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Convergence of adaptive 3D BEM for weakly singular integral equations based on isotropic mesh-refinement

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Abstract

We consider the adaptive lowest-order boundary element method (ABEM) based on isotropic mesh-refinement for the weakly-singular integral equation for the 3D Laplacian. The proposed scheme resolves both, possible singularities of the solution as well as of the given data. The implementation thus only deals with discrete integral operators, i.e. matrices. We prove that the usual adaptive mesh-refining algorithm drives the corresponding error estimator to zero. Under an appropriate saturation assumption which is observed empirically, the sequence of discrete solutions thus tends to the exact solution within the energy norm.

1 Introduction

The $(h - h/2)$-error estimation strategy is a well known technique to derive a-posteriori error estimates for the error $|||\phi - \Phi_\ell|||$ in the natural energy norm; see [HNW] in the context of ordinary differential equations, and the overview article of Bank [B] or the monograph [A] in the context of the finite element method: Let X_ℓ be a discrete subspace of the energy space H and let $\widehat{X}_\ell$ be its uniform refinement. With the corresponding Galerkin solutions Φ_ℓ and $\widehat{\Phi}_\ell$, the canonical $(h - h/2)$-*error estimator*

$$\eta_\ell := |||\widehat{\Phi}_\ell - \Phi_\ell||| \tag{1}$$

is a computable quantity [DLY] which can be used to estimate $|||\phi - \Phi_\ell|||$, where $\phi \in H$ denotes the exact solution.

For finite element methods (FEM), the energy norm, e.g. $|||\cdot||| = \|\nabla_\Gamma(\cdot)\|_{L_2(\Omega)}$ in case of the Poisson-Dirichlet problem, provides local information, which elements of the underlying mesh should be refined to decrease the error effectively. For boundary element methods (BEM), the energy norm $|||\cdot|||$ is (equivalent to) a fractional order (and possibly negative) Sobolev norm and does typically not provide local information directly. Besides the $(h - h/2)$ strategy, several a posteriori error estimators for the efficient numerical treatment of integral equations via adaptive mesh-refining algorithms are available in the literature, see [CF] and the references therein. For example, estimators of residual type are proposed and analyzed in [CS95, CS96, C96, C97, CMS, CMPS], where weighted Sobolev norms of integer order are used to localize norms of fractional order. In [F98, F00, F02], fractional order Sobolev norms are considered on overlapping subsets of a mesh, e.g. node patches. Furthermore, a projection of the residual onto multilevel functions may be used, as proposed in [MS, MSW], or other integral equations can be applied to estimate the error as in [SSt, S00]. Error estimators that account for p or hp-versions are even found in [CFS, H02, HMS01, HMS02]. In [EFGP, FP], localized variants of η_ℓ were introduced for certain weakly-singular and hypersingular integral equations. However, the mentioned list of works is by no means exhaustive.

Recently [FOP], convergence of some $(h - h/2)$-steered adaptive mesh-refinement has been proved for linear model problems in the context of FEM and BEM. In [AFP+], the concept of estimator reduction has been introduced to analyze convergence of anisotropic mesh-refinement steered by $(h - h/2)$-type or averaging-based error estimators for weakly-singular integral equations arising in 3D BEM. However, in [AFP+, FOP] it is assumed that the right-hand side of the integral equation is computed analytically. This assumption is relaxed in [AFGKMP] where, for the mixed boundary value problem with Dirichlet and Neumann conditions in 2D, the resolution of the given data becomes a part of the adaptive loop. In this last work [AFGKMP], the Dirichlet data is assumed to be in H^1, and is, by the Sobolev imbedding theorem, continuous. The approximation can thus be carried out by a nodal interpolant, and approximation estimates from [C97, EFGP] can be used to extract local information on the approximation error. Moreover, for the Neumann data, [AFGKMP] assumes additional regularity, and the Neumann data is then discretized by the L_2-projection onto piecewise constants. Whereas the discretization of the Neumann data transfers directly to 3D BEM, the discretization of the Dirichlet data does not. The reason is that H^1-functions on a 2D manifold (i.e. the boundary of a 3D domain) lack continuity and hence nodal interpolation is not allowed.

The aim of this present work is twofold: First, we extend the one-dimensional local approximation estimates for nodal interpolation from [C97, EFGP] to certain quasi-interpolation operators in two dimensions in Theorem 3. This theorem extends the approximation properties of quasi-interpolation operators on adaptive meshes to positive fractional order Sobolev spaces. It thus enables us to control the error that is induced by the approximation of the given data using appropriate *data oscillation* terms whick look similar to the 2D case of [AFGKMP]. Since we are interested in the local mesh size of adaptively refined meshes, it is not feasible to use the interpolation theorem in an easy way to obtain approximation estimates in fractional order Sobolev spaces, which are, in fact, interpolation spaces. Instead, the essential ingredient is an upper bound for the interpolation norm of weighted H^1-spaces.

To control the discretization error, we use the $(h-h/2)$-type error estimators from [FP]. We show that the sum of data oscillation and error estimator is a means to control energy error and data approximation in an efficient and, under the so-called *saturation assumption*, reliable way. For a Galerkin boundary element method in 3D, we then propose an adaptive algorithm which is steered by the sum of data oscillation and error estimator.

The second aim of this work is to show that the discrete solutions that are generated by the proposed adaptive algorithm converge to the exact solution. This is done by following the concept of *estimator reduction* from [AFP+]. With nothing but notational overhead, this allows to transfer the results of [AFGKMP] from 2D to 3D. Using results from [AFP], convergence of adaptive FEM-BEM coupling driven by $(h-h/2)$ estimators can be derived.

The remainder of this paper is organized as follows: Section 2 introduces the model problems as well as the integral formulations thereof. Then, the main results of this paper are summarized. In Section 3, we collect the preliminaries on Sobolev spaces, boundary discretization, and discrete spaces. Furthermore, we prove approximation properties of H^1-stable projections in fractional-order Sobolev spaces. In Section 4, we introduce the error estimators as well as the data oscillation. The adaptive algorithm is stated in Section 5, where we also recall the newest-vertex-bisection refinement. The main result of this paper is found in Theorem 15, which states the estimator reduction property and hence convergence of the proposed adaptive BEM algorithm. Finally, numerical experiments in Section 6 conclude the work.

2 Model Problem and analytical results

The aim of this section is to introduce the model problem, its integral formulations, and the Galerkin formulations. Afterwards, we give an overview on the main results contained in this work. For all stated properties of the integral operators involved, we refer to the literature, e.g., the monographs [HW, ML, SaS].

2.1. Continuous model problem. The considered model problem is the Laplace equation: For $\Omega \subset \mathbb{R}^3$ an open set with Lipschitz-boundary, it reads

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \Gamma := \partial\Omega, \end{aligned} \tag{2}$$

where the inhomogeneous Dirichlet data $g \in H^{1/2}(\Gamma)$ are given. The problem of finding u in equation (2) is equivalently stated as follows: Find $\phi \in H^{-1/2}(\Gamma)$ such that

$$V\phi = (K + 1/2)g \quad \text{on } \Gamma. \tag{3}$$

Here, V is the simple-layer potential and K is the double-layer potential which are formally defined by

$$V\psi(x) = \int_\Gamma \psi(y) G(x-y)\, d\Gamma(y),$$
$$Kv(x) = \oint_\Gamma v(y) \partial_{n(y)} G(x-y)\, d\Gamma(y),$$

where $G(\cdot)$ denotes the fundamental solution of the 3D Laplacian

$$G(z) = \frac{1}{4\pi} \frac{1}{|z|} \quad \text{for } z \in \mathbb{R}^3 \backslash \{0\}.$$

The solution ϕ of (3) is then the normal derivative of u, i.e., $\phi = \partial_n u$. Conversely, the solution of (3) provides a solution u of (2) by means of the representation formula, see e.g., [HW, Chapter 1.1].

Let $\langle \cdot\,,\cdot \rangle$ denote the L_2-scalar product which is extended to duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Note that $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is an elliptic and symmetric isomorphism between $H^{-1/2}(\Gamma)$ and its dual $H^{1/2}(\Gamma)$. It thus provides a scalar product defined by $\langle\!\langle \phi\,,\psi \rangle\!\rangle = \langle V\phi\,,\psi\rangle$. We denote by $|||\cdot||| := \langle\!\langle \cdot\,,\cdot \rangle\!\rangle^{1/2}$ the induced energy norm which is an equivalent norm on $H^{-1/2}(\Gamma)$. Therefore, the theorem of Riesz-Fischer (resp. the Lax-Milgram lemma) proves the existence and uniqueness of the solution $\phi \in H^{-1/2}(\Gamma)$ of the variational form

$$\langle\!\langle \phi\,,\psi \rangle\!\rangle = \langle (K+1/2)g\,,\psi\rangle \quad \text{for all } \psi \in H^{-1/2}(\Gamma). \tag{4}$$

2.2. Galerkin Discretization. We consider the lowest-order Galerkin discretization of (4) with the discrete space of piecewise constant functions $\mathcal{P}^0(\mathcal{T}_\ell)$. Here, $\mathcal{T}_\ell$ is a regular partition of Γ into triangles. Again, the Riesz-Fischer theorem provides a unique solution of the Galerkin formulation

$$\langle\!\langle \Phi_\ell^\star\,,\Psi_\ell \rangle\!\rangle = \langle (K+1/2)g\,,\Psi_\ell\rangle \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell). \tag{5}$$

The right-hand side in the preceding equation can in principle be computed by methods proposed in [CPS, SaS, S]. However, the right-hand side in (5) involves the double-layer potential operator K. To decouple the singularities of K and possible singularities of g and to enable the direct use of fast methods like hierarchical matrices or the fast multipole method, we additionally approximate g by some appropriate G_ℓ. To that end, we use the L_2-projection $\Pi_\ell : L_2(\Gamma) \to \mathcal{S}^1(\mathcal{T}_\ell)$ and define

$$G_\ell := \Pi_\ell g \in \mathcal{S}^1(\mathcal{T}_\ell),$$

where $\mathcal{S}^1(\mathcal{T}_\ell)$ is the space of piecewise affine, globally continuous functions on $\mathcal{T}_\ell$. Therefore, we restrict ourselves to the use conforming triangulations of Γ. Then, we replace g by G_ℓ and denote by Φ_ℓ the corresponding Galerkin solution of

$$\langle\!\langle \Phi_\ell\,,\Psi_\ell \rangle\!\rangle = \langle (K+1/2)G_\ell\,,\Psi_\ell\rangle \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell). \tag{6}$$

We stress that problem (6) is equivalent to a linear system $\mathbf{V}\mathbf{x} = (\frac{1}{2}\mathbf{M} + \mathbf{K})\mathbf{g}$, where $\mathbf{x}$ and $\mathbf{g}$ are the coefficient vectors of Φ_ℓ and G_ℓ, $\mathbf{V}$ and $\mathbf{K}$ are the Galerkin matrices for the discrete integral operators V and K, and $\mathbf{M}$ is a mass-type matrix.

