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Dual–Primal Boundary Element Tearing and Interconnecting Methods

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Abstract

The aim of this paper is to introduce the *dual primal boundary element tearing and interconnecting* (BETI–DP) method with Dirichlet and hypersingular boundary integral operator preconditioners. This extends the previous work on boundary element tearing and interconnecting (BETI) methods and on coupled finite and boundary element tearing and interconnecting (FETI/BETI) methods by U. Langer and O. Steinbach. As a natural continuation we present here the BETI–DP method and discuss few general choices of the dual spaces as needed in the formulation of the method. Moreover, we also analyze the use of the Dirichlet and of the hypersingular boundary integral operator preconditioner. We show that the condition number of the preconditioned BETI–DP system matrix behaves like the condition number of the system matrix of the corresponding dual-primal finite element tearing and interconnecting (FETI–DP) method. The numerical results presented confirm the theoretical estimates.

1 Introduction

The *finite element tearing and interconnecting* (FETI) method and its boundary element counterpart *boundary element tearing and interconnecting* (BETI) method are domain decomposition methods of iterative substructuring type. The local finite and boundary element spaces are given on each substructure separately to realize the local Dirichlet to Neumann maps. The global continuity of the primal variables is enforced by using Lagrange multipliers. This results in a saddle point problem which can be solved iteratively via its dual problem using a preconditioned conjugate gradient method.

The *dual-primal finite element tearing and interconnecting method* (FETI–DP) was introduced in [2]. The term *dual–primal* refers to the idea of enforcing some continuity constraints across the interface between the subdomains as in a primal method, while all other constraints are enforced by using Lagrange multipliers as in the dual method. The tearing part coincides for FETI and BETI as well as for FETI–DP and BETI–DP methods, while the major differences appear in the interconnecting part. The analysis of FETI–DP methods for two–dimensional second and fourth order elliptic boundary value problems was first considered in [16], and later in [1]. For three–dimensional boundary value problems, see [3, 9]. Recently, a pure algebraic formulation of FETI–DP which is independent of the underlying partial differential equation was given in [17].

In [11, 12] the BETI and coupled FETI/BETI methods were introduced. These results are based on the spectral equivalence inequalities of the discrete finite and boundary element Galerkin approximations of the continuous local Steklov–Poincaré operators and with the discrete hypersingular

boundary integral operator. Note that all constants in those spectral equivalence inequalities are independent of the discretization.
In this paper, following the ideas of BETI and adapting them to the *dual–primal* case we introduce and analyze the BETI–DP concept.
The rest of this paper is organized as follows: in the next section we present the BETI–DP formulation. Section 3 is dedicated to the analysis of the preconditioners used. Section 4 presents a more general approach for the three–dimensional case and a discussion of some choices to define dual and primal spaces in this general context. In Section 5 we present some numerical results and finally we sketch some conclusions in Section 6.

2 BETI–DP formulation

2.1 Model Problem and its Boundary Element Discretization

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with the boundary $\Gamma = \partial\Omega$ which is assumed to be polygonal for $d = 2$ or polyhedral for $d = 3$. As a model problem we consider the Dirichlet boundary value problem

$$-\mathrm{div}[\alpha(x)\nabla u(x)] \;=\; 0 \quad \text{for } x \in \Omega, \quad u(x) \;=\; g(x) \quad \text{for } x \in \Gamma. \tag{2.1}$$

We assume that there is given a nonoverlapping decomposition of Ω satisfying

$$\overline{\Omega} \;=\; \bigcup_{i=1}^{p} \overline{\Omega}_i,\; \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j,\; \Gamma_i = \partial\Omega_i,\; \Gamma_{ij} = \Gamma_i \cap \Gamma_j,\; \Gamma_S = \bigcup_{i=1}^{p} \Gamma_i.$$

In what follows we assume that the coefficient function $\alpha(x)$ is piecewise constant, i.e.,

$$\alpha(x) \;=\; \alpha_i > 0 \quad \text{for } x \in \Omega_i,\; i = 1, \ldots, p.$$

Thus, instead of the global boundary value problem (2.1) we now consider the local boundary value problems

$$-\alpha_i \Delta u_i(x) \;=\; 0 \quad \text{for } x \in \Omega_i, \quad u_i(x) \;=\; g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma, \tag{2.2}$$

together with transmission conditions on the internal coupling boundaries,

$$u_i(x) \;=\; u_j(x), \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) \;=\; 0 \quad \text{for } x \in \Gamma_{ij}, \tag{2.3}$$

where n_i is the unit outward normal vector with respect to Γ_i.
All solutions of the local boundary value problems (2.2) can be written by using the representation formulae [19]

$$u_i(x) \;=\; \int_{\Gamma_i} U^*(x,y) \frac{\partial}{\partial n_i} u_i(y) ds_y - \int_{\Gamma_i} \frac{\partial}{\partial n_i(y)} U^*(x,y) u_i(y) ds_y \quad \text{for } x \in \Omega_i, \tag{2.4}$$

where $U^*(x,y)$ is the fundamental solution of the Laplace operator,

$$U^*(x,y) \;=\; \begin{cases} -\dfrac{1}{2\pi} \log|x-y| & \text{for } d = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & \text{for } d = 3. \end{cases} \tag{2.5}$$

