10-t)^{3/4}.$$

The appearing integrals to compute the related right–hand side in (24) are calculated by using high order quadrature rules. The numerical results for the smooth solution u_1 are given in Table 3 where we observe unconditional stability and quadratic convergence in $\|\cdot\|_{L^2(Q)}$, as predicted by the error estimate (31). Moreover we have linear convergence when measuring the error in $|\cdot|_{H^1(Q)}$. Note that such an error estimate can be shown by using the $H^1(Q)$ projection, an inverse inequality, and the error estimate (31). For the singular solution u_2 the related results are given in Table 4 where we observe a reduced order of convergence in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$, respectively. These convergence rates correspond to the reduced Sobolev regularity $u_2 \in H^{5/4-\varepsilon}(Q)$, $\varepsilon > 0$.

Remark 2. The Galerkin–Petrov finite element method (24) seems to fulfill a kind of conservation of the total energy

$$E(t) := \frac{1}{2}\|\partial_t u(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla_x u(\cdot,t)\|^2_{L^2(\Omega)}, \quad t \in [0,T].$$

As illustration we consider a solution of the homogeneous wave equation, i.e.

$$u_3(x,t) = (\cos(\pi t) + \sin(\pi t))\sin(\pi x) \quad \text{for } (x,t) \in Q := (0,1)\times(0,10)$$

with the total energy

$$E(t) = \frac{\pi^2}{2} \quad \text{for } t \in [0,10].$$

Here, the initial condition $u_3(x,0) = \sin(\pi x)$, $x \in \Omega$, is treated via homogenization, while the initial condition $\partial_t u_3(x,0) = \pi \sin(\pi x)$, $x \in \Omega$, is incorporated in a weak sense. For the solution u_3 and for the mesh as given in Fig. 1 we compute the discrete total energy

$$E_h(t) := \frac{1}{2}\|\partial_t \tilde{u}_h(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla_x \tilde{u}_h(\cdot,t)\|^2_{L^2(\Omega)},\ t \in [0,T].$$

In Fig. 2 the difference

$$E_h(t) - E(t) = E_h(t) - \frac{\pi^2}{2} \quad \text{for } t \in [0,10]$$

is plotted pointwise for the refinement level with uniform mesh sizes $h_t = \frac{10}{6\cdot 2^{10}}$ and $h_x = \frac{1}{6\cdot 2^{10}}$. Note that $\partial_t \widetilde{u}_h$ is piecewise constant in time. Probably due to the used space–time approximation we observe some oscillations within the finite element accuracy, but no energy loss occurs.

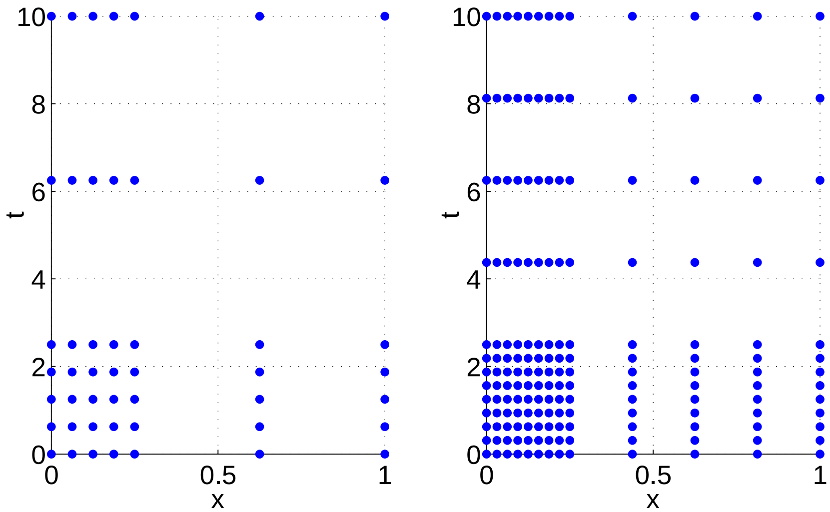

Fig. 1 Nonuniform meshes: Starting mesh and the mesh after one uniform refinement step.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\|u_1 - u_{1,h}\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	3.579e+00	0.00	1.289e+01	0.00
132	0.18750	0.03125	1.87500	0.31250	1.975e+00	0.86	9.849e+00	0.39
552	0.09375	0.01562	0.93750	0.15625	9.213e-01	1.10	6.534e+00	0.59
2256	0.04688	0.00781	0.46875	0.07812	6.829e-01	0.43	5.210e+00	0.33
9120	0.02344	0.00391	0.23438	0.03906	2.466e-01	1.47	2.848e+00	0.87
36672	0.01172	0.00195	0.11719	0.01953	7.029e-02	1.81	1.435e+00	0.99
147072	0.00586	0.00098	0.05859	0.00977	1.819e-02	1.95	7.159e-01	1.00
589056	0.00293	0.00049	0.02930	0.00488	4.588e-03	1.99	3.576e-01	1.00
2357760	0.00146	0.00024	0.01465	0.00244	1.149e-03	2.00	1.788e-01	1.00
9434112	0.00073	0.00012	0.00732	0.00122	2.875e-04	2.00	8.938e-02	1.00
37742592	0.00037	0.00006	0.00366	0.00061	7.189e-05	2.00	4.469e-02	1.00

Table 3 Numerical results of (24) for $Q = (0,1) \times (0,10)$ and for u_1.

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dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	7.836e+01	0.00	3.173e+02	0.00
132	0.18750	0.03125	1.87500	0.31250	2.166e+01	1.86	1.191e+02	1.41
552	0.09375	0.01562	0.93750	0.15625	5.487e+00	1.98	5.225e+01	1.19
2256	0.04688	0.00781	0.46875	0.07812	1.777e+00	1.63	2.696e+01	0.95
9120	0.02344	0.00391	0.23438	0.03906	6.476e-01	1.46	1.593e+01	0.76
36672	0.01172	0.00195	0.11719	0.01953	3.001e-01	1.11	1.076e+01	0.57
147072	0.00586	0.00098	0.05859	0.00977	1.393e-01	1.11	8.077e+00	0.41
589056	0.00293	0.00049	0.02930	0.00488	6.156e-02	1.18	6.452e+00	0.32
2357760	0.00146	0.00024	0.01465	0.00244	2.650e-02	1.22	5.308e+00	0.28
9434112	0.00073	0.00012	0.00732	0.00122	1.126e-02	1.23	4.423e+00	0.26
37742592	0.00037	0.00006	0.00366	0.00061	4.758e-03	1.24	3.704e+00	0.26

Table 4 Numerical results of (24) for $Q = (0,1) \times (0,10)$ and for u_2.

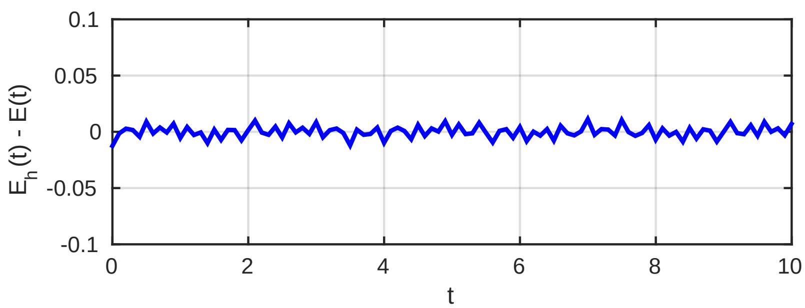

Fig. 2 Difference of the total energy $E(t) = \frac{\pi^2}{2}$ and $E_h(t)$ for the solution u_3 for a uniform mesh.

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