Remark. *A remarkable advantage of BEM over FEM is its possible high-order approximation of the solution u of* (2) *by means of the represenation formula. For smooth ϕ and lowest-order BEM, one can prove $\mathcal{O}(h^3)$ convergence for the pointwise error within Ω. However, this order is reduced to $\mathcal{O}(h^2)$ when using nodal interpolation to discretize g as in [AFGKMP]. Contrary, discretization of g by L_2-projection preserves $\mathcal{O}(h^3)$, see e.g., [St, Chapter 12.1].* $\square$

2.3. A-posteriori error estimation. The discretization error $|||\phi - \Phi_\ell^\star|||$ is measured using the $(h{-}h/2)$-type error estimators from [FP], where four different error estimators are provided. Lemma 6 recalls results from the latter work: Under the saturation assumption

$$|||\phi - \widehat{\Phi}_\ell^\star||| \le C_{\rm sat}|||\phi - \Phi_\ell^\star||| \quad \text{with some uniform constant } 0 < C_{\rm sat} < 1, \tag{7}$$

the proposed error estimators $\tau_\ell^\star$ provide lower and upper bounds of the Galerkin error,

$$C_{\rm eff}^{-1}\tau_\ell^\star \le |||\phi - \Phi_\ell^\star||| \le C_{\rm rel}\tau_\ell^\star, \tag{8}$$

where only the upper bound hinges on (7), and $C_{\rm eff}, C_{\rm rel} > 0$ depend additionally on the shape regularity of $\mathcal{T}_\ell$ and on Γ. Here, $\tau_\ell^\star$ denotes any of the proposed error estimators, and $\widehat{\Phi}_\ell^\star$ is the Galerkin solution of (5) with respect to the uniform refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$. Throughout, the upper index $\star$ denotes quantities wich are only of theoretical interest, but are *not* computed numerically. In a second step, we include the data approximation error, which stems from approximating the right-hand side g by G_ℓ. Under additional regularity $g \in H^1(\Gamma)$, we introduce some numerically computable data oscillation term $\mathrm{osc}_\ell := \|h_\ell^{1/2}\nabla_\Gamma(g - G_\ell)\|_{L_2(\Gamma)}$ which measures the local error of the data approximation $\|g - G_\ell\|_{H^{1/2}(\Gamma)}$. Then, the proposed error estimators τ_ℓ — now computed with respect to approximated Dirichlet data — fulfill equation (8) up to oscillation terms, i.e. it holds that

$$C_{\rm eff}^{-1}\tau_\ell \le |||\phi - \Phi_\ell||| + \mathrm{osc}_\ell \quad \text{and} \quad |||\phi - \Phi_\ell||| + \mathrm{osc}_\ell \le C_{\rm rel}\tau_\ell. \tag{9}$$

Throughout the work, the symbol $\lesssim$ abbreviates $\le$ up to a multiplicative constant which may only depend on shape regularity of the mesh $\mathcal{T}_\ell$. Moreover, $\simeq$ abbreviates the inequalities $\lesssim$ and $\gtrsim$.

Remark. *The saturation assumption* (7) *mainly states that the adaptive scheme has reached an asymptotic phase [FP, Section 5.2]. However, it may fail to hold in general, as was shown in [DN]. However, in [DN] it was shown that* (7) *holds if the data is sufficiently resolved, i.e., in the context of FEM there holds*

$$|||\phi - \widehat{\Phi}_\ell^\star||| \le C_{\rm sat}|||\phi - \Phi_\ell^\star||| + \mathrm{osc}_\ell,$$

so that small data oscillation osc_ℓ *implies the saturation assumption. This observation is another reason for the inclusion of the data oscillation into the adaptive algorithm. We stress that the saturation assumption is usually observed in experiments. A reason for this might be that we have to ensure* (7) *only for the sequence of meshes that is generated by the numerical algorithm.* $\square$

2.4. Adaptive mesh-refining algorithm. In Section 5, we introduce an adaptive mesh-refining algorithm which is steered by the local contributions of the error estimator

$$\gamma_\ell = \left(\widetilde{\mu}_\ell^2 + \mathrm{osc}_\ell^2\right)^{1/2}.$$

Here, $\widetilde{\mu}_\ell$ is a localized variant of the $(h - h/2)$-based error estimator from (1). Theorem 15 then guarantees that the adaptive algorithm leads to

$$\lim_{\ell \to \infty} \gamma_\ell = 0.$$

According to (9), the saturation assumption (7) for the non-perturbed problem thus yields convergence of the discrete Galerkin solutions Φ_ℓ to the exact solution ϕ.

3 Preliminairies

3.1. Sobolev spaces on the boundary. We assume throughout that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain. The usual Sobolev spaces are denoted by $L_2(\Omega)$ and $H^1(\Omega)$. Sobolev spaces with noninteger order are defined by use of the Sobolev-Slobodeckij seminorm. Sobolev spaces on the boundary $\Gamma := \partial\Omega$ are defined likewise by using a parametrization of Γ as a two-dimensional manifold, see [SaS]. Equivalently, noninteger order spaces can be defined as interpolation spaces. We use the K-method of interpolation, see [T], to that end. We stress that the definition of a noninteger order Sobolev space as interpolation space yields the same set of functions, but an equivalent norm. However, norm equivalence constants depend on the boundary Γ.

Furthermore, we will use weighted Sobolev spaces. For a weight function $w \in L_\infty(\Gamma)$ with $w > 0$ almost everywhere, we denote by $H^1(\Gamma, w)$ the space of all functions $u \in H^1(\Gamma)$ equipped with the norm

$$\|u\|_{H^1(\Gamma,w)}^2 := \|wu\|_{L_2(\Gamma)}^2 + \|w\nabla_\Gamma u\|_{L_2(\Gamma)}^2.$$

Here, ∇_Γ denotes the surface gradient. According to the properties of w, it holds $H^1(\Gamma, w) = H^1(\Gamma)$, but with a different norm. Throughout, we abbreviate the notation and write, e.g., H^s or $\|\cdot\|_{L_2}$ instead of $H^s(\Gamma)$ and $\|\cdot\|_{L_2(\Gamma)}$, respectively.

3.2. Discrete spaces. A mesh $\mathcal{T}_\ell$ on the boundary $\Gamma = \partial\Omega$ consists of flat triangles which are denoted by T. We assume that $\mathcal{T}_\ell$ is regular, i.e., it contains no hanging nodes.

Associated to every element T is an affine element map $F_T : \widehat{T} \to T$, where $\widehat{T}$ is a reference element. The volume area $|\cdot|$ defines the local mesh-width $h_\ell \in L_\infty$ by $h_\ell|_T := h_\ell(T) := |T|^{1/2}$, whereas $\rho_\ell(T)$ is the diameter of the largest ball that can be inscribed in T. A sequence of meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ is called *quasi-uniform* if there are global discretization parameters h_ℓ and ρ_ℓ such that $h_\ell(T) \simeq h_\ell$ and $\rho_\ell(T) \simeq \rho_\ell$ for all $T \in \mathcal{T}_\ell$. We call a sequence of meshes *locally quasi-uniform*, if the mesh-size $h_\ell(T)$ as well as the parameter $\rho_\ell(T)$ are comparable on adjacent elements, i.e. $h_\ell(T) \simeq h_\ell(T')$ and $\rho_\ell(T) \simeq \rho_\ell(T')$ for all $T, T' \in \mathcal{T}_\ell$ with $T \cap T' \neq \emptyset$. We define the so-called shape-regularity constant by

$$\sigma(\mathcal{T}_\ell) := \max_{T\in\mathcal{T}_\ell} \frac{h_\ell(T)}{\rho_\ell(T)}.$$

For $\gamma > 0$, we call a sequence of meshes γ-shape-regular if $\sigma(\mathcal{T}_\ell)$ is bounded uniformly by γ, i.e. $\sup_{\ell\in\mathbb{N}} \sigma(\mathcal{T}_\ell) \le \gamma$. In this context, we will also call a single mesh quasi-uniform, locally quasi-uniform, or γ-shape-regular. Finally, γ-shape-regular meshes are often referred to as isotropic meshes. We define $\mathcal{N}_\ell$ to be the set of nodes of a mesh $\mathcal{T}_\ell$. For $p \in \mathbb{N}_0$, polynomial spaces on the reference element are denoted by

$$\mathcal{P}^p(\widehat{T}) := \operatorname{span}\big\{x^i y^k \,:\, 0 \le i + k \le p\big\}.$$

Spaces of piecewise polynomials are denoted by

$$\begin{aligned}
\mathcal{P}^p(\mathcal{T}_\ell) &:= \big\{u \in L_\infty(\Gamma) \,:\, u \circ F_T \in \mathcal{P}^p(\widehat{T}) \text{ for all } T \in \mathcal{T}_\ell\big\},\\
\mathcal{S}^p(\mathcal{T}_\ell) &:= \mathcal{P}^p(\mathcal{T}_\ell) \cap C^0(\Gamma).
\end{aligned}$$

3.3. The L_2-projection. The L_2-projections $\pi_\ell : L_2 \to \mathcal{P}^0(\mathcal{T}_\ell)$ and $\Pi_\ell : L_2 \to \mathcal{S}^1(\mathcal{T}_\ell)$ are defined by

$$\begin{aligned}
(\pi_\ell \psi \,, \Psi_\ell)_{L_2} &= (\psi \,, \Psi_\ell)_{L_2} \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell),\\
(\Pi_\ell u \,, U_\ell)_{L_2} &= (u \,, U_\ell)_{L_2} \quad \text{for all } U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell).
\end{aligned}$$

Obviously, Π_ℓ is bounded in L_2, i.e.,

$$\|\Pi_\ell u\|_{L_2} \le \|u\|_{L_2}.$$

Since $\mathcal{S}^1(\mathcal{T}_\ell) \subset H^1$, one can ask if Π_ℓ is also stable in H^1, i.e.,

$$\|\Pi_\ell u\|_{H^1} \le C \|u\|_{H^1} \quad \text{for all } u \in H^1. \tag{10}$$

Since the L_2-norm and H^1-norm are equivalent on $\mathcal{S}^1(\mathcal{T}_\ell)$, the stability estimate (10) clearly holds with a constant $C > 0$ which a-priori depends on $\mathcal{T}_\ell$.

