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Regularization error estimates for distributed control problems in energy spaces

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Abstract

For tracking type distributed optimal control problems subject to second order elliptic partial differential equations we analyze the regularization error of the state u_ϱ und the target $\overline{u}$ with respect to the regularization parameter ϱ. The main focus is on the regularization in the energy space $H^{-1}(\Omega)$, but we also consider the regularization in $L^2(\Omega)$ for comparison. While there is no difference in the regularization error estimates when considering suitable target functions $\overline{u} \in H^1_0(\Omega)$, we obtain a higher order convergence in the relaxation parameter ϱ when considering the control in the enrgy space $H^{-1}(\Omega)$ which also affects the approximation of the target $\overline{u}$ by the state u_ϱ.

1 Introduction

Optimal control [7, 9, 16] and inverse problems [4, 13] subject to partial differential equations often involve some parameter dependent cost or regularization terms. Our particular interest is in the behavior of the solution approximating the target function when the regularization parameter tends to zero, see also [2] for a related discussion. For different choices of the underlying function space and depending on the regularity of the target function we prove different orders of convergence.

As a model problem we consider a tracking type functional to reach a given target $\overline{u} \in L^2(\Omega)$ subject to the Dirichlet boundary value problem for the Poisson equation with distributed control. While the control is often considered in $L^2(\Omega)$, we also consider the control in the energy space $H^{-1}(\Omega)$. It turns out that the order of convergence with respect to the regularization parameter can be quite different when considering the control either in $L^2(\Omega)$ or in $H^{-1}(\Omega)$. While there is no difference when the target function is sufficient regular, e.g., $\overline{u} \in C^\infty(\Omega)$, in the general case of some restricted regularity assumptions on $\overline{u}$, in the case of energy control we can prove a convergence order which is twice compared to the convergence order for the control in $L^2(\Omega)$, see Theorem 3.2 and Theorem 4.1, respectively. This different behavior is also reflected by the approximability of the target function $\overline{u}$ by the related states u_ϱ. In particular, considering the control in $L^2(\Omega)$ always results in a $H^1(\Omega)$ regularity which never allows the use of discontinuous controls. In contrast, the energy control approach to approximate a target function in $L^2(\Omega)$ gives $L^2(\Omega)$ regularity for the control, i.e. discontinuous controls are included.

Although optimal control problems either in $L^2(\Omega)$ or in $H^{-1}(\Omega)$ may include control constraints, regularization error estimates are less meaningful in this case. Thus we consider, as in inverse problems, neither control nor state constraints. Then, in both cases the optimality system can be reduced to a boundary value problem for the state, see (2.15) and (4.6), respectively. In particular, for the energy control problem we obtain a singular perturbed Dirichlet boundary value problem for the Poisson equation, while for the control in $L^2(\Omega)$ this is a singular perturbed problem for the BiLaplace operator.

This paper is structured as follows: In Section 2 we describe the distributed control problem with energy regularization including box constraints for the control. We derive the complementarity conditions for the constrained problem, and the optimality system in the unconstrained case. Regularization error estimates for the energy control are given in Theorem 3.2 in $L^2(\Omega)$ and in $H^{-1}(\Omega)$. For comparison we consider in Section 4 the case of the control in $L^2(\Omega)$ where the related results are given in Theorem 4.1. Since all regularization error estimates also depend on the regularity of the given target function, we study and discuss the behavior of the solutions of both approaches for different targets. These numerical results are given in Section 4.

2 Distributed control problem in the energy space

Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$. As a model problem we consider the distributed control problem to minimize the cost functional, for varying $\varrho \in \mathbb{R}_+$,

$$\mathcal{J}(u_\varrho, z_\varrho) = \frac{1}{2} \int_\Omega [u_\varrho(x) - \overline{u}(x)]^2 \, dx + \frac{1}{2} \varrho \, \|z_\varrho\|^2_{H^{-1}(\Omega)} \tag{2.1}$$

subject to the constraint

$$-\Delta u_\varrho(x) = z_\varrho(x) \quad \text{for } x \in \Omega, \quad u_\varrho(x) = 0 \quad \text{for } x \in \Gamma, \tag{2.2}$$

and where the control z_ϱ satisfies

$$z_\varrho \in \mathcal{U} := \Big\{ w \in H^{-1}(\Omega) : w \leq g \quad \text{in } H^{-1}(\Omega) \Big\}. \tag{2.3}$$

We assume $\overline{u} \in L_2(\Omega)$ and $g \in H^{-1}(\Omega)$. Note that $w \leq g$ in $H^{-1}(\Omega)$ is equivalent to

$$\langle w - g, \varphi \rangle_\Omega \leq 0 \quad \text{for all } \varphi \in H_0^1(\Omega), \quad \varphi \geq 0 \quad \text{in } \Omega.$$

The solution of the Dirichlet boundary value problem (2.2), i.e. the solution of the related variational problem to find $u_\varrho \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_\varrho(x) \cdot \nabla v(x) \, dx = \int_\Omega z_\varrho(x) \, v(x) \, dx \quad \text{for all } v \in H_0^1(\Omega), \tag{2.4}$$

induces an operator $\mathcal{H} : H^{-1}(\Omega) \to H_0^1(\Omega)$, which implies an equivalent norm in $H^{-1}(\Omega)$, i.e. the energy norm

$$\|z_\varrho\|^2_{H^{-1}(\Omega)} := \langle \mathcal{H} z_\varrho, z_\varrho \rangle_\Omega = \int_\Omega u_\varrho(x) z_\varrho(x) \, dx = \int_\Omega |\nabla u_\varrho(x)|^2 dx = \|\nabla u_\varrho\|^2_{L_2(\Omega)} \,. \tag{2.5}$$

By using $u_\varrho = \mathcal{H} z_\varrho$ we can write the cost functional (2.1) as the reduced cost functional

$$\widetilde{J}(z_\varrho) = \frac{1}{2} \langle \mathcal{H}^* \mathcal{H} z_\varrho, z_\varrho \rangle_\Omega - \langle \mathcal{H}^* \overline{u}, z_\varrho \rangle_\Omega + \frac{1}{2} \|\overline{u}\|^2_{L_2(\Omega)} + \frac{1}{2} \varrho \langle \mathcal{H} z_\varrho, z_\varrho \rangle_\Omega, \tag{2.6}$$

where $\mathcal{H}^* : L_2(\Omega) \to H_0^1(\Omega)$ is the adjoint operator of $\mathcal{H} : H^{-1}(\Omega) \to H_0^1(\Omega) \subset L_2(\Omega)$. For the application $\mathcal{H}^* \psi \in H_0^1(\Omega)$, $\psi \in L_2(\Omega)$, and for $\varphi \in H^{-1}(\Omega)$ we have

$$\langle \mathcal{H}^* \psi, \varphi \rangle_\Omega = \langle \psi, \mathcal{H} \varphi \rangle_\Omega = \langle \nabla p, \nabla \mathcal{H} \varphi \rangle_\Omega = \langle p, \varphi \rangle_\Omega,$$

where $p \in H_0^1(\Omega)$ is the unique solution of the Dirichlet boundary value problem

$$-\Delta p(x) = \psi(x) \quad \text{for } x \in \Omega, \quad p(x) = 0 \quad \text{for } x \in \Gamma. \tag{2.7}$$

To characterize the minimizer of the reduced cost functional (2.6) we introduce the self–adjoint and bounded operator

$$T_\varrho : \mathcal{H}^* \mathcal{H} + \varrho \mathcal{H} : H^{-1}(\Omega) \to H_0^1(\Omega)$$

and we define

$$f := \mathcal{H}^* \overline{u} \in H_0^1(\Omega).$$

Hence we can rewrite the reduced cost functional (2.6) as

$$\widetilde{J}(z_\varrho) = \frac{1}{2} \langle T_\varrho z_\varrho, z_\varrho \rangle_\Omega - \langle f, z_\varrho \rangle_\Omega + \frac{1}{2} \|\overline{u}\|^2_{L_2(\Omega)}. \tag{2.8}$$