On the boundary Γ_i the Cauchy data $[u_i, t_i]$ of the local boundary value problems verify the Calderon equations

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix} \tag{2.6}$$

where $t_i = n_i \cdot \nabla u_i$ is the normal derivative of u_i on Γ_i. The boundary integral operators for $x \in \Gamma_i$ are given as the single layer potential

$$(V_i t_i)(x) \;=\; \int\limits_{\Gamma_i} U^*(x,y) t_i(y) ds_y,$$

the double layer potential

$$(K_i u_i)(x) \;=\; \int\limits_{\Gamma_i} \frac{\partial}{\partial n_i(y)} U^*(x,y) u_i(y) ds_y,$$

the adjoint double layer potential

$$(K_i' t_i)(x) \;=\; \int\limits_{\Gamma_i} \frac{\partial}{\partial n_i(x)} U^*(x,y) t_i(y) ds_y,$$

and the hypersingular boundary integral operator

$$(D_i u_i)(x) \;=\; -\frac{\partial}{\partial n_i(x)} \int\limits_{\Gamma_i} \frac{\partial}{\partial n_i(y)} U^*(x,y) u_i(y) ds_y.$$

The properties of all boundary integral operators are well known (see for example [19]). In particular, the local single layer potential V_i is $H^{-1/2}(\Gamma_i)$–elliptic, in the two dimensional case we assume $diam(\Omega_i) < 1$. From (2.6) we then obtain the local Dirichlet to Neumann map

$$t_i(x) \;=\; \left[D_i + (\frac{1}{2}I + K_i') V_i^{-1} (\frac{1}{2}I + K_i) \right] u_i(x) \;=\; (S_i u_i)(x) \quad \text{for } x \in \Gamma_i, \tag{2.7}$$

where $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is the local Steklov–Poincaré operator.
Let $H^{1/2}(\Gamma_S) = \{u_{|\Gamma_S} : u \in H^1(\Omega)\}$ be the trace space of $H^1(\Omega)$ on the skeleton Γ_S, and let

$$H_0^{1/2}(\Gamma_S, \Gamma) = \{v \in H^{1/2}(\Gamma_S) \;:\; v = 0 \;\text{ on } \Gamma\}.$$

Then, for $\widehat{g} \in H^{1/2}(\Gamma_S)$ being an arbitrary but fixed extension of the given Dirichlet datum $g \in H^{1/2}(\Gamma)$ it remains to find $\widehat{u} \in H_0^{1/2}(\Gamma_S, \Gamma)$ such that $u_i = \widehat{u}_{|\Gamma_i} + \widehat{g}_{|\Gamma_i}$, and

$$\alpha_i (S_i u_i)(x) + \alpha_j (S_j u_j)(x) \;=\; 0 \quad \text{for } x \in \Gamma_{ij} \tag{2.8}$$

is satisfied along all local coupling boundaries Γ_{ij}. Hence we obtain a global variational problem to find $\widehat{u} \in H_0^{1/2}(\Gamma_S, \Gamma)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (S_i \widehat{u}_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x \;=\; -\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i (S_i \widehat{g}_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x \tag{2.9}$$

is satisfied for all $v \in H_0^{1/2}(\Gamma_S, \Gamma)$. Since the local Dirichlet to Neumann map (2.7) is defined in an implicit way, it is in general not possible to discretize the variational problem (2.9) in an exact manner. Hence we have to approximate the solution of all local Dirichlet boundary value problems which occur in the definition of the local Dirichlet to Neumann maps.
For $v_i \in H^{1/2}(\Gamma_i)$ the application of S_i is given by

$$(S_i v_i)(x) \;=\; (D_i v_i)(x) + (\frac{1}{2}I + K_i') w_i(x) \quad \text{for } x \in \Gamma_i,$$

where $w_i = V_i^{-1}(\frac{1}{2}I + K_i) v_i \in H^{-1/2}(\Gamma_i)$ is the unique solution of the variational problem

$$\langle V_i w_i, \tau_i \rangle_{\Gamma_i} \;=\; \langle (\frac{1}{2}I + K_i) v_i, \tau_i \rangle_{\Gamma_i} \quad \text{for all } \tau_i \in H^{-1/2}(\Gamma_i). \tag{2.10}$$