For quasi-uniform meshes $\mathcal{T}_\ell$, it is possible to show that C is independent of $\mathcal{T}_\ell$, see [BX]. We can exploit any Clément-type quasi-interpolation operator J_ℓ, e.g. the original operator from [Cl], to see

$$\begin{aligned}
\|\nabla_\Gamma \Pi_\ell u\|_{L_2} &\le \|\nabla_\Gamma(\Pi_\ell u - J_\ell u)\|_{L_2} + \|\nabla_\Gamma J_\ell u\|_{L_2}\\
&\lesssim h_\ell^{-1} \|\Pi_\ell u - J_\ell u\|_{L_2} + \|\nabla_\Gamma u\|_{L_2},
\end{aligned}$$

where we use the stability properties of J_ℓ and an inverse inequality. Here, h_ℓ is the global discretization parameter. Since Π_ℓ is a projection, we conclude

$$\|\Pi_\ell u - J_\ell u\|_{L_2} = \|\Pi_\ell(u - J_\ell u)\|_{L_2} \leq \|u - J_\ell u\|_{L_2} \lesssim h_\ell \|\nabla_\Gamma u\|_{L_2}.$$

Now, (10) follows, and the constant C does not depend on the sequence of meshes $\mathcal{T}_\ell$, but only on the quasi-uniformity constants.

Since this work is concerned with adaptive meshes, it is of interest to obtain (10) with a constant C which does not depend on the actual step of the adaptive algorithm even for locally quasi-uniform meshes. We then say that the L_2-projections Π_ℓ are uniformly stable in H^1. Likewise, we say that the L_2-projection is uniformly stable in H^β for $\frac{1}{2} \leq \beta \leq 1$. There are several works to this question, see [BPS, C01, C02, S01]. The analysis in this work does not only assume $X_{\ell+1} \subseteq X_\ell$, but also $\widehat{X}_{\ell+1} \subseteq \widehat{X}_\ell$, where the X_ℓ are the discrete spaces generated by the algorithm and the $\widehat{X}_\ell$ are the discrete spaces on the uniformly refined meshes. To ensure this, will use the classical Newest-Vertex-Bisection algorithm for local mesh-refinement. None of the above mentioned works can be used in this case, but a result of the recent work [KPP] states the H^1-stability of Π_ℓ.

Throughout, we will assume that we are dealing with a sequence of uniformly shape-regular meshes $\mathcal{T}_\ell$ which are obtained by successive refinement, i.e. $\mathcal{P}^0(\mathcal{T}_\ell) \subseteq \mathcal{P}^0(\mathcal{T}_{\ell+1})$.

3.4. Local inverse estimates in $\boldsymbol{H^{-1/2}}$ and $\boldsymbol{H^{1/2}}$. We will need an inverse estimate in $H^{-1/2}$ which even holds for quasi-uniform K-meshes, see [GHS, Theorem 3.6].

Lemma 1. *It holds that*

$$\|h_\ell^{1/2}\Psi_\ell\|_{L_2} \leq C_1 \|\Psi_\ell\|_{H^{-1/2}} \quad \textit{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell). \tag{11}$$

The constant $C_1 > 0$ depends solely on an upper bound of $\sigma(\mathcal{T}_\ell)$ and on Γ. □

Moreover, we shall need an inverse estimate in $H^{1/2}$, see [AKP].

Lemma 2. *It holds that*

$$\|h_\ell^{1/2}\nabla_\Gamma U_\ell\|_{L_2} \leq C_2 \|U_\ell\|_{H^{1/2}} \quad \textit{for all } U_\ell \in \mathcal{S}^1(\mathcal{T}_\ell). \tag{12}$$

The constant $C_2 > 0$ depends solely on Γ. □

3.5. Local approximation estimate in $\boldsymbol{H^{1/2}}$. In this section, we prove an approximation estimate in $H^{1/2}$, where —as in the prior Section 3.4— the emphasis is laid on the fact that the right-hand side involves the local mesh-size $h_\ell \in L_\infty$.

Theorem 3. *For each continuous projection $\mathbb{P}_\ell : H^\alpha \to \mathcal{S}^1(\mathcal{T}_\ell)$, it holds that*

$$\|(1-\mathbb{P}_\ell)g\|_{H^\alpha} \leq C_3 \min\left\{\|h_\ell^{1-\alpha}\nabla_\Gamma g\|_{L_2}, \|h_\ell^{1-\alpha}\nabla_\Gamma(1-\mathbb{P}_\ell)g\|_{L_2}\right\} \tag{13}$$

for all $g \in H^1$. The constant $C_3 > 0$ depends solely on $0 \leq \alpha < 1$, the γ-shape-regularity of $\mathcal{T}_\ell$, the operator norm of $\mathbb{P}_\ell : H^\alpha \to H^\alpha$, and on the boundary Γ.

Before we develop the proof of (13), we first state two possible choices for $\mathbb{P}_\ell$ in the following two remarks.

Remark. *The Scott-Zhang projection $\mathbb{P}_\ell$ from [SZ] can be defined in a way such that it is stable in both L_2 and H^1. By interpolation arguments, $\mathbb{P}_\ell$ then is also stable in H^α for $0 \le \alpha \le 1$, and its operator norm does depend solely on $\sigma(\mathcal{T}_\ell)$.* $\square$

Remark. *Suppose that the mesh-refinement ensures that the L_2-projection Π_ℓ onto $\mathcal{S}^1(\mathcal{T}_\ell)$ is uniformly H^β stable for some $0 < \beta < 1$ and that the stability estimate depends only on $\sigma(\mathcal{T}_\ell)$, e.g. newest vertex bisection is used throughout, cf. Section 5.1. By interpolation arguments, $\mathbb{P}_\ell = \Pi_\ell$ then is also stable in H^α for $0 \le \alpha < \beta$, and its operator norm does depend solely on $\sigma(\mathcal{T}_\ell)$.* $\square$

For the proof of Theorem 3, we define the locally averaged mesh-size function $\widetilde{h}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ by

$$\widetilde{h}_\ell(z) = \max\big\{h_\ell(T) \,:\, T \in \mathcal{T}_\ell \text{ with } z \in T\big\} \tag{14}$$

for all nodes $z \in \mathcal{N}_\ell$. According to uniform shape-regularity, $\widetilde{h}_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$ then is locally equivalent to the local mesh-size $h_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$.

Lemma 4. *It holds pointwise on Γ that*

$$C_5^{-1}\,\widetilde{h}_\ell \le h_\ell \le C_5\,\widetilde{h}_\ell \quad \textit{as well as} \quad C_6^{-1} \le |\nabla_\Gamma \widetilde{h}_\ell| \le C_6. \tag{15}$$

The constants $C_5, C_6 > 0$ depend only on the γ-shape-regularity of $\mathcal{T}_\ell$. $\square$

The heart of the proof of the approximation estimate (13) is the following estimate for the interpolation norm of $H^1(\widetilde{h}_\ell)$ and H^1.

Lemma 5. *For $u \in H^1$ and all $0 < \theta < 1$, it holds that*

$$C_4^{-1}\,\|u\|_{[H^1(\widetilde{h}_\ell),H^1]_\theta} \le \|\widetilde{h}_\ell^{1-\theta}\nabla_\Gamma u\|_{L_2} + \|\widetilde{h}_\ell^{-\theta} u\|_{L_2}. \tag{16}$$

The constants $C_4 > 0$ depends solely on θ and the surface area $|\Gamma|$ of Γ.

Proof. The interpolation norm of u is given by

$$\|u\|^2_{[H^1(\widetilde{h}_\ell),H^1]_\theta} = \int_0^\infty t^{-2\theta} K(t,u)^2 \frac{dt}{t}.$$

We first note that

$$K(t,u)^2 = \Big(\inf_{u=u_h+u_1} \|u_h\|_{H^1(\widetilde{h}_\ell)} + t\|u_1\|_{H^1}\Big)^2 \simeq \inf_{u=u_h+u_1} \|u_h\|^2_{H^1(\widetilde{h}_\ell)} + t^2\|u_1\|^2_{H^1} =: K_2(t,u)^2,$$

where the infimum is taken over all $u_h, u_1 \in H^1(\widetilde{h}_\ell) = H^1$. Therefore, we may consider K_2 instead of K. We proceed as for the interpolation of weighted L_2-spaces [T, Section 23] and choose the decomposition

$$u(x) = u_h(x) + u_1(x) := \psi(x)u(x) + \big(1 - \psi(x)\big)\, u(x), \quad \text{where} \quad \psi(x) := \frac{t^2}{\widetilde{h}_\ell^2(x) + t^2}.$$

We then have

$$\begin{aligned} K_2(t,u)^2 &\le \int_\Gamma |\psi u|^2 \widetilde{h}_\ell^2 + |\nabla_\Gamma(\psi u)|^2 \widetilde{h}_\ell^2 + t^2|(1-\psi)u|^2 + t^2 |\nabla_\Gamma\left((1-\psi)\,u\right)|^2\, d\Gamma \\ &= \int_\Gamma |\psi u|^2 \widetilde{h}_\ell^2 + t^2|(1-\psi)u|^2\, d\Gamma + \int_\Gamma |\nabla_\Gamma(\psi u)|^2 \widetilde{h}_\ell^2 + t^2 |\nabla_\Gamma\left((1-\psi)\,u\right)|^2\, d\Gamma. \end{aligned} \tag{17}$$

For the integrand of the second integral in (17), we compute

$$\begin{aligned} |\nabla_\Gamma(\psi u)|^2 \widetilde{h}_\ell^2 + t^2 |\nabla_\Gamma\left((1-\psi)\,u\right)|^2 &= |u\nabla_\Gamma \psi + \psi \nabla_\Gamma u|^2 \widetilde{h}_\ell^2 + t^2 |u \nabla_\Gamma(1-\psi) + (1-\psi)\nabla_\Gamma u|^2 \\ &= \widetilde{h}_\ell^2 |u\nabla_\Gamma\psi|^2 + t^2u^2|\nabla_\Gamma(1-\psi)|^2 + \widetilde{h}_\ell^2|\psi\nabla_\Gamma u|^2 + t^2|(1-\psi)\nabla_\Gamma u|^2 \\ &\quad + 2\widetilde{h}_\ell^2 u\psi\nabla_\Gamma\psi\cdot\nabla_\Gamma u + 2t^2u(1-\psi)\nabla_\Gamma(1-\psi)\cdot\nabla_\Gamma u. \end{aligned}$$