Since $\mathcal{U} \subset H^{-1}(\Omega)$ is convex and closed, and since T_ϱ is self–adjoint and $H^{-1}(\Omega)$–elliptic, i.e. for all $z \in H^{-1}(\Omega)$ implying $u = \mathcal{H} z \in H_0^1(\Omega)$ we have

$$\langle T_\varrho z, z \rangle_\Omega = \varrho \, \langle \mathcal{H} z, z \rangle_\Omega + \|\mathcal{H} z\|^2_{L_2(\Omega)} = \varrho \, \|\nabla u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} =: \|u\|^2_{H^1(\Omega), \varrho}, \tag{2.9}$$

the minimization of (2.8) is equivalent to solving a variational inequality to find $z_\varrho \in \mathcal{U}$ such that

$$\langle T_\varrho z_\varrho, w - z_\varrho \rangle_\Omega \geq \langle f, w - z_\varrho \rangle_\Omega \quad \text{for all } w \in \mathcal{U}. \tag{2.10}$$

Note that $\| \cdot \|^2_{H^1(\Omega),\varrho}$ defines, for $\varrho > 0$, an equivalent norm in $H^1(\Omega)$. Since (2.10) is an elliptic variational inequality of the first kind, we can use standard arguments as given, for example in [5, 9], to establish unique solvability of the variational inequality (2.10). For $z_\varrho \in H^{-1}(\Omega)$ being the unique solution of the variational inequality (2.10) we rewrite

$$T_\varrho z_\varrho - f = \mathcal{H}^*(\mathcal{H} z_\varrho - \overline{u}) + \varrho \mathcal{H} z_\varrho = p_\varrho + \varrho u_\varrho,$$

where $p_\varrho \in H^1_0(\Omega)$ is the unique solution of the adjoint boundary value problem

$$-\Delta p_\varrho(x) = u_\varrho(x) - \overline{u}(x) \quad \text{for } x \in \Omega, \quad p_\varrho(x) = 0 \quad \text{for } x \in \Gamma. \tag{2.11}$$

By using the energy norm in $H^{-1}(\Omega)$ we finally obtain

$$\|u_\varrho - \overline{u}\|^2_{H^{-1}(\Omega)} = \langle \mathcal{H}(u_\varrho - \overline{u}), u_\varrho - \overline{u} \rangle_\Omega = \langle p_\varrho, u_\varrho - \overline{u} \rangle_\Omega = \int_\Omega \nabla p_\varrho \cdot \nabla p_\varrho \, dx = \|\nabla p_\varrho\|^2_{L_2(\Omega)}. \tag{2.12}$$

Recall that the complementarity conditions of the variational inequality (2.10) are given as

$$p_\varrho + \varrho u_\varrho \leq 0 \quad \text{in } H^1_0(\Omega), \quad z_\varrho \leq g \quad \text{in } H^{-1}(\Omega), \quad [p_\varrho + \varrho u_\varrho][z_\varrho - g] = 0 \quad \text{in } \Omega \text{ a.e.} \tag{2.13}$$

In the particular case $z_\varrho \in H^{-1}(\Omega)$, i.e. no constraints on the control z_ϱ, we conclude the optimality condition

$$p_\varrho(x) + \varrho u_\varrho(x) = 0 \quad \text{for } x \in \Omega, \tag{2.14}$$

and for the adjoint boundary value problem (2.11) we obtain

$$-\varrho \Delta u_\varrho(x) + u_\varrho(x) = \overline{u}(x) \quad \text{for } x \in \Omega, \quad u_\varrho(x) = 0 \quad \text{for } x \in \Gamma. \tag{2.15}$$

The variational formulation of the Dirichlet boundary value problem (2.15) reads to find $u_\varrho \in H^1_0(\Omega)$ such that

$$\varrho \int_\Omega \nabla u_\varrho(x) \cdot \nabla v(x) \, dx + \int_\Omega u_\varrho(x) \, v(x) \, dx = \int_\Omega \overline{u}(x) \, v(x) \, dx \quad \text{for all } v \in H^1_0(\Omega). \tag{2.16}$$

Note that for $\varrho \to 0$ (2.15) is a singular perturbed boundary value problem, see, e.g., [6, 8], implying boundary layers. Finally, the control is given, by using $-\Delta u_\varrho = z_\varrho$, as

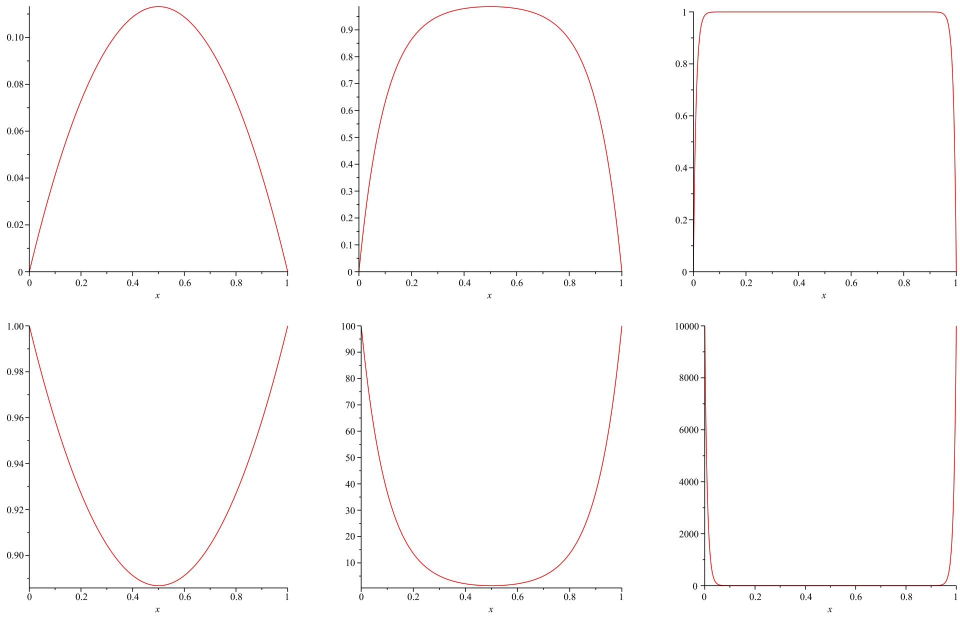
$$z_\varrho = \frac{1}{\varrho}(\overline{u} - u_\varrho) \in L_2(\Omega). \tag{2.17}$$

Obviously, higher regularity of $\overline{u}$ will imply higher regularity of z_ϱ as well, e.g. $\overline{u} \in H^1(\Omega)$ implies $z_\varrho \in H^1(\Omega)$.

Example 2.1 *For $n = 1$ and $\Omega = (0,1)$ we choose $\overline{u} = 1$ and we consider the Dirichlet boundary value problem* (2.15),

$$-\varrho u_\varrho''(x) + u_\varrho(x) = 1 \quad \textit{for } x \in (0,1), \quad u_\varrho(0) = u_\varrho(1) = 0. \tag{2.18}$$

For different values of ϱ the state u_ϱ and the related control z_ϱ are given in Fig. 1. Already for $\varrho = 10^{-4}$ we observe a rather good approximation u_ϱ of $\overline{u}$ while the control z_ϱ is concentrated near to the boundary points. Although the control is not identically zero between, the values inside and near to the boundary differ by magnitudes, and hence may be neglected.



$\varrho = 1$ $\qquad$ $\varrho = 10^{-2}$ $\qquad$ $\varrho = 10^{-4}$

Figure 1: State and control for $\Omega = (0,1), \overline{u} = 1$, control in $H^{-1}(0,1)$.