Let

$$Z_{i,h} \;=\; \mathrm{span}\{\psi_k^i\}_{k=1}^{N_i} \subset H^{-1/2}(\Gamma_i)$$

be a conforming trial space, for example the space of piecewise constant functions ψ_k^i with respect to a local quasi uniform and regular boundary mesh with average mesh size h_i. The Galerkin variational problem of (2.10) is to find $w_{i,h} \in Z_{i,h}$ such that

$$\langle V_i w_{i,h}, \tau_{i,h}\rangle_{\Gamma_i} \;=\; \langle (\frac{1}{2}I + K_i)v_i, \tau_{i,h}\rangle_{\Gamma_i} \quad \text{for all } \tau_{i,h} \in Z_{i,h}.$$

The solution of this problem is uniquely determined and satisfies the a priori error estimate [7]

$$\|w_i - w_{i,h}\|_{H^{-1/2}(\Gamma_i)} \;\leq\; c_i \inf_{\tau_{i,h}\in Z_{i,h}} \|w_i - \tau_{i,h}\|_{H^{-1/2}(\Gamma_i)}.$$

Hence we can define an approximate Steklov–Poincaré operator $\widetilde{S}_i$ as

$$(\widetilde{S}_i v_i)(x) \;=\; (D_i v_i)(x) + (\frac{1}{2}I + K_i')w_{i,h}(x) \quad \text{for } x \in \Gamma_i. \tag{2.11}$$

Now, the perturbed variational problem of (2.9) is to find $\widehat{u} \in H_0^{1/2}(\Gamma_S, \Gamma)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i(\widetilde{S}_i \widehat{u}_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x \;=\; -\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i(\widetilde{S}_i \widehat{g}_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x \tag{2.12}$$

is satisfied for all $v \in H_0^{1/2}(\Gamma_S, \Gamma)$. Let W_h be a boundary element subspace of $H_0^{1/2}(\Gamma_S, \Gamma)$, e.g.,

$$W_h \;=\; \mathrm{span}\{\varphi_n\}_{n=1}^{M} \subset H_0^{1/2}(\Gamma_S, \Gamma),$$

of piecewise linear basis functions φ_n which are defined with respect to a globally quasi uniform and regular mesh with average mesh size h_S. The spaces

$$W_{i,h} \;=\; \mathrm{span}\{\varphi_n^i\}_{n=1}^{M_i}$$

denote the restrictions of W_h onto the local subdomain boundaries Γ_i, $i = 1, \ldots, p$. The resulting Galerkin variational formulation of (2.12) is to find $u_h \in W_h$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i(\widetilde{S}_i u_{h|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x \;=\; -\sum_{i=1}^{p} \int_{\Gamma_i} \alpha_i(\widetilde{S}_i \widehat{g}_{|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x \tag{2.13}$$

is satisfied for all $v_h \in W_h$. This variational problem has a unique solution $u_h \in W_h$ which satisfies the a priori error estimate, see, e.g., [20],

$$\|\widehat{u} - u_h\|_{H^{1/2}(\Gamma_S)} \;\leq\; c_1 \inf_{v_h \in W_h} \|\widehat{u} - v_h\|_{H^{1/2}(\Gamma_S)} + c_2 \sum_{i=1}^{p} \inf_{\tau_{i,h}\in Z_{i,h}} \|S_i \widehat{u}_{|\Gamma_i} - \tau_{i,h}\|_{H^{-1/2}(\Gamma_i)}.$$

When assuming $\widehat{u} \in H^2(\Gamma_S)$ and $S_i \widehat{u}_{|\Gamma_i} \in H^1_{\mathrm{pw}}(\Gamma_i)$ for $i = 1, \ldots, p$, we then obtain the a priori estimate [20]

$$\|\widehat{u} - u_h\|_{H^{1/2}(\Gamma_S)} \;\leq\; c_1\, h_S^{3/2} \|\widehat{u}\|_{H^2(\Gamma_S)} + c_2 \sum_{i=1}^{p} h_i^{3/2} \|S_i \widehat{u}_{|\Gamma_i}\|_{H^1_{\mathrm{pw}}(\Gamma_i)}.$$