The last line in the preceding equation adds up to zero. This is seen from

$$2\widetilde{h}_\ell^2 u\psi\nabla_\Gamma\psi\cdot\nabla_\Gamma u + 2t^2u(1-\psi)\nabla_\Gamma(1-\psi)\cdot\nabla_\Gamma u = 2u(\widetilde{h}_\ell^2\psi - t^2(1-\psi))\nabla_\Gamma\psi\cdot\nabla_\Gamma u$$

and $\widetilde{h}_\ell^2\psi - t^2(1-\psi) = 0$ by definition of ψ. Therefore, (17) becomes

$$\begin{aligned} K_2(t,u)^2 \le \int_\Gamma |\psi u|^2\widetilde{h}_\ell^2 + t^2|(1-\psi)u|^2\, d\Gamma &+ \int_\Gamma \widetilde{h}_\ell^2|u\nabla_\Gamma\psi|^2 + t^2u^2|\nabla_\Gamma(1-\psi)|^2\, d\Gamma \\ &+ \int_\Gamma \widetilde{h}_\ell^2|\psi\nabla_\Gamma u|^2 + t^2|(1-\psi)\nabla_\Gamma u|^2\, d\Gamma, \end{aligned}$$

whence

$$\begin{aligned} \|u\|^2_{[H^1(\widetilde{h}_\ell),H^1]_\theta} \lesssim &\int_0^\infty t^{-2\theta} \int_\Gamma |\psi u|^2\widetilde{h}_\ell^2 + t^2|(1-\psi)u|^2\, d\Gamma\, \frac{dt}{t} \\ &+ \int_0^\infty t^{-2\theta}\int_\Gamma \widetilde{h}_\ell^2|u\nabla_\Gamma\psi|^2 + t^2u^2|\nabla_\Gamma(1-\psi)|^2\, d\Gamma\, \frac{dt}{t} \\ &+ \int_0^\infty t^{-2\theta}\int_\Gamma \widetilde{h}_\ell^2|\psi\nabla_\Gamma u|^2 + t^2|(1-\psi)\nabla_\Gamma u|^2\, d\Gamma\, \frac{dt}{t}. \end{aligned} \tag{18}$$

Let us compute the three parts separately. Together with the identity

$$|\psi u|^2\widetilde{h}_\ell^2 + t^2|(1-\psi)u|^2 = u^2(t^2\widetilde{h}_\ell^2)/(\widetilde{h}_\ell^2 + t^2),$$

substitution $t = s\widetilde{h}_\ell$, and Fubini's theorem, the first double integral becomes

$$\begin{aligned}\int_0^\infty t^{-2\theta} \int_\Gamma |\psi u|^2 \widetilde{h}_\ell^2 + t^2 |(1-\psi)u|^2 \, d\Gamma \, \frac{dt}{t} &= \int_\Gamma u^2 \int_0^\infty t^{-2\theta} \frac{t^2 \widetilde{h}_\ell^2}{\widetilde{h}_\ell^2 + t^2} \frac{dt}{t} \, d\Gamma \\ &= \Big(\int_0^\infty \frac{s^{-2\theta+1}}{s^2+1} ds \Big) \Big(\int_\Gamma u^2 \widetilde{h}_\ell^{-2\theta+2} \, d\Gamma \Big).\end{aligned}$$

Exactly the same arguments apply for the third double integral in (18) and show

$$\int_0^\infty t^{-2\theta} \int_\Gamma \widetilde{h}_\ell^2 |\psi \nabla_\Gamma u|^2 + t^2 |(1-\psi)\nabla_\Gamma u|^2 \, d\Gamma \, \frac{dt}{t} = \Big(\int_0^\infty \frac{s^{-2\theta+1}}{s^2+1} ds \Big) \Big(\int_\Gamma |\nabla_\Gamma u|^2 \widetilde{h}_\ell^{-2\theta+2} \, d\Gamma \Big).$$

Let us now compute the second double integral in (18). Using the identity

$$-\nabla_\Gamma(1-\psi) = \nabla_\Gamma \psi = -\frac{2t^2 \widetilde{h}_\ell \nabla_\Gamma \widetilde{h}_\ell}{(\widetilde{h}_\ell^2 + t^2)^2},$$

the substitution $t = s\widetilde{h}_\ell$, and Fubini's theorem, we see

$$\begin{aligned}\int_0^\infty t^{-2\theta} \int_\Gamma \widetilde{h}_\ell^2 |u \nabla_\Gamma \psi|^2 + t^2 u^2 |\nabla_\Gamma(1-\psi)|^2 \, d\Gamma \, \frac{dt}{t} &= \int_\Gamma u^2 \int_0^\infty t^{-2\theta} \frac{4t^4 \widetilde{h}_\ell^2 |\nabla_\Gamma \widetilde{h}_\ell|^2}{(\widetilde{h}_\ell^2 + t^2)^3} \frac{dt}{t} \, d\Gamma \\ &= 4 \Big(\int_0^\infty \frac{s^{-2\theta+3}}{(s^2+1)^3} ds \Big) \Big(\int_\Gamma u^2 \widetilde{h}_\ell^{-2\theta} |\nabla_\Gamma \widetilde{h}_\ell|^2 \, d\Gamma \Big).\end{aligned}$$

Finally, recall that $|\nabla_\Gamma \widetilde{h}_\ell| \simeq 1$ from Lemma 4. Using the estimates for the three parts on the right-hand side of (18), we thus arrive at

$$\|u\|_{[H^1(\widetilde{h}_\ell), H^1]_\theta} \lesssim \|\widetilde{h}_\ell^{1-\theta} u\|_{L_2} + \|\widetilde{h}_\ell^{1-\theta} \nabla_\Gamma u\|_{L_2} + \|\widetilde{h}_\ell^{-\theta} u\|_{L_2}.$$

Now, $\widetilde{h}_\ell \lesssim 1$ concludes the proof of (16). $\square$

Proof of Theorem 3. Let J_ℓ denote an arbitrary Clément-type quasi-interpolation operator which satisfies a local first-order approximation property

$$\|h_\ell^\beta (1 - J_\ell) g\|_{L_2} \lesssim \|h_\ell^{1+\beta} \nabla_\Gamma g\|_{L_2} \tag{19}$$

as well as local stability in H^1

$$\|h_\ell^\beta \nabla_\Gamma (1 - J_\ell) g\|_{L_2} \lesssim \|h_\ell^\beta \nabla_\Gamma g\|_{L_2}, \tag{20}$$

for all $g \in H^1$ and all $\beta \in \mathbb{R}$. A valid choice is, e.g., the Scott-Zhang projection from [SZ]. For this choice, the constants in (19)–(20) depend only on the γ-shape-regularity of $\mathcal{T}_\ell$. For $\beta = 0$, the estimates (19)–(20) and $h_\ell \simeq \widetilde{h}_\ell$ give

$$\|(1 - J_\ell) g\|_{L_2} \lesssim \|g\|_{H^1(\widetilde{h}_\ell)} \quad \text{as well as} \quad \|(1 - J_\ell) g\|_{H^1} \lesssim \|g\|_{H^1}$$

for all $g \in H^1 = H^1(\widetilde{h}_\ell)$. Using the interpolation theorem for the operator $(1 - J_\ell) : [H^1(\widetilde{h}_\ell), H^1]_\alpha \to H^\alpha$ and Lemma 5, we see

$$\|(1 - J_\ell)g\|_{H^\alpha} \lesssim \|g\|_{[H^1(\widetilde{h}_\ell),H^1]_\alpha} \lesssim \|\widetilde{h}_\ell^{1-\alpha}\nabla_\Gamma g\|_{L_2} + \|\widetilde{h}_\ell^{-\alpha} g\|_{L_2}$$

for all $g \in H^1$. Moreover, the projection property $\mathbb{P}_\ell J_\ell g = J_\ell g$ and stability of $\mathbb{P}_\ell$ yield

$$\|(1 - \mathbb{P}_\ell)g\|_{H^\alpha} \le \|(1 - J_\ell)g\|_{H^\alpha} + \|\mathbb{P}_\ell(1 - J_\ell)g\|_{H^\alpha} \lesssim \|(1 - J_\ell)g\|_{H^\alpha}$$

for all $g \in H^1$. Combining the last two estimates and using the identity $(1 - \mathbb{P}_\ell)(1 - J_\ell) = (1 - \mathbb{P}_\ell)$ and $(1 - J_\ell)g \in H^1$, we may bootstrap this results to see

$$\|(1 - \mathbb{P}_\ell)g\|_{H^\alpha} = \|(1 - \mathbb{P}_\ell)(1 - J_\ell)g\|_{H^\alpha} \lesssim \|\widetilde{h}_\ell^{1-\alpha}\nabla_\Gamma(1 - J_\ell)g\|_{L_2} + \|\widetilde{h}_\ell^{-\alpha}(1 - J_\ell)g\|_{L_2}$$

for all $g \in H^1$. By use of $h_\ell \simeq \widetilde{h}_\ell$ and the estimates (19)–(20), we thus arrive at

$$\|(1 - \mathbb{P}_\ell)g\|_{H^\alpha} \lesssim \|\widetilde{h}_\ell^{1-\alpha}\nabla_\Gamma(1 - J_\ell)g\|_{L_2} + \|\widetilde{h}_\ell^{-\alpha}(1 - J_\ell)g\|_{L_2} \lesssim \|\widetilde{h}_\ell^{1-\alpha}\nabla_\Gamma g\|_{L_2}.$$

In addition, we now may bootstrap this estimate via $(1-\mathbb{P}_\ell)^2 = (1-\mathbb{P}_\ell)$ and $(1-\mathbb{P}_\ell)g \in H^1$ to see

$$\|(1 - \mathbb{P}_\ell)g\|_{H^\alpha} = \|(1 - \mathbb{P}_\ell)(1 - \mathbb{P}_\ell)g\|_{H^\alpha} \lesssim \|\widetilde{h}_\ell^{1-\alpha}\nabla_\Gamma(1 - \mathbb{P}_\ell)g\|_{L_2}.$$

Altogether, the combination of the last two estimates concludes the proof. ☐

4 A-posteriori error estimation

Recall that $\widehat{\Phi}_\ell^\star$ is the Galerkin solution (5) with respect to the uniform refinement $\widehat{\mathcal{T}}_\ell$ of $\mathcal{T}_\ell$. Likewise, $\widehat{\Phi}_\ell$ is the Galerkin solution (6) with respect to $\widehat{\mathcal{T}}_\ell$, computed from the same right-hand side as Φ_ℓ, i.e.:

$$\begin{aligned}\langle\!\langle \Phi_\ell, \Psi_\ell \rangle\!\rangle &= \langle (K + 1/2)G_\ell, \Psi_\ell \rangle \quad \text{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell),\\ \langle\!\langle \widehat{\Phi}_\ell, \widehat{\Psi}_\ell \rangle\!\rangle &= \langle (K + 1/2)G_\ell, \widehat{\Psi}_\ell \rangle \quad \text{for all } \widehat{\Psi}_\ell \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell).\end{aligned} \tag{21}$$

We recall the following two results from [FP, Proposition 1.1]:

Lemma 6. *With $\eta_\ell^\star := |||\widehat{\Phi}_\ell^\star - \Phi_\ell^\star|||$, it holds that*

$$\eta_\ell^\star \le |||\phi - \Phi_\ell^\star|||. \tag{22}$$

Moreover, the estimate

$$|||\phi - \Phi_\ell^\star||| \le C_{\mathrm{rel}}\eta_\ell^\star \tag{23}$$

is equivalent to the saturation assumption (7) *with $C_{\mathrm{sat}} = \big(1 - C_{\mathrm{rel}}^{-2}\big)^{1/2}$.* ☐

Lemma 7. *The following four a-posteriori error estimators*

$$\begin{array}{rclrcl} \eta_\ell &:=& |||\widehat{\Phi}_\ell - \Phi_\ell||| & \widetilde{\eta}_\ell &:=& |||(1-\pi_\ell)\widehat{\Phi}_\ell||| \\ \mu_\ell &:=& \|h_\ell^{1/2}(\widehat{\Phi}_\ell - \Phi_\ell)\|_{L_2} & \widetilde{\mu}_\ell &:=& \|h_\ell^{1/2}(1-\pi_\ell)\widehat{\Phi}_\ell\|_{L_2}. \end{array} \tag{24}$$

satisfy the equivalence estimates

$$\eta_\ell \le \widetilde{\eta}_\ell \le C_7\sigma(\mathcal{T}_\ell)\widetilde{\mu}_\ell \quad \textit{and} \quad \widetilde{\mu}_\ell \le \mu_\ell \le C_8\eta_\ell. \tag{25}$$

The constants $C_7, C_8 > 0$ depend solely on Γ. □

We are now in position to prove estimate (9).