3 Regularization error estimates for energy control

Although the regularization parameter $\varrho > 0$ is required to ensure well posedness of the optimal control problem (2.1)–(2.3), and in particular the $H^{-1}(\Omega)$–ellipticity (2.9) of T_ϱ, we are interested in the approximability of the target $\overline{u}$ by the state u_ϱ, in particular in the error $\|u_\varrho - \overline{u}\|_{L_2(\Omega)}$. Let us start to consider the optimal control problem without constraints, where we finally have to solve the variational problem (2.16), i.e. for $\varrho \in \mathbb{R}_+$ we consider $u_\varrho \in H_0^1(\Omega)$ to be the unique solution of the variational formulation

$$\varrho \int_\Omega \nabla u_\varrho(x) \cdot \nabla v(x)\, dx + \int_\Omega u_\varrho(x)\, v(x)\, dx = \int_\Omega \overline{u}(x)\, v(x)\, dx \quad \text{for all } v \in H_0^1(\Omega). \tag{3.1}$$

Lemma 3.1 *For any $\varrho > 0$, there exists a unique solution $u_\varrho \in H_0^1(\Omega)$ of the variational formulation* (3.1) *satisfying*

$$\|u_\varrho\|_{L_2(\Omega)} \le \|\overline{u}\|_{L_2(\Omega)}, \quad \sqrt{\varrho}\, \|\nabla u_\varrho\|_{L_2(\Omega)} \le \|\overline{u}\|_{L_2(\Omega)}. \tag{3.2}$$

Proof. Unique solvability of the variational formulation (3.1) follows from the Lax–Milgram lemma due to the ellipticity of T_ϱ, see (2.9). By chosing $v = u_\varrho$ and applying the Cauchy–Schwarz inequality this gives

$$\varrho\, \|\nabla u_\varrho\|^2_{L_2(\Omega)} + \|u_\varrho\|^2_{L_2(\Omega)} = \langle \overline{u}, u_\varrho \rangle_{L_2(\Omega)} \le \|\overline{u}\|_{L_2(\Omega)} \|u_\varrho\|_{L_2(\Omega)}, \tag{3.3}$$

from which the estimates (3.2) follow. ■

When assuming some regularity of the target $\overline{u} \in L^2(\Omega)$ in some higher order interpolation spaces, see, e.g., [10, 12, 15], we can also derive regularization error estimates for $u_\varrho - \overline{u}$ in $L_2(\Omega)$, and in $H^{-1}(\Omega)$, respectively.

Theorem 3.2 *Let $u_\varrho \in H_0^1(\Omega)$ be the unique solution of the variational formulation* (3.1)*. Assume $\overline{u} \in H_0^s(\Omega) := [L_2(\Omega), H_0^1(\Omega)]_s$ for some $s \in [0, 1]$. Then there hold the estimates*

$$\|u_\varrho - \overline{u}\|_{H^{-1}(\Omega)} \le c\, \varrho^{(1+s)/2}\, \|\overline{u}\|_{H^s(\Omega)} \tag{3.4}$$

and

$$\|u_\varrho - \overline{u}\|_{L_2(\Omega)} \le c\, \varrho^{s/2}\, \|\overline{u}\|_{H^s(\Omega)}\, . \tag{3.5}$$

The error estimate (3.5) *remains true if we have $\overline{u} \in H_0^1(\Omega) \cap H^s(\Omega)$ for some $s \in (1, 2]$.*

Proof. By using the energy norm (2.12), the optimality condition (2.14) in $H_0^1(\Omega)$, and the stability estimate (3.2), we have

$$\|u_\varrho - \overline{u}\|^2_{H^{-1}(\Omega)} = \|\nabla p_\varrho\|^2_{L_2(\Omega)} = \varrho^2\, \|\nabla u_\varrho\|^2_{L_2(\Omega)} \le \varrho\, \|\overline{u}\|^2_{L_2(\Omega)},$$

i.e.

$$\|u_\varrho - \overline{u}\|_{H^{-1}(\Omega)} \le \sqrt{\varrho}\, \|\overline{u}\|_{L_2(\Omega)}\, . \tag{3.6}$$

From the variational formulation (3.1) we obtain

$$\varrho \int_\Omega \nabla u_\varrho(x) \cdot \nabla v(x)\, dx = \int_\Omega [\overline{u}(x) - u_\varrho(x)]\, v(x)\, dx \quad \text{for all } v \in H_0^1(\Omega), \tag{3.7}$$

and in particular for $v = u_\varrho \in H_0^1(\Omega)$ we then conclude

$$\begin{aligned} \varrho\, \|\nabla u_\varrho\|^2_{L_2(\Omega)} &= \varrho \int_\Omega \nabla u_\varrho(x) \cdot \nabla u_\varrho(x)\, dx = \int_\Omega [\overline{u}(x) - u_\varrho(x)]\, u_\varrho(x)\, dx \\ &= \int_\Omega [\overline{u}(x) - u_\varrho(x)]\, \overline{u}(x)\, dx - \int_\Omega [u_\varrho(x) - \overline{u}(x)][u_\varrho(x) - \overline{u}(x)]\, dx, \end{aligned}$$

and therefore,

$$\varrho \, \|\nabla u_\varrho\|^2_{L_2(\Omega)} + \|u_\varrho - \overline{u}\|^2_{L_2(\Omega)} = \int_\Omega [\overline{u}(x) - u_\varrho(x)] \, \overline{u}(x) \, dx \leq \|u_\varrho - \overline{u}\|_{L_2(\Omega)} \|\overline{u}\|_{L_2(\Omega)}$$

follows, i.e. we have

$$\|u_\varrho - \overline{u}\|_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}. \tag{3.8}$$

Now we assume $\overline{u} \in H^1_0(\Omega)$, then we can choose $v = \overline{u} - u_\varrho \in H^1_0(\Omega)$ as a test function in (3.7) to obtain

$$\begin{aligned} \|u_\varrho - \overline{u}\|^2_{L_2(\Omega)} &= \int_\Omega [\overline{u}(x) - u_\varrho(x)][\overline{u}(x) - u_\varrho(x)] \, dx \\ &= \varrho \int_\Omega \nabla u_\varrho(x) \cdot \nabla [\overline{u}(x) - u_\varrho(x)] \, dx \\ &= \varrho \int_\Omega \nabla \overline{u}(x) \cdot \nabla [\overline{u}(x) - u_\varrho(x)] \, dx - \varrho \int_\Omega \nabla [\overline{u}(x) - u_\varrho(x)] \cdot \nabla [\overline{u}(x) - u_\varrho(x)] \, dx, \end{aligned}$$

and therefore,

$$\begin{aligned} \|u_\varrho - \overline{u}\|^2_{L_2(\Omega)} + \varrho \|\nabla (u_\varrho - \overline{u})\|^2_{L_2(\Omega)} &= \varrho \int_\Omega \nabla \overline{u}(x) \cdot \nabla [\overline{u}(x) - u_\varrho(x)] \, dx \\ &\leq \varrho \, \|\nabla \overline{u}\|_{L_2(\Omega)} \|\nabla (\overline{u} - u_\varrho)\|_{L_2(\Omega)} \end{aligned} \tag{3.9}$$

follows. From this we conclude

$$\|\nabla (u_\varrho - \overline{u})\|_{L_2(\Omega)} \leq \|\nabla \overline{u}\|_{L_2(\Omega)},$$

as well as

$$\|u_\varrho - \overline{u}\|_{L_2(\Omega)} \leq \sqrt{\varrho} \, \|\nabla \overline{u}\|_{L_2(\Omega)}. \tag{3.10}$$