The Galerkin variational problem (2.13) is equivalent to a linear system of algebraic equations,

$$\sum_{i=1}^{p} \alpha_i A_i^\top \widetilde{S}_{i,h}^{\mathrm{BEM}} A_i \underline{u} \;=\; \sum_{i=1}^{p} A_i^\top \underline{f}_i, \tag{2.14}$$

where the connectivity matrices $A_i \in \mathbb{R}^{M_i \times M}$ map the global vector $\underline{v} \in \mathbb{R}^M$ originating from the global discretization on Γ_S onto their local components $\underline{v}_i \in \mathbb{R}^{M_i}$ corresponding to the local discretization on Γ_i. In (2.14), the discrete approximate Steklov–Poincaré operator is given as

$$\widetilde{S}_{i,h}^{\text{BEM}} = D_{i,h} + (\frac{1}{2}M_{i,h}^T + K_{i,h}^T)V_{i,h}^{-1}(\frac{1}{2}M_{i,h} + K_{i,h})$$

with the local boundary element matrices

$$\begin{aligned} V_{i,h}[\ell,k] &= \langle V_i\psi_k^i, \psi_\ell^i\rangle_{\Gamma_i}, \\ D_{i,h}[m,n] &= \langle D_i\varphi_n^i, \varphi_m^i\rangle_{\Gamma_i}, \\ K_{i,h}[\ell,n] &= \langle K_i\varphi_n^i, \psi_\ell^i\rangle_{\Gamma_i}, \\ M_{i,h}[\ell,n] &= \langle \varphi_n^i, \psi_\ell^i\rangle_{\Gamma_i}, \end{aligned}$$

for $k,\ell = 1,...,N_i$ and $m,n = 1,...,M_i$.

2.2 Tearing

The linear system (2.14) is equivalent to the solution of the minimization problem

$$\widetilde{J}(\underline{u}) = \min_{\underline{v}\in\mathbb{R}^M} \widetilde{J}(\underline{v}), \tag{2.15}$$

where the linear functional $\widetilde{J}$ is given as

$$\widetilde{J}(\underline{v}) = \sum_{i=1}^{p} \left[\frac{1}{2}\alpha_i(\widetilde{S}_{i,h}^{\text{BEM}} A_i\underline{v}, A_i\underline{v}) - (f_i, A_i\underline{v})\right].$$

By introducing local vectors $\underline{v}_i = A_i\underline{v} \in \mathbb{R}^{M_i}$ we obtain

$$\overline{J}(\underline{v}_1,\dots,\underline{v}_p) = \sum_{i=1}^{p} \left[\frac{1}{2}(\alpha_i\widetilde{S}_{i,h}^{\text{BEM}}\underline{v}_i, \underline{v}_i) - (\underline{f}_i, \underline{v}_i)\right] \tag{2.16}$$

to be minimized subject to some continuity constraints across the interface. With

$$W = \prod_{i=1}^{p} W_i, \quad W_i = \mathbb{R}^{M_i}, \quad S = \operatorname{diag}\left(\alpha_i\widetilde{S}_{i,h}^{\text{BEM}}\right)_{i=1\dots p}$$

the functional (2.16) can be rewritten as

$$\overline{J}(\hat{\underline{v}}) = \frac{1}{2}(S\hat{\underline{v}}, \hat{\underline{v}}) - (\underline{f}, \hat{\underline{v}}), \quad \hat{\underline{v}} := (\underline{v}_1,\dots,\underline{v}_p)^\top \in W. \tag{2.17}$$

2.3 Interconnecting

It remains to impose the constraints that correspond to the continuity of the primal variables $\underline{u}$ across the interface, i.e. $u_i(x) = u_j(x)$ for $x \in \Gamma_{ij}$. In the classical tearing and interconnecting FETI/BETI approach the global continuity of the potentials $\underline{u}_i$ is enforced by the constraints

$$\sum_{i=1}^{p} B_i\underline{u}_i = \underline{0}$$

interconnecting the local potential vectors across the subdomain boundaries. Each row of the matrix $B = (B_1,\dots,B_p)$ is connected with a pair of matching nodes, and with entries 1, −1, or 0.

The basic idea of BETI–DP is to consider the degrees of freedom corresponding to some *strong connectivity points* as global degrees of freedom. Let

$$\underline{u}_C = \begin{pmatrix} u_{C,1} \\ \vdots \\ u_{C,M_C} \end{pmatrix}$$

be the vector of degrees of freedom corresponding to the strong connectivity points, where M_C is the total number of the strong connectivity points. By $W_C = \mathbb{R}^{M_C}$ we denote the vector space which is associated with the degrees of freedoms in the strong connectivity points. Let $R^i_C : W_C \to W_i$ be the matrix operator between the Euclidean spaces W_C and W_i in such a way that $R^i_C \underline{u}_C = \underline{u}_{C,i} \in W_i$ for $\underline{u}_C \in W_C$. After reordering we have

$$\underline{u}_i = \begin{pmatrix} \underline{u}_{R,i} \\ \underline{u}_{C,i} \end{pmatrix}$$

where $\underline{u}_{R,i}$ corresponds to all degrees of freedom which are not associated with strong connectivity points. Now the continuity conditions have to be enforced only on the remainder degrees of freedom,

$$\sum_{i=1}^{p} B_{R,i}\underline{u}_{R,i} = \underline{0}.$$

Then we have to solve the following constrained minimization problem to find $\underline{\hat{u}} \in W$ such that

$$\overline{J}(\underline{\hat{u}}) = \min_{\underline{\hat{v}}\in W,\ \sum_{i=1}^{p} B_{R,i}\underline{v}_{R,i}=\underline{0},\ R_{C,i}\underline{v}_C=\underline{v}_{C,i},i=1,\ldots,p} \overline{J}(\underline{\hat{v}}). \tag{2.18}$$