Theorem 8. *Assume that the sequence of meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ allows for a sequence of L_2-projections Π_ℓ which is uniformly H^β-stable for some $\beta > 1/2$. Define the data oscillations by*

$$\mathrm{osc}_\ell := \|h_\ell^{1/2}\nabla_\Gamma(1-\Pi_\ell)g\|_{L_2}.$$

Let $\tau_\ell \in \{\eta_\ell, \widetilde{\eta}_\ell, \mu_\ell, \widetilde{\mu}_\ell\}$. Then, there is a constant $C_9 > 0$ which depends solely on Γ and γ-shape-regularity of $\mathcal{T}_\ell$, such that we have efficiency

$$C_9^{-1}\tau_\ell \le |||\phi - \Phi_\ell||| + \mathrm{osc}_\ell. \tag{26}$$

Under the saturation assumption (7)*, we have reliability*

$$C_{10}^{-1}|||\phi - \Phi_\ell||| \le \tau_\ell + \mathrm{osc}_\ell, \tag{27}$$

where $C_{10} > 0$ depends solely on Γ, γ, and C_{sat}.

Proof. The proof follows along the lines of [AFGKMP] but is now transferred to 3D. By Lemma 7, it suffices to consider $\tau_\ell = \eta_\ell$. According to the best-approximation property of the Galerkin scheme, we have $|||\phi - \Phi_\ell^\star||| \le |||\phi - \Phi_\ell|||$ as well as $|||\widehat{\Phi}_\ell^\star - \Phi_\ell^\star||| \le |||\widehat{\Phi}_\ell^\star - \Phi_\ell|||$ respectively $|||\widehat{\Phi}_\ell - \Phi_\ell||| \le |||\widehat{\Phi}_\ell - \Phi_\ell^\star|||$. Using the triangle inequality and Lemma 6, we obtain

$$\begin{aligned} \eta_\ell = |||\Phi_\ell - \widehat{\Phi}_\ell||| &\le \eta_\ell^\star + |||\widehat{\Phi}_\ell^\star - \widehat{\Phi}_\ell||| \\ &\le |||\phi - \Phi_\ell^\star||| + |||\widehat{\Phi}_\ell^\star - \widehat{\Phi}_\ell||| \\ &\le |||\phi - \Phi_\ell||| + |||\widehat{\Phi}_\ell^\star - \widehat{\Phi}_\ell||| \\ &\lesssim |||\phi - \Phi_\ell||| + \|(K + \tfrac{1}{2})(g - G_\ell)\|_{H^{1/2}} \\ &\lesssim |||\phi - \Phi_\ell||| + \|g - G_\ell\|_{H^{1/2}}, \end{aligned}$$

where the first $\lesssim$ estimate follows from stability of Galerkin schemes. Finally, Theorem 3 applied for $\mathbb{P}_\ell = \Pi_\ell$ and $\alpha = 1/2$ shows estimate (26). In the same manner, the triangle inequality and Lemma 6 together with the saturation assumption (7) show

$$\begin{aligned} |||\phi - \Phi_\ell||| &\le |||\phi - \Phi_\ell^\star||| + |||\Phi_\ell^\star - \Phi_\ell||| \\ &\lesssim \eta_\ell^\star + |||\Phi_\ell^\star - \Phi_\ell||| \\ &\le \eta_\ell + |||\Phi_\ell - \Phi_\ell^\star||| + |||\widehat{\Phi}_\ell - \widehat{\Phi}_\ell^\star||| \\ &\lesssim \eta_\ell + \|g - G_\ell\|_{H^{1/2}}. \end{aligned}$$

Another application of Theorem 3 shows the desired result. □

Remark. *Theorem 8 also holds if Φ_ℓ and $\widehat{\Phi}_\ell$ are computed with different right-hand sides, namely*

$$\begin{aligned} \langle\!\langle \Phi_\ell\,, \Psi_\ell \rangle\!\rangle &= \langle (K+1/2)G_\ell\,, \Psi_\ell\rangle \quad \textit{for all } \Psi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell) \\ \langle\!\langle \widehat{\Phi}_\ell\,, \widehat{\Psi}_\ell \rangle\!\rangle &= \langle (K+1/2)\widehat{G}_\ell\,, \widehat{\Psi}_\ell\rangle \quad \textit{for all } \widehat{\Psi}_\ell \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell) \end{aligned}$$

To see this, note that

$$\|g - \widehat{G}_\ell\|_{H^{1/2}(\Gamma)} \le \|g - G_\ell\|_{H^{1/2}(\Gamma)} + \|G_\ell - \widehat{G}_\ell\|_{H^{1/2}(\Gamma)} \lesssim \|g - G_\ell\|_{H^{1/2}(\Gamma)},$$

since

$$\|G_\ell - \widehat{G}_\ell\|_{H^{1/2}(\Gamma)} = \|\widehat{\Pi}_\ell(G_\ell - g)\|_{H^{1/2}(\Gamma)} \lesssim \|g - G_\ell\|_{H^{1/2}(\Gamma)}.$$

□

5 Adaptive Mesh-refining algorithm

In this chapter, we introduce the adaptive algorithm. For the local mesh-refinement, we use the newest-vertex bisection algorithm see e.g. [V, Chapter 4] as well as Figure 1. The properties of the resulting mesh-refinement are collected in Section 5.1. Section 5.2 deals with the a-priori convergence of computed discrete solutions as well as the a-priori convergence of the approximated data. In Section 5.3, we finally state the adaptive algorithm. The main result of this work is stated in Theorem 15: It states that the overall error estimator γ_ℓ is contractive up to the norm of the difference of two successive Galerkin solutions and the difference of two successive data approximations. Together with the a-priori convergence results from Section 5.2, we obtain that the adaptive algorithm drives the overall error estimator to zero. Under the saturation assumption (7), Theorem 8 finally yields convergence of the computed discrete solutions towards the exact solution.

5.1. Newest Vertex Bisection. The newest vertex bisection algorithm is an edge-based refinement algorithm. For a given initial mesh $\mathcal{T}_0$, one choses for every element $T \in \mathcal{T}_0$ a so-called *reference edge*. Given a mesh $\mathcal{T}_\ell$ with a subset $\mathcal{E}_\ell$ of its edges, one performs the following steps to obtain $\mathcal{T}_{\ell+1}$:

Algorithm 9 (NVB). *Input: mesh $\mathcal{T}_\ell$, set of marked edges $\mathcal{E}_\ell^{(0)} := \mathcal{E}_\ell$, counter $i := 0$. Output: refined mesh $\mathcal{T}_{\ell+1}$*

(i) Define $\mathcal{U}^{(i)} := \bigcup\limits_{\substack{T,T'\in\mathcal{T}_\ell \\ \overline{T}\cap\overline{T'}\in\mathcal{E}_\ell^{(i)}}} \Big\{ e \in \mathcal{E}_\ell \setminus \mathcal{E}_\ell^{(i)} \mid e \textit{ reference edge of } T \textit{ or } T' \Big\}$

(ii) If $\mathcal{U}^{(i)} \neq \emptyset$, define $\mathcal{E}_\ell^{(i+1)} := \mathcal{E}_\ell^{(i)} \cup \mathcal{U}^{(i)}$, increase counter $i \mapsto i+1$ and goto (i).

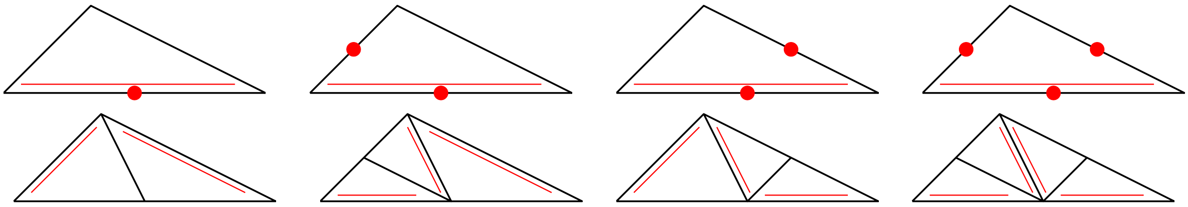

Figure 1: For each triangle $T \in \mathcal{T}_\ell$, there is one fixed *reference edge*, indicated by the double line (left, top). Refinement of T is done by bisecting the reference edge, where its midpoint becomes a new node. The reference edges of the son triangles $T' \in \mathcal{T}_{\ell+1}$ are opposite to this newest vertex (left, bottom). To avoid hanging nodes, one proceeds as follows: We assume that certain edges of T, but at least the reference edge, are marked for refinement (top). Using iterated newest vertex bisection, the element is then split into 2, 3, or 4 son triangles (bottom). If all elements are refined by three bisections (right, bottom), we obtain the so-called uniform bisec(3)-refinement which is denoted by $\widehat{\mathcal{T}}_\ell$.

(iii) With $\mathcal{E}_\ell^{(i)}$ being the marked edges, refine each element $\mathcal{T} \in \mathcal{T}_\ell$ according to the rules shown in Figure 1.

In the next lemma, we collect some properties of the newest-vertex bisection algorithm.

Lemma 10. *Consider a coarse mesh $\mathcal{T}_0$ and a sequence of meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}}$ generated by algorithm NVB. Let $\widehat{\mathcal{T}}_\ell$ denote the uniform bisec(3)-refinement of $\mathcal{T}_\ell$. Then, there holds the following:*

(i) is obtained by successive refinement, i.e., $\mathcal{P}^0(\mathcal{T}_\ell) \subseteq \mathcal{P}^0(\mathcal{T}_{\ell+1})$,

(ii) the fine meshes are obtained by successive refinement, i.e., $\mathcal{P}^0(\widehat{\mathcal{T}}_\ell) \subseteq \mathcal{P}^0(\widehat{\mathcal{T}}_{\ell+1})$,

(iii) is uniformly shape regular,

(iv) each element $T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$ which is refined, is the union of its sons $T' \in \mathcal{T}_{\ell+1}$, i.e. $T = \bigcup \{T' \in \mathcal{T}_{\ell+1} : T' \subseteq T\}$. Moreover, there is a constant $0 < q < 1$ with $h_{\ell+1}(T') \le q\, h_\ell(T)$ for all sons $T' \in \mathcal{T}_{\ell+1}$ of a refined element $T \in \mathcal{T}_\ell \backslash \mathcal{T}_{\ell+1}$.