Moreover, the energy norm (2.12), the optimality condition (2.14), and the equality (3.3) imply

$$\begin{aligned} \|u_\varrho - \overline{u}\|^2_{H^{-1}(\Omega)} &= \varrho^2 \, \|\nabla u_\varrho\|^2_{L_2(\Omega)} = \varrho \, \langle \overline{u} - u_\varrho, u_\varrho \rangle_{L_2(\Omega)} \\ &= \varrho \Big[\langle \overline{u} - u_\varrho, u_\varrho - \overline{u} \rangle_{L_2(\Omega)} + \langle \overline{u} - u_\varrho, \overline{u} \rangle_{L_2(\Omega)} \Big], \end{aligned}$$

i.e.

$$\|u_\varrho - \overline{u}\|^2_{H^{-1}(\Omega)} + \varrho \|u_\varrho - \overline{u}\|^2_{L_2(\Omega)} = \varrho \, \langle \overline{u} - u_\varrho, \overline{u} \rangle_{L_2(\Omega)}.$$

In particular for $\overline{u} \in H^1_0(\Omega)$ we further obtain from the variational formulation of the adjoint boundary value problem (2.11)

$$\begin{aligned} \|u_\varrho - \overline{u}\|^2_{H^{-1}(\Omega)} + \varrho \|u_\varrho - \overline{u}\|^2_{L_2(\Omega)} &= \varrho \, \langle \overline{u} - u_\varrho, \overline{u} \rangle_\Omega = -\varrho \int_\Omega \nabla p_\varrho(x) \cdot \nabla \overline{u}(x) \, dx \\ &\leq \varrho \, \|\nabla p_\varrho\|_{L_2(\Omega)} \|\nabla \overline{u}\|_{L_2(\Omega)} = \varrho \, \|u_\varrho - \overline{u}\|_{H^{-1}(\Omega)} \|\nabla \overline{u}\|_{L_2(\Omega)}. \end{aligned}$$

Hence we conclude

$$\|u_\varrho - \overline{u}\|_{H^{-1}(\Omega)} \leq \varrho \, \|\nabla \overline{u}\|_{L_2(\Omega)}. \tag{3.11}$$

Now, using an interpolation argument, the estimate (3.4) follows from (3.6) and (3.11), while (3.5) follows from (3.8) and (3.10).

Finally we consider the case $\overline{u} \in H^1_0(\Omega) \cap H^2(\Omega)$. From (3.9) we then conclude, using integration by parts,

$$\begin{aligned}
\|u_\varrho - \overline{u}\|^2_{L_2(\Omega)} + \varrho \|\nabla(u_\varrho - \overline{u})\|^2_{L_2(\Omega)} &= \varrho \int_\Omega \nabla \overline{u}(x) \cdot \nabla[\overline{u}(x) - u_\varrho(x)] \, dx \\
&= \varrho \int_\Omega [-\Delta \overline{u}(x)] \, [\overline{u}(x) - u_\varrho(x)] \, dx \\
&\leq \varrho \, \|\Delta \overline{u}\|_{L_2(\Omega)} \|u_\varrho - \overline{u}\|_{L_2(\Omega)},
\end{aligned}$$

i.e.

$$\|u_\varrho - \overline{u}\|_{L_2(\Omega)} \leq \varrho \, \|\Delta \overline{u}\|_{L_2(\Omega)}. \tag{3.12}$$

Using an interpolation argument for $\overline{u} \in H^1_0(\Omega) \cap H^s(\Omega)$, $s \in (1, 2)$, concludes the proof. ■

Remark 3.1 *While the error estimate* (3.5) *remains valid for* $\overline{u} \in H^1_0(\Omega) \cap H^s(\Omega)$ *and* $s \in (1, 2]$, *this is not the case for the estimate* (3.4). *This behavior is also observed in our numerical experiments.*

In the case of a convex polygonal ($d = 2$) or polyhedral ($d = 3$) bounded domain Ω, or if $\Gamma = \partial\Omega$ is smooth, we can conclude $u_\varrho \in H^2(\Omega)$. However, in the related norm estimate there will be some dependency on the regularization parameter ϱ.

Corollary 3.3 *Assume that* $u_\varrho \in H^2(\Omega)$ *is the unique solution of the variational problem* (2.16). *Assume* $\overline{u} \in H^s(\Omega)$ *for some* $s \in [0, 1]$, *or* $\overline{u} \in H^1_0(\Omega) \cap H^s(\Omega)$ *for some* $s \in (1, 2]$. *Then,*

$$\|\Delta u_\varrho\|_{L_2(\Omega)} \leq c \, \varrho^{(s-2)/2} \, \|\overline{u}\|_{H^s(\Omega)} \, . \tag{3.13}$$

Proof. By using (2.15) and (3.8) we first have

$$\varrho \, \|\Delta u_\varrho\|_{L_2(\Omega)} = \|u_\varrho - \overline{u}\|_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)},$$

and with (3.12) we have

$$\varrho \, \|\Delta u_\varrho\|_{L_2(\Omega)} = \|u_\varrho - \overline{u}\|_{L_2(\Omega)} \leq \varrho \, \|\Delta \overline{u}\|_{L_2(\Omega)} \leq \varrho \, \|\overline{u}\|_{H^2(\Omega)}.$$

Now the assertion follows from an interpolation argument. ■

Now we are in a position to state some regularity results for the unconstrained control $z_\varrho \in H^{-1}(\Omega)$.

Corollary 3.4 *Assume $\overline{u} \in H_0^1(\Omega)$. Then we have*

$$\|z_\varrho\|_{H^{-1}(\Omega)} \leq c\,\|\overline{u}\|_{H^1(\Omega)}$$

with a constant c independent of ϱ. If we assume $\overline{u} \in H_0^1(\Omega) \cap H^2(\Omega)$ we further obtain

$$\|z_\varrho\|_{L_2(\Omega)} \leq c\,\|\overline{u}\|_{H^2(\Omega)}$$

with a constant c independent of ϱ.

Proof. By using the optimality condition (2.17) and the error estimate (3.4) we obtain

$$\|z_\varrho\|_{H^{-1}(\Omega)} = \varrho^{-1}\|u_\varrho - \overline{u}\|_{H^{-1}(\Omega)} \leq c\varrho^{(s-1)/2}\|\overline{u}\|_{H^s(\Omega)}$$

if we assume $\overline{u} \in H^s(\Omega)$ for some $s \in [0,1]$. In particular for $s = 1$ we conclude the desired estimate. The second estimate follows by using the optimality condition (2.17) and the error estimate (3.5). ∎

Remark 3.2 *In fact, for $\overline{u} \in H_0^1(\Omega)$ and $\varrho = 0$ we obtain $u_0 = \overline{u}$ and $z_0 = -\Delta\overline{u} \in H^{-1}(\Omega)$ as optimal solution. In particular, $\overline{u} \in H_0^1(\Omega)$ ensures that $\overline{u}$ is in the image of the solution operator $\mathcal{H} : H^{-1}(\Omega) \to H_0^1(\Omega)$.*

Finally we turn back to the optimal control problem with constraints. It is obvious that in general we can not expect such kind of error estimates as given in Theorem 3.2. However, we can bound the state u_ϱ as follows.