Recall that the *primal space* W_C is the subspace of W which corresponds to all global degrees of freedom in the strong connectivity points. In addition, we introduce the *dual spaces* $W_{\Delta,i}$ which correspond to the local remainders $\underline{v}_{R,i}$. Hence we introduce

$$\widetilde{W} = W_C \oplus \prod_{i=1}^{p} W_{\Delta,i} = W_C \oplus W_\Delta,$$

where we have used the isomorphism

$$\underline{\hat{v}} = \begin{pmatrix} \underline{v}_1 \\ \vdots \\ \underline{v}_p \end{pmatrix} \in W, \quad \underline{v}_i = \begin{pmatrix} \underline{v}_{R,i} \\ \underline{v}_{C,i} \end{pmatrix}, \quad \underline{v}_{R,i} \in W_{\Delta,i}, \quad \underline{v}_{C,i} = R^i_C \underline{v}_C, \quad \underline{v}_C \in W_C,$$

in particular

$$\underline{\widetilde{v}} = (\underline{v}_C, \underline{v}_{R,1}, \ldots, \underline{v}_{R,p})^\top \in \widetilde{W}.$$

Therefore, we have to find $\underline{\widetilde{u}} \in \widetilde{W} \leftrightarrow \underline{\hat{u}} \in W$ such that

$$\overline{J}(\underline{\hat{u}}) = \min_{\underline{\hat{v}}\in W \leftrightarrow \underline{\widetilde{v}}\in\widetilde{W}:\ \sum_{i=1}^{p} B_{R,i}\underline{v}_{R,i}=\underline{0}} \overline{J}(\underline{\hat{v}}).$$

After introducing Lagrange multipliers $\underline{\lambda} \in \mathbb{R}^{\dim W_\Delta}$ the resulting linear system is given by

$$\begin{pmatrix} \alpha_1 \widetilde{S}^{RR}_{1,h} & & & \alpha_1 \widetilde{S}^{RC}_{1,h} R^1_C & B^{1,\top}_R \\ & \ddots & & \vdots & \vdots \\ & & \alpha_p \widetilde{S}^{RR}_{p,h} & \alpha_p \widetilde{S}^{RC}_{p,h} R^p_C & B^{p,\top}_R \\ \alpha_1 R^{1,\top}_C \widetilde{S}^{RC,\top}_{1,h} & \ldots & \alpha_p R^{p,\top}_C \widetilde{S}^{RC,\top}_{p,h} & \sum_{i=1}^{p} \alpha_i {R^i_C}^T \widetilde{S}^{CC}_{i,h} R^i_C & \\ B^1_R & \ldots & B^p_R & & \end{pmatrix} \begin{pmatrix} \underline{u}_{R,1} \\ \vdots \\ \underline{u}_{R,p} \\ \underline{u}_C \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_{R,1} \\ \vdots \\ \underline{f}_{R,p} \\ \sum_{i=1}^{p} R^{i,\top}_C \underline{f}_{C,i} \\ \underline{0} \end{pmatrix} \tag{2.19}$$

where the block partitioning of the discrete Steklov–Poincaré operators $\widetilde{S}_{i,h}^{\text{BEM}}$ corresponds to the reordering of $\underline{u}_i = (\underline{u}_{R,i}^\top, \underline{u}_{C,i}^\top)^\top$. In compact form we can write

$$\begin{pmatrix} S_{RR} & S_{RC}R_C & B_R^\top \\ (S_{RC}R_C)^\top & S_{CC} & 0 \\ B_R & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_R \\ \underline{u}_C \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_R \\ \underline{f}_C \\ \underline{0} \end{pmatrix}. \tag{2.20}$$

After eliminating the primal variables $\underline{u}_R$ and $\underline{u}_C$ we obtain the Schur complement system

$$F\underline{\lambda} = \underline{g} \tag{2.21}$$

where

$$F = B_R \widetilde{S}^{-1} B_R^\top \tag{2.22}$$

is symmetric and positive definite, and

$$\widetilde{S} = S_{RR} - S_{RC}R_C S_{CC}^{-1}(S_{RC}R_C)^\top. \tag{2.23}$$

Hence we can solve (2.21) by using a preconditioned conjugate gradient method.