(v) allows for uniformly H^1-stable L_2-projections onto $\mathcal{S}^1(\mathcal{T}_\ell)$.

Proof. It is well known that the NVB algorithm fulfills assumptions (i)-(iv). In the recent work [KPP] it is shown that the newest-vertex bisection algorithm does even allow for uniformly H^1-stable L_2-projections onto $\mathcal{S}^1(\mathcal{T}_\ell)$. ∎

5.2. A-priori convergence of Galerkin solutions. The following elementary result has already been proved in [BV, Lemma 6.1]. It actually states that the orthogonal projections on any sequence of nested subspaces of some Hilbert space *a-priori* converge strongly in the norm of the underlying space.

Lemma 11. *Let H be a Hilbert space and $X_\ell \subset X_{\ell+1}$ be a sequence of nested closed subspaces of H. Let $\mathbb{P}_\ell : H \to X_\ell$ be the orthogonal projection onto X_ℓ and $x \in H$. Then, the limit $x_\infty := \lim_\ell \mathbb{P}_\ell x \in H$ exists and belongs to the closure of $X_\infty := \bigcup_{\ell=0}^\infty X_\ell$ with respect to H.* $\square$

Before we prove in Proposition 13 below that our adaptive algorithm for the lowest-order Galerkin BEM (5)–(6) leads to a-priori convergent sequences of $\Phi_\ell^\star, \Phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$, we first prove the a-priori convergence of the approximated data $G_\ell = \Pi_\ell g$.

Lemma 12. *Suppose that the sequence of meshes $\mathcal{T}_\ell$ satisfies Assumptions (i) and (v) of Lemma 10. Then, for given $g \in H^\beta$ and $G_\ell := \Pi_\ell g$, the L_2-limit $g_\infty := \lim_\ell G_\ell$ exists and satisfies $g_\infty \in H^\beta$. Moreover, there holds weak convergence*

$$G_\ell \rightharpoonup g_\infty \quad \textit{as } \ell \to \infty \tag{28}$$

in H^β as well as strong convergence

$$\lim_{\ell\to\infty} \|G_\ell - g_\infty\|_{H^\alpha} = 0 \tag{29}$$

for any $0 \le \alpha < \beta$.

Proof. According to Lemma 11, the limit $g_\infty := \lim_{\ell\to\infty} G_\ell$ exists in L_2. Fix $0 < \alpha < \beta$. According to uniform H^β-stability, the sequence (G_ℓ) is a bounded sequence in H^β. Therefore, there is a weakly convergent subsequence (G_{ℓ_k}) of (G_ℓ) such that

$$G_{\ell_k} \rightharpoonup \widetilde{g}_\infty \in H^\beta \quad \text{as } k \to \infty$$

with a certain limit $\widetilde{g}_\infty \in H^\beta$. The Rellich theorem now proves that the subsequence (G_{ℓ_k}) of (G_ℓ) satisfies even strong convergence

$$G_{\ell_k} \to \widetilde{g}_\infty \in H^\alpha.$$

In particular, this provides convergence in L_2, and the uniqueness of limits yields equality $g_\infty = \widetilde{g}_\infty$. First, we thus obtain that $g_\infty \in H^\beta$.

Second, we can apply the same argument to see that each subsequence (G_{ℓ_k}) of (G_ℓ) has a subsequence $(G_{\ell_{k_j}})$ which converges weakly to g_∞ in H^β. We argue by contradiction to see that this implies weak convergence (28) of the entire sequence: Assume that (G_ℓ) does not converge weakly to g_∞. By definition, there is some functional $\psi \in H^{-\beta}$, some scalar $\varepsilon > 0$, and some subsequence (G_{ℓ_k}) of (G_ℓ) such that

$$|\psi(G_{\ell_k}) - \psi(g_\infty)| \ge \varepsilon$$

Consequently, each subsequence $(G_{\ell_{k_j}})$ of (G_{ℓ_k}) satisfies this estimate as well and thus cannot converge weakly to g_∞. This, however, yields a contradiction and proves $G_\ell \rightharpoonup g_\infty$ weakly in H^β. Finally, the Rellich compactness theorem even predicts $G_\ell \to g_\infty$ strongly in H^α for $\alpha < \beta$. $\square$

With the aid of the last two lemmata, we can show that any adaptive algorithm for the solution of (5) or (6) converges a-priori.

Proposition 13. *Suppose that the sequence of meshes $\mathcal{T}_\ell$ satisfies Assumptions (i) and (v) of Lemma 10. Let $\Phi_\ell^\star$ and Φ_ℓ be the Galerkin solutions of* (5) *and* (6)*. Then, there exist limits $\phi_\infty^\star, \phi_\infty \in H^{-1/2}$ such that*

$$\lim_{\ell\to 0} |||\Phi_\ell^\star - \phi_\infty^\star||| = 0 = \lim_{\ell\to 0} |||\Phi_\ell - \phi_\infty|||.$$

The same holds under Assumptions (ii) and (v) of Lemma 10 for the fine mesh solutions $\widehat{\Phi}_\ell^\star$ and $\widehat{\Phi}_\ell$, and the limits are denoted by $\widehat{\phi}_\infty^\star$ and $\widehat{\phi}_\infty$. However, none of the limits $\phi_\infty^\star, \phi_\infty, \widehat{\phi}_\infty^\star$, and $\widehat{\phi}_\infty \in H^{-1/2}$ coincide in general.

Proof of a-priori convergence of $\Phi_\ell^\star$ resp. $\widehat{\Phi}_\ell^\star$. Recall that the simple-layer potential V is linear, elliptic, continuous, and symmetric. Therefore, the Galerkin projection which maps ϕ to $\Phi_\ell^\star$ resp. $\widehat{\Phi}_\ell^\star$ is the orthogonal projection with respect to the energy scalar product $\langle\langle\cdot\,,\cdot\rangle\rangle$. Consequently, Lemma 11 applies and provides limits $\phi_\infty^\star$ and $\widehat{\phi}_\infty^\star$. ☐

Proof of a-priori convergence of Φ_ℓ and $\widehat{\Phi}_\ell$. Since Φ_ℓ and $\widehat{\Phi}_\ell$ are computed with respect to the ℓ-dependent right hand side G_ℓ, we note that Φ_ℓ and $\widehat{\Phi}_\ell$ are not the orthogonal projections of ϕ. Hence, we must not use Lemma 11 directly. Recall that the approximated data converge to some limit g_∞ by Lemma 12. Denote by $\Phi_{\infty,\ell}^\star$ the Galerkin solution to (5) with right-hand side g_∞. We may use Lemma 11 to see that the limit $\Phi_{\infty,\ell}^\star \to \phi_\infty$ exists in $H^{-1/2}$. Moreover, there holds

$$|||\phi_\infty - \Phi_\ell||| \le |||\phi_\infty - \Phi_{\infty,\ell}^\star||| + |||\Phi_{\infty,\ell}^\star - \Phi_\ell|||.$$

Now, the first term tends to zero by definition, and the second term is bounded by stability of Galerkin schemes

$$|||\Phi_{\infty,\ell}^\star - \Phi_\ell||| \lesssim \|(1/2+K)(g_\infty - G_\ell)\|_{H^{1/2}} \lesssim \|g_\infty - G_\ell\|_{H^{1/2}}$$

and thus tends to zero by Lemma 12. This proves a-priori convergence of Φ_ℓ with limit ϕ_∞ in $H^{-1/2}$. The same argument applies to the sequence of Galerkin solutions $\widehat{\Phi}_\ell$. ☐

5.3. Convergent adaptive algorithm. The adaptive algorithm which we introduce below, is steered by the local refinement indicators

$$\gamma_\ell(T)^2 := \|h_\ell^{1/2}(1-\pi_\ell)\widehat{\Phi}_\ell\|_{L_2(T)}^2 + \|h_\ell^{1/2}\nabla_\Gamma(g - G_\ell)\|_{L_2(T)}^2 = \widetilde{\mu}_\ell(T)^2 + \mathrm{osc}_\ell(T)^2. \tag{30}$$

We stress that by use of the above choice for

$$\gamma_\ell^2 = \sum_{T\in\mathcal{T}_\ell} \gamma_\ell(T)^2, \tag{31}$$

the computation of the coarse-mesh solution Φ_ℓ is avoided, and only $\widehat{\Phi}_\ell$ has to be computed. The adaptive algorithm reads as follows:

Algorithm 14. *Input: Initial mesh $\mathcal{T}_0$, parameter $\theta \in (0,1)$, counter $\ell := 0$.*

(i) Obtain $\widehat{\mathcal{T}}_\ell$ by uniform bisec(3)-refinement of $\mathcal{T}_\ell$, see Figure 1.

(ii) Compute solution $\widehat{\Phi}_\ell$ of (6) with respect to $\widehat{\mathcal{T}}_\ell$.

(iii) Compute refinement indicators $\gamma_\ell(T)$ for all $T \in \mathcal{T}_\ell$.

(iv) Choose a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ with minimal cardinality such that

$$\sum_{T \in \mathcal{M}_\ell} \gamma_\ell(T)^2 \geq \theta \sum_{T \in \mathcal{T}_\ell} \gamma_\ell(T)^2 \tag{32}$$

with some fixed parameter $0 < \theta < 1$.

(v) Set $\mathcal{E}_\ell$ to be the set of reference edges of the elements in $\mathcal{M}_\ell$.

(vi) Refine mesh $\mathcal{T}_\ell$ according to Algorithm 9 and obtain $\mathcal{T}_{\ell+1}$.

(vii) Update counter $\ell := \ell + 1$ and goto (i).

Now, we state the main result of this section.

Theorem 15. *Algorithm 14 guarantees the existence of constants $0 < \kappa < 1$ and $C_{11} > 0$ such that*

$$\gamma_{\ell+1} \leq \kappa\, \gamma_\ell + C_{11} \left(\|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell\|_{H^{-1/2}}^2 + \|\Pi_{\ell+1} g - \Pi_\ell g\|_{H^{1/2}}^2 \right)^{1/2} \tag{33}$$

for all $\ell \in \mathbb{N}_0$. In particular, this implies estimator convergence

$$\lim_{\ell \to 0} \gamma_\ell = 0. \tag{34}$$

The constant $\kappa = 1 - (1-q)\theta$ depends on the adaptivity parameter θ and the constant $0 < q < 1$ from the mesh-refinement of Lemma 10. The constant $C_{11} > 0$ depends solely on the initial mesh $\mathcal{T}_0$ and on Γ.