Lemma 3.5 *Let $(u_\varrho, p_\varrho, z_\varrho) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega)$ be the unique solution of the optimality system* (2.2), (2.11), *and* (2.13). *Then there hold*

$$\|u_\varrho\|_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}, \quad \sqrt{\varrho}\,\|\nabla u_\varrho\|_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}.$$

Proof. From the optimality condition (2.10) we first find

$$\langle p_\varrho + \varrho u_\varrho, w - z_\varrho\rangle_\Omega \geq 0 \quad \text{for all } w \in \mathcal{U},$$

or, by using (2.4) for $v = p_\varrho + \varrho u_\varrho \in H_0^1(\Omega)$,

$$\begin{aligned}
0 &\leq \langle p_\varrho + \varrho u_\varrho, w\rangle_\Omega - \langle p_\varrho + \varrho u_\varrho, z_\varrho\rangle_\Omega \\
&= \langle p_\varrho + \varrho u_\varrho, w\rangle_\Omega - \langle \nabla u_\varrho, \nabla p_\varrho + \varrho\nabla u_\varrho\rangle_\Omega \\
&= \langle p_\varrho + \varrho u_\varrho, w\rangle_\Omega - \varrho\|\nabla u_\varrho\|^2_{L_2(\Omega)} - \langle \nabla p_\varrho, \nabla u_\varrho\rangle_\Omega \\
&= \langle p_\varrho + \varrho u_\varrho, w\rangle_\Omega - \varrho\|\nabla u_\varrho\|^2_{L_2(\Omega)} - \langle u_\varrho - \overline{u}, u_\varrho\rangle_\Omega,
\end{aligned}$$

i.e.

$$\varrho\|\nabla u_\varrho\|^2_{L_2(\Omega)} + \|u_\varrho\|^2_{L_2(\Omega)} \leq \langle p_\varrho + \varrho u_\varrho, w\rangle_\Omega + \langle \overline{u}, u_\varrho\rangle_\Omega \quad \text{for all } w \in \mathcal{U}.$$

In particular for $w = g - z_\varrho \in \mathcal{U}$ we conclude, by the complementarity condition,

$$\varrho\|\nabla u_\varrho\|^2_{L_2(\Omega)} + \|u_\varrho\|^2_{L_2(\Omega)} \leq \langle \overline{u}, u_\varrho\rangle_\Omega \leq \|\overline{u}\|_{L_2(\Omega)}\|u_\varrho\|_{L_2(\Omega)},$$

from which we further conclude

$$\|u_\varrho\|_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}, \quad \varrho\|\nabla u_\varrho\|^2_{L_2(\Omega)} \leq \|\overline{u}\|^2_{L_2(\Omega)}.$$

∎

4 Distributed control in $L_2(\Omega)$

Instead of the energy space $H^{-1}(\Omega)$, the space $L_2(\Omega)$ is often considered for distributed control problems, i.e. the cost functional reads

$$\mathcal{J}(u,z) = \frac{1}{2}\int_\Omega [u(x)-\overline{u}(x)]^2\,dx + \frac{1}{2}\varrho\,\|z\|^2_{L_2(\Omega)} \tag{4.1}$$

subject to the constraint (2.2),

$$-\Delta u(x) = z(x) \quad \text{for } x\in\Omega, \quad u(x)=0 \quad \text{for } x\in\Gamma.$$

For simplicity in the presentation we do not include box constraints which can be handled by using a suitable projection operator, see, e.g., [3, 16]. By using $u=\mathcal{H}z$ we can write the cost functional (4.1) in its reduced form,

$$\widetilde{J}(z) = \frac{1}{2}\langle \mathcal{H}^*\mathcal{H}z, z\rangle_\Omega - \langle \mathcal{H}^*\overline{u}, z\rangle_\Omega + \frac{1}{2}\|\overline{u}\|^2_{L_2(\Omega)} + \frac{1}{2}\varrho\langle z,z\rangle_{L_2(\Omega)}, \tag{4.2}$$

and where the adjoint state $p=\mathcal{H}^*(\mathcal{H}z-\overline{u})\in H^1_0(\Omega)$ still solves the adjoint Dirichlet boundary value problem (2.11),

$$-\Delta p(x) = u(x)-\overline{u}(x) \quad \text{for } x\in\Omega, \quad p(x)=0 \quad \text{for } x\in\Gamma.$$

From minimizing the reduced cost functional (4.2) we now obtain the optimality condition

$$p(x) + \varrho\, z(x) = 0 \quad \text{for } x\in\Omega. \tag{4.3}$$

Using the optimality condition (4.3) we obtain the variational formulation of the primal Dirichlet boundary value problem (2.2): Find $u\in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \frac{1}{\varrho}\int_\Omega p(x)v(x)\,dx = 0 \quad \text{for all } v\in H^1_0(\Omega). \tag{4.4}$$

On the other hand, the adjoint $p\in H^1_0(\Omega)$ is the unique solution of

$$\int_\Omega \nabla p(x)\cdot\nabla q(x)\,dx = \int_\Omega [u(x)-\overline{u}(x)]q(x)\,dx \quad \text{for all } q\in H^1_0(\Omega). \tag{4.5}$$

Using standard arguments we can ensure unique solvability of the reduced optimality system (4.4) and (4.5). In fact, using the optimality condition (4.3) within the primal Dirichlet boundary value problem (2.2) this gives

$$p(x) = \varrho\Delta u(x) \quad \text{for } x\in\Omega.$$

Inserting this into the adjoint Dirichlet boundary value problem (2.11) this finally gives

$$\varrho\Delta^2 u(x) + u(x) = \overline{u}(x) \quad \text{for } x\in\Omega, \quad u(x)=0, \quad \Delta u(x)=0 \quad \text{for } x\in\Gamma. \tag{4.6}$$

Note that (4.6) is a singularly perturbed problem for the biharmonic operator with Dirichlet boundary conditions, in contrast to the singularly perturbed problem (2.15) for the Laplace operator in case of the energy control.

As in Theorem 3.2 we have the following result:

Theorem 4.1 *Assume* $\overline{u} \in H_0^s(\Omega) := [L_2(\Omega), H_0^1(\Omega)]_S$ *for some* $s \in [0,1]$. *Then there hold the estimates*

$$\|u - \overline{u}\|_{H^{-1}(\Omega)} \leq \, c\,\varrho^{(1+s)/4}\,\|\overline{u}\|_{H^s(\Omega)}, \tag{4.7}$$

and

$$\|u - \overline{u}\|_{L_2(\Omega)} \leq \, c\,\varrho^{s/4}\,\|\overline{u}\|_{H^s(\Omega)}. \tag{4.8}$$

If we assume $z \in H_0^1(\Omega) \cap H^2(\Omega)$ *we further have the error estimate*

$$\|u - \overline{u}\|_{L_2(\Omega)} \leq \varrho\,\|\Delta z\|_{L_2(\Omega)}. \tag{4.9}$$

Proof. By using (2.12) and the optimality condition (4.3) we first have

$$\|u - \overline{u}\|^2_{H^{-1}(\Omega)} = \|\nabla p\|^2_{L_2(\Omega)} = \varrho^2\,\|\nabla z\|^2_{L_2(\Omega)}\,. \tag{4.10}$$

From the variational formulation (4.5), using the optimality condition (4.3), i.e. $p = -\varrho z$, and choosing $q = -z$ this gives

$$\varrho \int_\Omega \nabla z(x) \cdot \nabla z(x)\,dx + \int_\Omega u(x)\,z(x)\,dx = \int_\Omega \overline{u}(x)\,z(x)\,dx. \tag{4.11}$$

Since $u \in H_0^1(\Omega)$ is the solution of the primal problem (2.2) this is equivalent to

$$\varrho\,\|\nabla z\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}\|z\|_{L_2(\Omega)}.$$

For $z \in H_0^1(\Omega)$ we have, by considering the primal problem (2.2),

$$\|z\|^2_{L_2(\Omega)} = \int_\Omega z(x)\,z(x)\,dx = \int_\Omega \nabla u(x) \cdot \nabla z(x)\,dx \leq \|\nabla u\|_{L_2(\Omega)}\|\nabla z\|_{L_2(\Omega)}.$$

Hence,

$$\|\nabla u\|^2_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}\|z\|_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}\|\nabla u\|^{1/2}_{L_2(\Omega)}\|\nabla z\|^{1/2}_{L_2(\Omega)},$$

i.e.