3 Preconditioning

As in FETI–DP [2, 9] we may introduce the **Dirichlet preconditioner**

$$M = \sum_{i=1}^{p} \alpha_i B_R^i \widetilde{S}_{i,h}^{RR} B_R^{i,\top}. \tag{3.1}$$

Using ideas and spectral equivalence inequalities as proved in [12] we may also define the **hypersingular boundary integral operator preconditioner**

$$M = \sum_{i=1}^{p} \alpha_i B_R^i D_{i,h}^{RR} B_R^{i,\top}. \tag{3.2}$$

For large variations in the coefficients α_i we introduce the **scaled Dirichlet preconditioner**

$$M = \sum_{i=1}^{p} \alpha_i C_{\alpha,i} B_R^i \widetilde{S}_{i,h}^{RR} B_R^{i,\top} C_{\alpha,i}^\top \tag{3.3}$$

and the **scaled hypersingular boundary integral operator preconditioner**

$$M = \sum_{i=1}^{p} \alpha_i C_{\alpha,i} B_R^i D_{i,h}^{RR} B_R^{i,\top} C_{\alpha,i}^\top \tag{3.4}$$

where $C_{\alpha,i}$ are diagonal scaling matrices as defined in [9].
To prove spectral equivalence inequalities of the above defined preconditioners M with the Schur complement matrix F as defined in (2.22) we first introduce finite element approximations of the local Steklov–Poincaré operators,

$$\widetilde{S}_{i,h}^{\text{FEM}} = K_{CC,i} - K_{CI,i} K_{II,i}^{-1} K_{IC,i}, \tag{3.5}$$

where the finite element stiffness matrix $K_{II,i}$ corresponds to interior degrees of freedom with respect to some quasi–uniform auxiliary finite element discretization within the subdomain Ω_i, see also [11].

Lemma 3.1 *The local boundary element approximations* $\widetilde{S}_{i,h}^{BEM}$ *and the local finite element approximations* $S_{i,h}^{FEM}$ *are spectrally equivalent to the exact Galerkin matrices* $S_{i,h}$ *of the local Steklov–Poincaré operators* S_i *and to the boundary element stiffness matrices* $D_{i,h}$ *of the local hypersingular boundary integral operators* D_i*, i.e.,*

$$\widetilde{S}_{i,h}^{BEM} \simeq \widetilde{S}_{i,h}^{FEM} \simeq S_{i,h} \simeq D_{i,h}, \quad i = 1, \ldots, p.$$

Here, $A \simeq B$ *means that the matrices* A *and* B *are spectrally equivalent with spectral equivalence constants which are independent of discretization parameters.*

Since the system matrix F of the BETI–DP method as defined in (2.22) differs from the system matrix of the FETI–DP approach only in the approximation $\widetilde{S}_{i,h}^{\mathrm{BEM/FEM}}$ of the local Steklov–Poincaré operators S_i, all spectral equivalence inequalities of FETI–DP transfer to the BETI–DP approach. Therefore, from Lemma 3.1 and the FETI-DB anlysis given in [9, 16], we immediately obtain the following theorem:

Theorem 3.1 *Let* F *be the system matrix of the BETI–DP approach as defined in* (2.22)*. Let* M *be the Dirichlet preconditioner* (3.1)*. Then, for* $d = 2$*, we have*

$$\mathit{cond}_2(MF) = \frac{\lambda_{\max}(MF)}{\lambda_{\min}(MF)} \le C \left(1 + \log \frac{H}{h}\right)^2, \tag{3.6}$$

whereas, for $d = 3$*, we only have*

$$\mathit{cond}_2(MF) = \frac{\lambda_{\max}(MF)}{\lambda_{\min}(MF)} \le C \left(1 + \log \frac{H}{h}\right)^2 \frac{H}{h}, \tag{3.7}$$

where $H/h = \max H_i/h_i$ *and* C *is a positive constant which does not depend on the subdomain size* H_i *and on the local mesh sizes* h_i*.*

Obviously, the hypersingular boundary integral operator preconditioner (3.2) yields the same asymptotical bounds for $\mathrm{cond}_2(MF)$ as given in Theorem 3.1. In order to get rid of the dependence of the constant C on possible large jumps of the coefficient $\alpha(\cdot)$, one should use the scaled versions (3.3) and (3.4) of the preconditioners (3.1) and (3.2), respectively.

4 A More General Approach for the 3D Case

4.1 Dual–Primal Spaces

As we have seen in Theorem 3.1, the condition number of the preconditioned system when using the Dirichlet preconditioner (3.1) behaves not optimal in the three–dimensional case. Therefore, the algorithm seems not to be very competitive. This is the reason why we have to modify the FETI/BETI–DP algorithms.