Proof. We consider $\gamma_{\ell+1}$. The triangle inequality yields

$$\begin{aligned}
\gamma_{\ell+1}^2 &= \|h_{\ell+1}^{1/2}(1-\pi_{\ell+1})\widehat{\Phi}_{\ell+1}\|_{L_2}^2 + \|h_{\ell+1}^{1/2}\nabla_\Gamma(g - \Pi_{\ell+1} g)\|_{L_2}^2 \\
&\leq \left(\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1})\widehat{\Phi}_\ell\|_{L_2} + \|h_{\ell+1}^{1/2}(1-\pi_{\ell+1})(\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell)\|_{L_2} \right)^2 + \|h_{\ell+1}^{1/2}\nabla_\Gamma(g - \Pi_{\ell+1} g)\|_{L_2}^2 \\
&\leq \left(\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1})\widehat{\Phi}_\ell\|_{L_2} + \|h_{\ell+1}^{1/2}(\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell)\|_{L_2} \right)^2 + \|h_{\ell+1}^{1/2}\nabla_\Gamma(g - \Pi_{\ell+1} g)\|_{L_2}^2,
\end{aligned} \tag{35}$$

where we have used the elementwise estimate

$$\|(1-\pi_{\ell+1})(\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell)\|_{L_2(T)} \leq \|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell\|_{L_2(T)} \quad \text{for all } T \in \mathcal{T}_{\ell+1}$$

in the second step. We now use Young's inequality and the inverse inequality of Lemma 1 to see, for arbitrary $\delta > 0$,

$$\begin{aligned}\gamma_{\ell+1}^2 &\le (1+\delta)\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1})\widehat{\Phi}_\ell\|_{L_2}^2 + C_1^2(1+\delta^{-1})\|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{H^{-1/2}}^2 \\ &\quad + \|h_{\ell+1}^{1/2}\nabla_\Gamma(g-\Pi_{\ell+1}g)\|_{L_2}^2.\end{aligned}$$

By use of the inverse inequality of Lemma 2, we proceed analogously for the oscillation term,

$$\begin{aligned}\|h_{\ell+1}^{1/2}\nabla_\Gamma(g-\Pi_{\ell+1}g)\|_{L_2}^2 &\le \big(\|h_{\ell+1}^{1/2}\nabla_\Gamma(g-\Pi_\ell g)\|_{L_2} + \|h_{\ell+1}^{1/2}\nabla_\Gamma(\Pi_{\ell+1}g-\Pi_\ell g)\|_{L_2}\big)^2 \\ &\le (1+\delta)\|h_{\ell+1}^{1/2}\nabla_\Gamma(g-\Pi_\ell g)\|_{L_2}^2 + C_2^2(1+\delta^{-1})\|\Pi_{\ell+1}g-\Pi_\ell g\|_{H^{1/2}}^2,\end{aligned}$$

so that ultimately

$$\begin{aligned}\gamma_{\ell+1}^2 &\le (1+\delta)\Big[\|h_{\ell+1}^{1/2}(1-\pi_{\ell+1})\widehat{\Phi}_\ell\|_{L_2}^2 + \|h_{\ell+1}^{1/2}\nabla_\Gamma(g-\Pi_\ell g)\|_{L_2}^2\Big] \\ &\quad + C_{11}^2(1+\delta^{-1})\Big[\|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{H^{-1/2}}^2 + \|\Pi_{\ell+1}g-\Pi_\ell g\|_{H^{1/2}}^2\Big]\end{aligned}$$

with $C_{11} = \max\{C_1, C_2\}$. Now, we consider the first term of (35) $\mathcal{T}_\ell$-elementwise,

$$\Delta(T) := \|h_{\ell+1}^{1/2}(1-\pi_{\ell+1})\widehat{\Phi}_\ell\|_{L_2(T)}^2 + \|h_{\ell+1}^{1/2}\nabla_\Gamma(g-\Pi_\ell g)\|_{L_2(T)}^2 \qquad \text{for } T \in \mathcal{T}_\ell.$$

For $T \in \mathcal{M}_\ell$, there holds $h_{\ell+1}|_T \le q h_\ell|_T$ and $(1-\pi_{\ell+1}\widehat{\Phi}_\ell) = 0$, and so

$$\Delta(\mathcal{M}_\ell) \le q \sum_{T\in\mathcal{M}_\ell} \|h_\ell^{1/2}\nabla_\Gamma(g-\Pi_\ell g)\|_{L_2(T)}^2 \le q\gamma_\ell(\mathcal{M}_\ell)^2.$$

Contrary, for $T \in \mathcal{T}_\ell \setminus \mathcal{M}_\ell$ it holds that

$$\Delta(T) \le \|h_\ell^{1/2}(1-\pi_\ell)\widehat{\Phi}_\ell\|_{L_2(T)}^2 + \|h_\ell^{1/2}\nabla_\Gamma(g-\Pi_\ell g)\|_{L_2(T)}^2 = \gamma_\ell(T)^2,$$

as $h_\ell \le h_{\ell+1}$ and $\pi_{\ell+1}$ being a better approximation as π_ℓ. Hence, $\Delta(\mathcal{T}_\ell \backslash \mathcal{M}_\ell) \le \gamma_\ell(\mathcal{T}_\ell \backslash \mathcal{M}_\ell)^2$. Splitting the first term into marked and non-marked elements, we have

$$\begin{aligned}\Delta(\mathcal{T}_\ell) = \Delta(\mathcal{M}_\ell) + \Delta(\mathcal{T}_\ell \setminus \mathcal{M}_\ell) \le q\gamma_\ell(\mathcal{M}_\ell)^2 + \gamma_\ell(\mathcal{T}_\ell \setminus \mathcal{M}_\ell)^2 &= \gamma_\ell^2 + (q-1)\gamma_\ell(\mathcal{M}_\ell)^2 \\ &\le \big(1+(q-1)\theta\big)\gamma_\ell^2,\end{aligned}$$

from which we conclude, for all $\delta > 0$,

$$\gamma_{\ell+1}^2 \le (1+\delta)\big(1+(q-1)\theta\big)\gamma_\ell^2 + C_{11}^2(1+\delta^{-1})\Big[\|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{H^{-1/2}}^2 + \|\Pi_{\ell+1}g-\Pi_\ell g\|_{H^{1/2}}^2\Big].$$

We now choose $\kappa := \big(1+(q-1)\theta\big) < 1$. Optimizing the choice of the free parameter $\delta > 0$, we see that the previous estimate is equivalent to

$$\gamma_{\ell+1} \le \kappa\,\gamma_\ell + C_{11}\big(\|\widehat{\Phi}_{\ell+1}-\widehat{\Phi}_\ell\|_{H^{-1/2}}^2 + \|\Pi_{\ell+1}g-\Pi_\ell g\|_{H^{1/2}}^2\big)^{1/2}.$$

Since $\mathcal{P}^0(\widehat{\mathcal{T}}_\ell) \subseteq \mathcal{P}^0(\widehat{\mathcal{T}}_{\ell+1})$ by Lemma 10, we can apply Proposition 13 to $(\widehat{\Phi}_\ell)_{\ell\in\mathbb{N}}$ and obtain $\|\widehat{\Phi}_{\ell+1} - \widehat{\Phi}_\ell\|^2_{H^{-1/2}} \to 0$ as $\ell \to \infty$. Lemma 12 reveals $\|\Pi_{\ell+1} g - \Pi_\ell g\|^2_{H^{1/2}} \to 0$. We thus infer the perturbed contraction

$$\gamma_{\ell+1} \leq \kappa\gamma_\ell + o(1).$$

This is the estimator reduction from [AFP+, Lemma 2.3]. Again, it follows from elementary calculus that $\gamma_\ell \to 0$ as $\ell \to \infty$. ☐

6 Numerical Experiments

6.1. Specification of the problem. We consider the three-dimensional L-shaped domain depicted in Fig. 2. It is composed of three cubes with side length 0.5 and a common edge in the z-axis. The uniform initial mesh $\mathcal{T}_0$ consists of 56 rectangular triangles. We choose

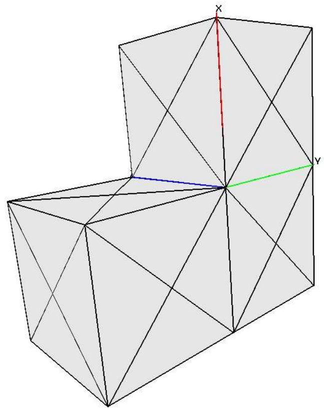

Figure 2: Initial mesh of the L–shaped domain.

the constant extension (in z-direction) of the singular solution of the two-dimensional L-shaped domain as solution of the considered three-dimensional Dirichlet boundary value problem (2):

$$u(r, \varphi, z) = r^{2/3} \sin(2/3\,\varphi) \tag{36}$$

in cylindrical coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$. Thus, we have a non-smooth conormal derivative ϕ along the reentrant edge, but the trace g of u is zero on the adjacent faces.

6.2. Computation of Galerkin solution and preconditioning. The computations were performed by an implementation [OSW] of the Galerkin boundary element method based on semi-analytic integration formulae [RS, Appendix C.2], the fast multipole method [GR], and an artificial multilevel preconditioner [S03] for the Galerkin matrix of the simple-layer potential. The latter is a modification of the BPX preconditioner [BPX, FS]. In case of a sequence of uniformly refined meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$, the preconditioner for the mesh $\mathcal{T}_L$ is based on the weighted sum

$$A^s = \sum_{\ell=0}^{L} h_\ell^{-2s}(\pi_\ell - \pi_{\ell-1})$$

of L_2 projections $\pi_\ell : L_2 \to \mathcal{P}^0(\mathcal{T}_\ell)$ (and $\pi_{-1} = 0$). Due to the spectral equivalence inequalities [O98, Theorem 2]

$$c_1\|w\|^2_{H^{-1/2}} \le \langle A^{-1/2}w, w\rangle_\Gamma \le c_2 L_2 \|w\|^2_{H^{-1/2}} \qquad \text{for all } w \in \mathcal{P}^0(\mathcal{T}_L), \tag{37}$$

the operator $A^{-1/2}$ is a suitable preconditioner for the simple-layer potential operator. The inversion of the Galerkin matrix of $A^{-1/2}$ can be avoided due to the identity

$$\left(A_h^{-1/2}\right)^{-1} = M_h^{-1} A_h^{1/2} M_h^{-1}$$

which involves the inversion of the diagonal mass matrices M_h only, where

$$A_h^{1/2}[i,j] = \langle A^{1/2}\Psi_j\,, \Psi_i\rangle \qquad \text{and} \qquad M_h[i,j] = \langle \Psi_k\,, \Psi_\ell\rangle.$$

This preconditioner has been extended to adaptively refined meshes [O06, p. 69] by an extension to a uniform mesh. In practice, the artificial multilevel preconditioner does not utilize a sequence of nested meshed constructed by uniform refinement, but a sequence of artificial spaces constructed by the geometrical clustering of the finest mesh as used for the fast multipole method.