$$\|\nabla u\|^{3/2}_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}\|\nabla z\|^{1/2}_{L_2(\Omega)}.$$

With this we further conclude

$$\varrho\,\|\nabla z\|^2_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}\|z\|_{L_2(\Omega)} \leq \|\overline{u}\|_{L_2(\Omega)}\|\nabla u\|^{1/2}_{L_2(\Omega)}\|\nabla z\|^{1/2}_{L_2(\Omega)} \leq \|\overline{u}\|^{4/3}_{L_2(\Omega)}\|\nabla z\|^{2/3}_{L_2(\Omega)},$$

i.e.

$$\varrho\,\|\nabla z\|^{4/3}_{L_2(\Omega)} \leq \|\overline{u}\|^{4/3}_{L_2(\Omega)}, \quad \varrho^{3/2}\,\|\nabla z\|^2_{L_2(\Omega)} \leq \|\overline{u}\|^2_{L_2(\Omega)}.$$

From (4.10) we now conclude

$$\|u - \overline{u}\|^2_{H^{-1}(\Omega)} = \varrho^2\,\|\nabla z\|^2_{L_2(\Omega)} \leq \varrho^{1/2}\,\|\overline{u}\|^2_{L_2(\Omega)},$$

i.e.

$$\|u - \overline{u}\|_{H^{-1}(\Omega)} \leq \varrho^{1/4}\,\|\overline{u}\|_{L_2(\Omega)}. \tag{4.12}$$

For $\overline{u} \in H^1_0(\Omega)$ we find from (4.11)

$$\begin{aligned} \varrho \, \|\nabla z\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)} &= \int_\Omega z(x) \, \overline{u}(x) \, dx \\ &= \int_\Omega \nabla u(x) \cdot \nabla \overline{u}(x) \, dx \leq \|\nabla u\|_{L_2(\Omega)} \|\nabla \overline{u}\|_{L_2(\Omega)}, \end{aligned}$$

from which we conclude

$$\|\nabla u\|_{L_2(\Omega)} \leq \|\nabla \overline{u}\|_{L_2(\Omega)}, \quad \varrho \, \|\nabla z\|^2_{L_2(\Omega)} \leq \|\nabla \overline{u}\|^2_{L_2(\Omega)}.$$

Together with (4.10) this gives

$$\|u - \overline{u}\|^2_{H^{-1}(\Omega)} = \varrho^2 \, \|\nabla z\|^2_{L_2(\Omega)} \leq \varrho \, \|\nabla \overline{u}\|^2_{L_2(\Omega)},$$

i.e.

$$\|u - \overline{u}\|_{H^{-1}(\Omega)} \leq \varrho^{1/2} \, \|\nabla \overline{u}\|_{L_2(\Omega)}. \tag{4.13}$$

By using an interpolation argument, the estimate (4.7) follows from (4.12) and (4.13).

Next we derive a bound for the primal solution u in $L_2(\Omega)$: By chosing $q = u$ in (4.5) this gives

$$\begin{aligned} \|u\|^2_{L_2(\Omega)} &= \int_\Omega \nabla p(x) \cdot \nabla u(x) \, dx + \int_\Omega \overline{u}(x) \, u(x) \, dx \\ &\leq \|\nabla p\|_{L_2(\Omega)} \|\nabla u\|_{L_2(\Omega)} + \|\overline{u}\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)}, \end{aligned}$$

and by using

$$\|\nabla u\|^2_{L_2(\Omega)} = \int_\Omega \nabla u(x) \cdot \nabla u(x) \, dx = \int_\Omega z(x) \, u(x) \, dx \leq \|z\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)}$$

we further obtain

$$\begin{aligned} \|u\|^2_{L_2(\Omega)} &\leq \|\nabla p\|_{L_2(\Omega)} \|z\|^{1/2}_{L_2(\Omega)} \|u\|^{1/2}_{L_2(\Omega)} + \|\overline{u}\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)} \\ &= \varrho \, \|\nabla z\|_{L_2(\Omega)} \|z\|^{1/2}_{L_2(\Omega)} \|u\|^{1/2}_{L_2(\Omega)} + \|\overline{u}\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)} \\ &\leq \varrho \, \|\nabla z\|^{5/4}_{L_2(\Omega)} \|\nabla u\|^{1/4}_{L_2(\Omega)} \|u\|^{1/2}_{L_2(\Omega)} + \|\overline{u}\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)} \\ &\leq \|\overline{u}\|^{3/2}_{L_2(\Omega)} \|u\|^{1/2}_{L_2(\Omega)} + \|\overline{u}\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)}, \end{aligned}$$

which is equivalent to

$$\left(\frac{\|u\|^{1/2}_{L_2(\Omega)}}{\|\overline{u}\|^{1/2}_{L_2(\Omega)}} \right)^3 - \frac{\|u\|^{1/2}_{L_2(\Omega)}}{\|\overline{u}\|^{1/2}_{L_2(\Omega)}} - 1 \leq 0,$$

and from which we obtain

$$\frac{\|u\|^{1/2}_{L_2(\Omega)}}{\|\overline{u}\|^{1/2}_{L_2(\Omega)}} \leq c_0 = \frac{1}{6} \frac{12 + (108 + 12\sqrt{69})^{2/3}}{(108 + 12\sqrt{69})^{1/3}} \approx 1.324718.$$

Hence we conclude the estimate

$$\|u-\overline{u}\|_{L_2(\Omega)} \leq (1+c_0)\,\|\overline{u}\|_{L_2(\Omega)}. \tag{4.14}$$

For $\overline{u} \in H^1_0(\Omega)$ we finally consider the variational formulation of the adjoint problem to obtain, for $q = u - \overline{u}$, and by using the previous estimates,

$$\begin{aligned}
\|u-\overline{u}\|^2_{L_2(\Omega)} &= \int_\Omega [u(x)-\overline{u}(x)][u(x)-\overline{u}(x)]\,dx = \int_\Omega \nabla p(x)\cdot\nabla[u(x)-\overline{u}(x)]\,dx \\
&\leq \|\nabla p\|_{L_2(\Omega)}\|\nabla(u-\overline{u})\|_{L_2(\Omega)} = \varrho\,\|\nabla z\|_{L_2(\Omega)}\|\nabla(u-\overline{u})\|_{L_2(\Omega)} \\
&\leq 2\,\varrho^{1/2}\,\|\nabla\overline{u}\|^2_{L_2(\Omega)},
\end{aligned}$$

i.e.

$$\|u-\overline{u}\|_{L_2(\Omega)} \leq \sqrt{2}\,\varrho^{1/4}\,\|\nabla\overline{u}\|_{L_2(\Omega)}. \tag{4.15}$$

Finally, (4.8) follows by an interpolation argument from (4.14) and (4.15).