As discussed in subsection 2.3 we have to solve the minimization problem to find $\underline{\widetilde{u}} \in \widetilde{W} \leftrightarrow \underline{\hat{u}} \in W$ such that

$$\overline{J}(\underline{\hat{u}}) = \min_{\underline{\hat{v}} \in W \leftrightarrow \underline{\widetilde{v}} \in \widetilde{W} : \sum_{i=1}^{p} B_{R,i} \underline{v}_{R,i} = \underline{0}} \overline{J}(\underline{\hat{v}})$$

where

$$\widetilde{W} = W_C \oplus W_\Delta, \quad W_\Delta = \prod_{i=1}^{p} W_{\Delta,i}.$$

In particular, W_C is the primal space which corresponds to all global degrees of freedom in the strong connectivity points while the dual spaces $W_{\Delta,i}$ cover the remaining degrees of freedom which are linked by nodal constraints. Instead of the strong connectivity points we now may

consider any other definition of the primal space W_C and therefore of the implicitly derived dual spaces $W_{\Delta,i}$.
After eliminating the primal variables we end up to find the dual variables $\underline{u}_\Delta$ and the associated Lagrange multipliers $\underline{\lambda} \in V = \text{range} B_\Delta$ from

$$\begin{pmatrix} \widetilde{S}_\Delta & B_\Delta^\top \\ B_\Delta & \end{pmatrix} \begin{pmatrix} \underline{u}_\Delta \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_\Delta \\ \underline{0} \end{pmatrix}$$

where, as before, B_Δ is the matrix describing the nodal continuity of the dual variables $\underline{u}_{\Delta,i}$, and $\widetilde{S}_\Delta$ is the associated Schur complement. Since the latter is invertible we obtain the Schur complement system

$$F\underline{\lambda} = \underline{g}$$

with

$$F = B_\Delta \widetilde{S}_\Delta^{-1} B_\Delta^\top, \quad \underline{g} = B_\Delta \widetilde{S}_\Delta^{-1} \underline{f}_\Delta. \tag{4.1}$$

In order to construct appropriate preconditioners for the Schur complement F we have to define suitable matrices $\widetilde{S}_{\Delta,i}$ and $\widetilde{D}_{\Delta,i}$ by the restriction of $\widetilde{S}_{i,h}$ and $\widetilde{D}_{i,h}$ onto the local dual spaces $W_{\Delta,i}$. In the original method (Choice A) this was done by a simple elimination of the rows and columns which correspond to the primal space W_C. For more details on the realization of this approach we refer to [21] and [9]. Now we are able to introduce the modified **scaled Dirichlet preconditioner**

$$M = \sum_{i=1}^{p} C_{\Delta,i} B_R^i \widetilde{S}_{\Delta,i} B_R^{i,\top} C_{\Delta,i}^\top \tag{4.2}$$

and the modified **scaled hypersingular boundary integral operator preconditioner**

$$M = \sum_{i=1}^{p} C_{\Delta,i} B_R^i \widetilde{D}_{\Delta,i} B_R^{i,\top} C_{\Delta,i}^\top \tag{4.3}$$

where $C_{\Delta,i}$ are suitable diagonal scaling matrices.

4.2 Some Choices for the Dual–Primal Spaces

Now we discuss few choices for the Dual–Primal spaces for BETI–DP as it was originally done in [9] for the FETI–DP method. For each subdomain Ω_i we denote by F^{ij}, ε^{ik}, and v^{il} the faces, edges and vertices, respectively.

4.2.1 Choice A: Corner Points

The primal space W_C is spanned by vectors which are one in all vertices and zero in all remaining nodes. The local spaces $W_{\Delta,i}$ are then spanned by local vectors from W_i which are zero in the vertices. This first choice corresponds to the original FETI/BETI–DP method.

4.2.2 Choice B: Corner Points and Edges

The primal space W_C is spanned by vectors which correspond to the nodal basis functions from W_h which take the value one in the vertices, and to cutoff functions which are one along the edges. The dual subspaces $W_{\Delta,i}$ are then spanned by local vectors from W_i which correspond to basis functions which vanish in the vertices, and to functions with zero average on each edge, i.e.,

$$\bar{u}_{\varepsilon^{ik}} = \frac{1}{|\varepsilon^{ik}|} \int_{\varepsilon^{ik}} u(x) ds_x = 0. \tag{4.4}$$

For both the **Dirichlet** and the **hypersingular boundary integral operator preconditioners**, the following result holds:

Theorem 4.1 *Let F be the system matrix of the modified BETI–DP approach as defined in* (4.1)*. Let M be the scaled Dirichlet preconditioner* (4.2) *or the scaled hypersingular boundary integral operator preconditioner* (4.3)*. Then the following estimate of the condition number of the preconditioned system is valid:*

$$cond_2(MF) \leq C(1 + log(H/h))^2, \tag{4.5}$$

where the constant C is again independent of H_i, h_i, and the coefficient jumps.