For $z \in H^1_0(\Omega)\cap H^2(\Delta)$ we obtain from the optimality condition $p + \varrho z = 0$ and by applying integration by parts,

$$\begin{aligned}
\|u-\overline{u}\|^2_{L_2(\Omega)} &= \int_\Omega [u(x)-\overline{u}(x)][u(x)-\overline{u}(x)]\,dx = \int_\Omega \nabla p(x)\cdot\nabla[u(x)-\overline{u}(x)]\,dx \\
&= \varrho\int_\Omega \nabla z(x)\cdot\nabla[\overline{u}(x)-u(x)]\,dx = \varrho\int_\Omega [-\Delta z(x)]\,[\overline{u}(x)-u(x)]\,dx \\
&\leq \varrho\,\|\Delta z\|_{L_2(\Omega)}\|u-\overline{u}\|_{L_2(\Omega)},
\end{aligned}$$

and therefore, (4.9) follows. ■

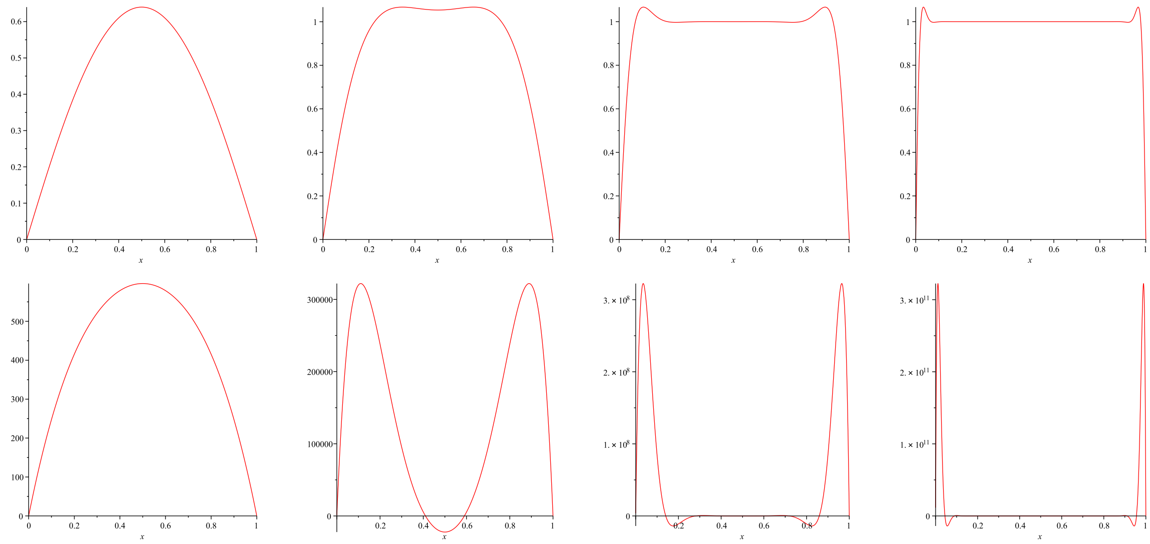
Example 4.1 *For $n = 1$ and $\Omega = (0,1)$ we chose $\overline{u} = 1$ and we consider the Dirichlet boundary value problem* (4.6)*,*

$$\varrho u''(x) + u(x) = 1 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0. \tag{4.16}$$

For different values of ϱ the state u and the related control z are given in Fig. 2. When comparing these results with those for the energy control, see Example 2.1, one observes that a comparable solution is obtained for $\varrho = 10^{-8}$ which corresponds to the theoretical estimates. However, in the case of the L_2 control both the state and the control show some oscillations near to the boundary points.

5 Numerical results

In this section we provide some numerical experiments in order to confirm the theoretical results of Theorem 3.2 and Theorem 4.1, and to give a comparison when considering the control either in the energy space $H^{-1}(\Omega)$ or in $L_2(\Omega)$. Since additional constraints on the



$\varrho = 10^{-2}$ $\varrho = 10^{-4}$ $\varrho = 10^{-6}$ $\varrho = 10^{-8}$

Figure 2: State and control for $\Omega = (0,1), \overline{u} = 1$, control in $L_2(0,1)$.

control do not affect such a comparison, we consider unconstrained problems only. For simplicity we consider the two–dimensional domain $\Omega = (0,1)^2$, and for the discretization we use an adaptive finite element approach to neglect the influence of the discretization error. In fact, let $V_h := S_h^1(\Omega) \cap H_0^1(\Omega)$ be a conforming finite element space of piecewise linear and continuous functions, which is defined with respect to a family of locally quasi–uniform, admissible and shape regular finite elements of mesh size h.

As a first example we consider the control problem (2.1) subject to the Poisson equation (2.2) but with inhomogeneous Dirichlet boundary conditions $u_\varrho(x) = 1$ for $x \in \Gamma$, and the target

$$\overline{u}(x) = \begin{cases} 2 & \text{for } x \in (\frac{1}{4}, \frac{3}{4})^2, \\ 1 & \text{for } x \in \Omega \backslash (\frac{1}{4}, \frac{3}{4})^2, \end{cases} \qquad \overline{u} \in H^{1/2-\varepsilon}(\Omega), \quad \varepsilon > 0.$$

Note that we can easily derive homogeneous Dirichlet boundary conditions by subtracting 1 from u_ϱ and $\overline{u}$, respectively. For the difference $u - \overline{u}$ we can expect, by using Theorem 3.2, an order of convergence of 1.5 when considering the energy norm $\|u - \overline{u}\|^2_{H^{-1}(\Omega)}$, and of 0.5 when considering $\|u - \overline{u}\|^2_{L_2(\Omega)}$, see Table 1 for the numerical results, and Fig. 3 for the state and the control. If we consider the control $z \in L_2(\Omega)$, by using Theorem 4.1, we obtain the orders 0.75 and 0.25 only, see Table 2 and Fig. 4.

As a second example we consider a piecewise linear function $\overline{u} \in H^{3/2-\varepsilon}(\Omega)$, $\varepsilon > 0$, which is one in the mid point $(\frac{1}{2}, \frac{1}{2})$, zero in all corner points, and piecewise linear else. Here we can apply the estimate (3.4) for $s = 1$ only, but the estimate (3.5) for all $s < \frac{3}{2}$. Hence we can expect the orders 2.0 and 1.5, respectively, see Table 3. Correspondingly, when considering the control $z \in L_2(\Omega)$, we expect the orders, by Theorem 4.1, to be 1.25 and 0.75, see Table 4.

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	7.99492 –3		2.33624 –1	
10^{-1}	4.05982 –3	0.29	1.49893 –1	0.19
10^{-2}	3.72475 –4	1.04	5.00162 –2	0.48
10^{-3}	1.45381 –5	1.41	1.58114 –2	0.50
10^{-4}	4.87268 –7	1.47	5.00000 –3	0.50
10^{-5}	1.56841 –8	1.49	1.58114 –3	0.50
10^{-6}	4.98727 –10	1.50	5.00000 –4	0.50
10^{-7}	1.57987 –11	1.50	1.58114 –4	0.50
10^{-8}	4.99872 –13	1.50	5.00000 –5	0.50
Theory		1.5		0.5

Table 1: Approximation of $\overline{u} \in H^{1/2-\varepsilon}(\Omega)$ for control $z \in H^{-1}(\Omega)$.

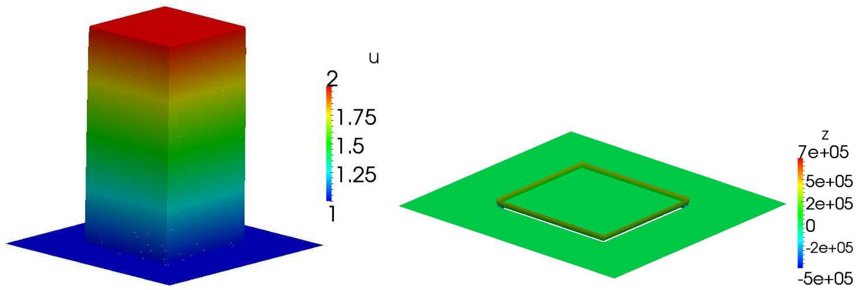


Figure 3: State u and control $z \in H^{-1}(\Omega)$, $\varrho = 10^{-6}$.

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	8.74344 –3		2.49152 –1	
10^{-1}	8.37405 –3	0.02	2.41801 –1	0.01
10^{-2}	5.72646 –3	0.17	1.88957 –1	0.11
10^{-3}	1.05087 –3	0.74	9.15551 –2	0.31
10^{-4}	1.77164 –4	0.77	5.32357 –2	0.24
10^{-5}	3.14348 –5	0.75	2.98236 –2	0.25
10^{-6}	5.59017 –6	0.75	1.67705 –2	0.25
10^{-7}	9.94088 –7	0.75	9.43075 –3	0.25
10^{-8}	1.76777 –7	0.75	5.30330 –3	0.25
Theory		0.75		0.25

Table 2: Approximation of $\overline{u} \in H^{1/2-\varepsilon}(\Omega)$ for control $z \in L_2(\Omega)$.

For the next example we consider the smooth target function

$$\overline{u}(x) = \sin \pi x_1 \, \sin \pi x_2, \quad \overline{u} \in H^1_0(\Omega) \cap H^2(\Omega)$$

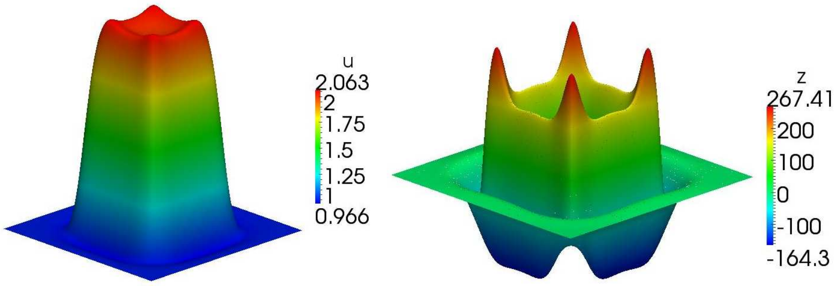


Figure 4: State u and control $z \in L_2(\Omega)$, $\varrho = 10^{-6}$.

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	1.88752 –3		3.77961 –1	
10^{-1}	9.19206 –4	0.31	1.86380 –2	0.31
10^{-2}	5.78152 –5	1.20	1.39486 –3	1.13
10^{-3}	8.65836 –7	1.82	4.47214 –5	1.49
10^{-4}	9.57574 –9	1.96	1.41421 –6	1.50
10^{-5}	9.86584 –11	1.99	4.47214 –8	1.50
10^{-6}	9.95757 –13	2.00	1.41421 –9	1.50
10^{-7}	9.98658 –15	2.00	4.47214 –11	1.50
10^{-8}	9.99576 –17	2.00	1.41421 –12	1.50
Theory		2.0		1.5

Table 3: Approximation of $\overline{u} \in H^{3/2-\varepsilon}(\Omega)$ for control $z \in H^{-1}(\Omega)$.

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	2.07270 –3		4.14567 –2	
10^{-1}	1.98052 –3	0.02	3.96370 –2	0.02
10^{-2}	1.32035 –3	0.18	2.66030 –2	0.17
10^{-3}	1.66385 –4	0.90	3.79982 –3	0.85
10^{-4}	4.71747 –6	1.55	4.40375 –4	0.94
10^{-5}	2.82461 –7	1.22	8.91439 –5	0.69
10^{-6}	1.58115 –8	1.25	1.58114 –5	0.75
10^{-7}	8.89140 –10	1.25	2.81171 –6	0.75
10^{-8}	5.00000 –11	1.25	5.00000 –7	0.75
Theory		1.25		0.75

Table 4: Approximation of $\overline{u} \in H^{3/2-\varepsilon}(\Omega)$ for control $z \in L_2(\Omega)$.

where we can expect second order convergence in all cases, see Table 5 and Table 6.

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	1.14732 –2		2.26472 –1	
10^{-1}	5.57970 –3	0.31	1.10139 –1	0.31
10^{-2}	3.44189 –4	1.21	6.79402 –3	1.21
10^{-3}	4.74560 –6	1.86	9.36745 –5	1.86
10^{-4}	4.91538 –8	1.98	9.70257 –7	1.98
10^{-5}	4.93285 –10	2.00	9.73706 –9	2.00
10^{-6}	4.93461 –12	2.00	9.74052 –11	2.00
10^{-7}	4.93478 –14	2.00	9.74087 –13	2.00
10^{-8}	4.93480 –16	2.00	9.74091 –15	2.00
Theory		2.0		2.0

Table 5: Approximation of $\overline{u} \in C^\infty(\Omega)$ for control $z \in H^{-1}(\Omega)$.

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	1.26004 –2		2.48722 –1	
10^{-1}	1.20392 –2	0.02	2.37645 –1	0.02
10^{-2}	8.02014 –3	0.18	1.58311 –1	0.18
10^{-3}	9.95696 –4	0.91	1.96543 –2	0.91
10^{-4}	1.78127 –5	1.75	3.51608 –4	1.75
10^{-5}	1.90788 –7	1.97	3.76601 –6	1.97
10^{-6}	1.92128 –9	2.00	3.79246 –8	2.00
10^{-7}	1.92263 –11	2.00	3.79512 –10	2.00
10^{-8}	1.92276 –13	2.00	3.79538 –12	2.00
Theory		2.0		2.0

Table 6: Approximation of $\overline{u} \in C^\infty(\Omega)$ for control $z \in L_2(\Omega)$.

As a last example we consider the smooth target function

$$\overline{u}(x) = 1 + \sin \pi x_1 \, \sin \pi x_2$$

wich does not satisfy the zero boundary condition. Hence we can apply all error estimates for $s < \frac{1}{2}$ only, and we expect the same orders of convergence as in the first example, see Table 7 and Table 8.

6 Conclusions

In this paper we have considered regularization error estimates $\|u_\varrho - \overline{u}\|$ of the optimal state u_ϱ and the target $\overline{u}$ of distributed control problems subject to the Poisson equation with the control either in $L^2(\Omega)$, or in the energy space $H^{-1}(\Omega)$. While in the case of a

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	8.06523 –2		1.89526 –0	
10^{-1}	3.99099 –2	0.31	1.06681 –0	0.25
10^{-2}	2.95004 –3	1.13	2.28887 –1	0.67
10^{-3}	7.82848 –5	1.58	6.36429 –2	0.56
10^{-4}	2.15759 –6	1.56	2.00041 –2	0.50
10^{-5}	6.48289 –8	1.52	6.32460 –3	0.50
10^{-6}	2.01584 –9	1.51	2.00000 –3	0.50
10^{-7}	6.34040 –11	1.50	6.32456 –4	0.50
10^{-8}	2.00158 –12	1.50	2.00000 –4	0.50
Theory		1.5		0.5

Table 7: Approximation of $\overline{u} \in C^\infty(\Omega)$, $\overline{u} \notin H^1_0(\Omega)$, for control $z \in H^{-1}(\Omega)$.

ϱ	$\|u_h - \overline{u}\|^2_{H^{-1}(\Omega)}$	eoc	$\|u_h - \overline{u}\|^2_{L_2(\Omega)}$	eoc
1	8.84281 –2		2.05175 –0	
10^{-1}	8.45700 –2	0.02	1.97536 –0	0.02
10^{-2}	5.69295 –2	0.17	1.42743 –0	0.14
10^{-3}	8.42752 –3	0.83	4.49599 –1	0.50
10^{-4}	7.84224 –4	1.03	2.14435 –1	0.32
10^{-5}	1.26548 –4	0.79	1.19310 –1	0.25
10^{-6}	2.23688 –5	0.75	6.70822 –2	0.25
10^{-7}	3.97643 –6	0.75	3.77230 –2	0.25
10^{-8}	7.07108 –7	0.75	2.12132 –2	0.25
Theory		0.75		0.25

Table 8: Approximation of $\overline{u} \in C^\infty(\Omega)$, $\overline{u} \notin H^1_0(\Omega)$, for control $z \in L_2(\Omega)$.

suitable target function $\overline{u} \in H^1_0(\Omega)$ there is no difference in the estimates when considering the control in $L^2(\Omega)$ or in $H^{-1}(\Omega)$, in all other cases we obtain higher order convergence in the relaxation parameter ϱ when considering the control in the energy space $H^{-1}(\Omega)$, which also affects the approximability of the target $\overline{u}$ by the state u_ϱ.

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