4.2.3 Choice C: Corner Points, Edges and Faces

The primal space W_C is spanned by vectors which correspond to nodal basis functions which are one in the vertices, and to cutoff functions which take the value one along the edges and over the faces. The dual subspaces $W_{\Delta,i}$ are then spanned by local vectors which correspond to basis functions which vanish in the vertices, and to functions that have zero average over each edge and each face, i.e., beside (4.4) we also have

$$\bar{u}_{F^{il}} = \frac{1}{|F^{il}|} \int_{F^{il}} u(x) dS_x \,. \tag{4.6}$$

For both the **scaled Dirichlet** and the **scaled hypersingular boundary integral operator preconditioners**, the same results as formulated in Theorem 4.1 are valid.
We immediately see that the number of primal constraints enforced in choices B and C is much larger than the one corresponding to choice A. However, we have a much better bound of the spectral condition number. As it was already shown in [9] we have to distinguish between the primal constraints in the strong connectivity points and the remainders of the primal constraints from choices B and C. The last ones are called *optional* constraints because they were introduced not to guarantee the solvability of the subproblems (this was already done by the corner points provided that in each subdomain there is a certain number of corner points which is at least equal to the dimension of the subproblem kernel) but to improve the convergence rate. The optional constraints can be handled as corner points after a suitable change of basis, but also by introducing an additional set of Lagrange multipliers which are computed exactly in each iteration step to enforce the required optional constraints of the primal space (see [21], [9] and [17]).
Several other choices can be found in [21] and [9] for the FETI–DP method, and as we have already seen, they are suitable for BETI–DP as well.

5 Numerical Results

As an example we consider the Dirichlet boundary value problem

$$-\text{div}[\alpha(x)\nabla u(x)] = 0 \quad \text{for } x \in \Omega, \quad u(x) = x_1 + x_2 \quad \text{for } x \in \Gamma = \partial\Omega,$$

where $\Omega = (0,1)^2$ is divided into 9 subdomains Ω_i with piecewise constant coefficients $\alpha(x) = \alpha_i$ for $x \in \Omega_i$, $i = 1, \ldots, 9$. Note that $\Omega_5 = (1/3, 2/3)^2$ is a floating subdomain.
Here we consider only the iterative solution of the linear system (2.21) by using a preconditioned conjugate gradient scheme with a relative error reduction of $\varepsilon = 10^{-7}$ as stopping criteria. As preconditioners we consider the Dirichlet preconditioner (3.1), the hypersingular boundary integral operator preconditioner (3.2), the scaled Dirichlet preconditioner (3.3), and the scaled hypersingular boundary integral operator preconditioner (3.4).

6 Conclusions

The main benefit of the BETI–DP approach is that all local subproblems are invertible, i.e., there is no need to characterize the kernels of the local Steklov–Poincaré operators. Moreover, we can use standard preconditioned conjugate gradient schemes in parallel without using any projection

N_i	(3.1)	(3.2)	(3.3)	(3.4)
40	8	12	8	12
60	8	12	8	12
80	9	13	9	13
104	9	13	9	13

Table 1: Number of CG iterations, no jumps: $\alpha_i = 1$ for $i = 1, \ldots, 9$.

N_i	(3.1)	(3.2)	(3.3)	(3.4)
40	15	19	14	17
60	15	21	15	17
80	16	22	16	18
104	16	23	16	19

Table 2: Number of CG iterations, small jumps: $\alpha_5 = 0.1$, $\alpha_i = 1$ for $i \neq 5$.

as used in standard BETI methods. The use of the hypersingular boundary integral operator preconditioners (3.2) and (3.4) may not yield so effective bounds as the Dirichlet preconditioners (3.1) and (3.3), but the application of (3.2) and (3.4) only require some matrix by vector multiplications instead of solving local boundary value problems as in (3.2) and (3.4).
As it is known from previous papers on BETI and coupled FETI/BETI, the method formulation is strictly analytical based on suitable Dirichlet to Neumann maps. In the recent work of Mandel and Tezaur [17] the FETI and FETI–DP methods were investigated in a strictly algebraic formulation which is totally independent of the underlying partial differential equation. The advantage of BETI and BETI–DP is that once we have proven spectral equivalence inequalities of the finite and boundary element approximations of the local Steklov–Poincaré operators, all results from FETI can be applied for BETI. This also leads to the further use of coupled FETI/BETI–DP as well as using the hypersingular boundary integral operator preconditioner for FETI methods, see, e.g., [13].

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N_i	(3.1)	(3.2)	(3.3)	(3.4)
40	29	47	18	21
60	30	55	18	21
80	31	64	19	22
104	32	67	21	23

Table 3: Number of CG iterations, large jumps: $\alpha_5 = 0.001$, $\alpha_i = 1$ for $i \neq 5$.

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