6.3. Computation of upper error bound. Reliability of the proposed error estimators is (for the non-perturbed problem) equivalent to the saturation assumption (7), see (27). However, since we prescribe the exact solution $\phi \in L_2(\Gamma)$, we can compute a reliable error bound err_ℓ. To that end, remember first that $\pi_\ell : L_2(\Gamma) \to \mathcal{P}^0(\mathcal{T}_\ell)$ is the $L_2(\Gamma)$-orthogonal projection. Using the triangle inequality and the best approximation property of the Galerkin solution $\Phi_\ell^\star$ with respect to the energy norm $|||\cdot|||$ yields

$$|||\phi - \Phi_\ell||| \le |||\phi - \Phi_\ell^\star||| + |||\Phi_\ell^\star - \Phi_\ell||| \le |||\phi - \pi_\ell\phi||| + |||\Phi_\ell^\star - \Phi_\ell|||.$$

Now, the second term is bounded by osc_ℓ as in the proof of Theorem 8. To bound the first term, we use the approximation estimate [CP06, Theorem 4.1] for π_ℓ and see

$$|||\phi - \pi_\ell\phi||| \simeq \|\phi - \pi_\ell\phi\|_{H^{-1/2}(\Gamma)} \lesssim \|h_\ell^{1/2}(\phi - \pi_\ell\phi)\|_{L_2(\Gamma)} \le \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L_2(\Gamma)},$$

where we used the $\mathcal{T}_\ell$-piecewise best approximation property of π_ℓ in the last step. Setting $\mathrm{err}_\ell := \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L_2(\Gamma)}$, we see

$$|||\phi - \Phi_\ell||| \lesssim \mathrm{err}_\ell + \mathrm{osc}_\ell, \tag{38}$$

and hence the right-hand side provides reliable feedback on the energy error.

6.4. Uniform refinement. In the case of uniform refinement, the sequence of meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ is obtained by uniform bisec(3)-refinement, i.e. $\mathcal{T}_{\ell+1}$ is a refinement of $\mathcal{T}_\ell$, where all edges are bisected. The $\Phi_\ell \in \mathcal{P}^0(\mathcal{T}_\ell)$ are the solutions of the Galerkin formulations (6) with right-hand side $G_\ell \in \mathcal{S}^1(\mathcal{T}_\ell)$, whereas $\widehat{\Phi}_\ell$ are the solutions of the Galerkin formulations (6) on the uniformly refined mesh $\widehat{\mathcal{T}}_\ell = \mathcal{T}_{\ell+1}$ with the same right-hand side, see (21). We define

$$\begin{aligned}
\mathrm{err}_{\mathrm{unif},\ell} &:= \mathrm{err}_\ell = \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L_2(\Gamma)}\\
\mathrm{osc}_{\mathrm{unif},\ell} &:= \mathrm{osc}_\ell = \|h_\ell^{1/2}\nabla_\Gamma(g - G_\ell)\|_{L_2(\Gamma)}\\
\widetilde{\mu}_{\mathrm{unif},\ell} &:= \mu_\ell = \|h_\ell^{1/2}(1-\pi_\ell)\widehat{\Phi}_\ell)\|_{L_2(\Gamma)}.
\end{aligned}$$

Carefully note that the computation of Φ_ℓ is, in principle, avoided by Algorithm 14. In fact, we compute it only to obtain the reliable error bound $\mathrm{err}_{\mathrm{unif},\ell}$.

6.5. Adaptive refinement. In the case of adaptive refinement, the sequence of meshes $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ is obtained by employing Algorithm 14. In fact, there is no need to store the meshes $\mathcal{T}_\ell$, since the computation of Φ_ℓ is avoided. We merely need to store $\widehat{\mathcal{T}}_\ell$, consequently we approximate the right-hand side g on these finer meshes. Altogether, the Galerkin solution $\widehat{\Phi}_\ell \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell)$ is obtained by solving

$$\langle\!\langle \widehat{\Phi}_\ell, \widehat{\Psi}_\ell \rangle\!\rangle = \langle (K+1/2)\widehat{G}_\ell, \widehat{\Psi}_\ell\rangle \quad \text{for all } \widehat{\Psi}_\ell \in \mathcal{P}^0(\widehat{\mathcal{T}}_\ell). \tag{39}$$

We define

$$\begin{aligned}
\mathrm{err}_{\mathrm{adap},\ell} &:= \mathrm{err}_\ell = \|\widehat{h}_\ell^{1/2}(\phi - \widehat{\Phi}_\ell)\|_{L_2(\Gamma)}\\
\mathrm{osc}_{\mathrm{adap},\ell} &:= \mathrm{osc}_\ell = \|\widehat{h}_\ell^{1/2}\nabla_\Gamma(g - \widehat{G}_\ell)\|_{L_2(\Gamma)}\\
\widetilde{\mu}_{\mathrm{adap},\ell} &:= \mu_\ell = \|h_\ell^{1/2}(1-\pi_\ell)\widehat{\Phi}_\ell)\|_{L_2(\Gamma)}.
\end{aligned}$$

The same computation that resulted in (38) yields the reliable error bound

$$|||\phi - \widehat{\Phi}_\ell||| \lesssim \mathrm{err}_{\mathrm{adap},\ell} + \mathrm{osc}_{\mathrm{adap},\ell}.$$

6.6. Comparison of uniform and adaptive approach. First of all, we compare the rate of convergence for uniform and adaptive approach by plotting the involved quantities over the number of degrees of freedom. For uniform refinement, we plot $\mathrm{err}_{\mathrm{unif},\ell}$, $\widetilde{\mu}_{\mathrm{unif},\ell}$, and $\mathrm{osc}_{\mathrm{unif},\ell}$ over the number of boundary elements $\#\mathcal{T}_\ell$, where $(\mathcal{T}_\ell)_{\ell\in\mathbb{N}_0}$ is the sequence of uniform meshes. For adaptive refinement, we plot $\mathrm{err}_{\mathrm{adap},\ell}$, $\widetilde{\mu}_{\mathrm{adap},\ell}$, and $\mathrm{osc}_{\mathrm{adap},\ell}$ over the number of boundary elements $\#\widehat{\mathcal{T}}_\ell$, where $(\widehat{\mathcal{T}}_\ell)_{\ell\in\mathbb{N}_0}$ is the sequence of temporary, bisec(3)-refined meshes $\widehat{\mathcal{T}}_\ell$ of the meshes $\mathcal{T}_\ell$ generated by Algorithm 14. Note that, in case of uniform mesh refinement, the optimal order of convergence of lowest-order Galerkin boundary element methods is $\mathcal{O}(h^{3/2}) \simeq \mathcal{O}(\#\mathcal{T}_\ell^{-3/4})$. However, the example is chosen in such a way that uniform mesh refinement can be predicted to exhibit a reduced order of convergence.

Furthermore, we plot the reliable error bounds $\mathrm{err}_{\mathrm{unif},\ell}$ and $\mathrm{err}_{\mathrm{adap},\ell}$ over the time that is consumed for their computation. Since the adaptive algorithm depends on the whole history of computed solutions, the time consumption is measured differently for the uniform and the adaptive approach:

- For uniform mesh-refinement, $t_{\mathrm{unif},\ell}$ is the time elapsed for ℓ uniform mesh-refinements of the initial mesh $\mathcal{T}_0$, the assembly of the Galerkin data with respect to $\mathcal{T}_\ell$, and the computation of the Galerkin solution Φ_ℓ with respect to $\mathcal{T}_\ell$.

For adaptive mesh-refinement, the computational time is defined in an inductive manner:

- We define $t_{\mathrm{adap},-1} := 0$.
- For $\ell \geq 0$, $t_{\mathrm{adap},\ell}$ is the sum of the previous steps $t_{\mathrm{adap},\ell-1}$ plus the time elapsed for the uniform refinement of $\mathcal{T}_\ell$ to obtain $\widehat{\mathcal{T}}_\ell$, the assembly of the Galerkin data with respect to $\widehat{\mathcal{T}}_\ell$, the computation of the Galerkin solution and the local contributions of the error indicators, the marking step, and the local refinement of $\mathcal{T}_\ell$ to obtain $\mathcal{T}_{\ell+1}$.

6.7. Discussion of the numerical experiments. In Fig. 3 and Fig. 4, we compare the errors of approximations obtained by uniform mesh refinement and by use of the adaptive approach of Algorithm 14 with parameters $\theta = 0.4$ and $\theta = 0.5$, respectively. We end up for $\mathcal{T}_\ell$ with 114,912 triangles after 30 adaptive refinement steps for $\theta = 0.4$ and with 123,134 triangles after 26 adaptive refinement steps for $\theta = 0.5$. In both cases, we observe a large adaptivity ratio of $h_{\max}/h_{min} \approx 362$.

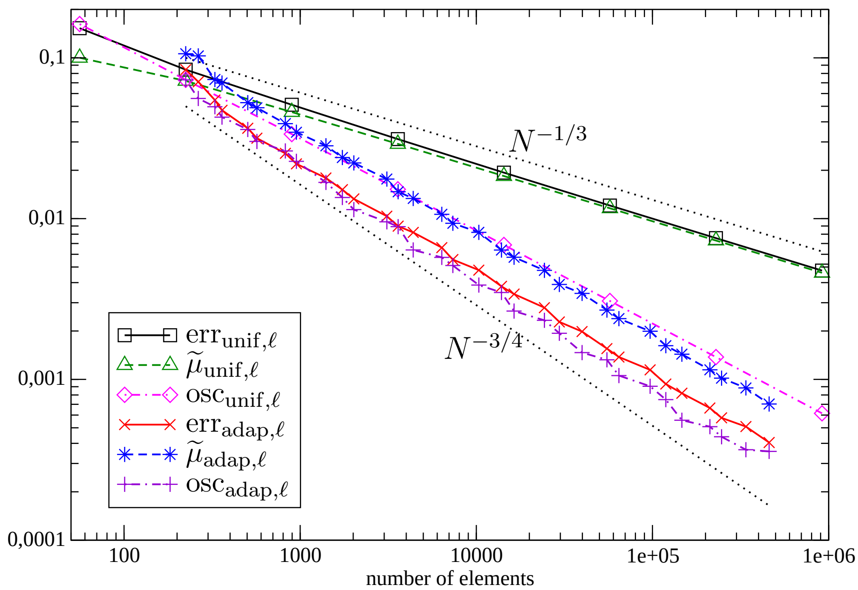

Figure 3: Error plots for uniform and the adaptive refinement with parameter $\theta = 0.4$.

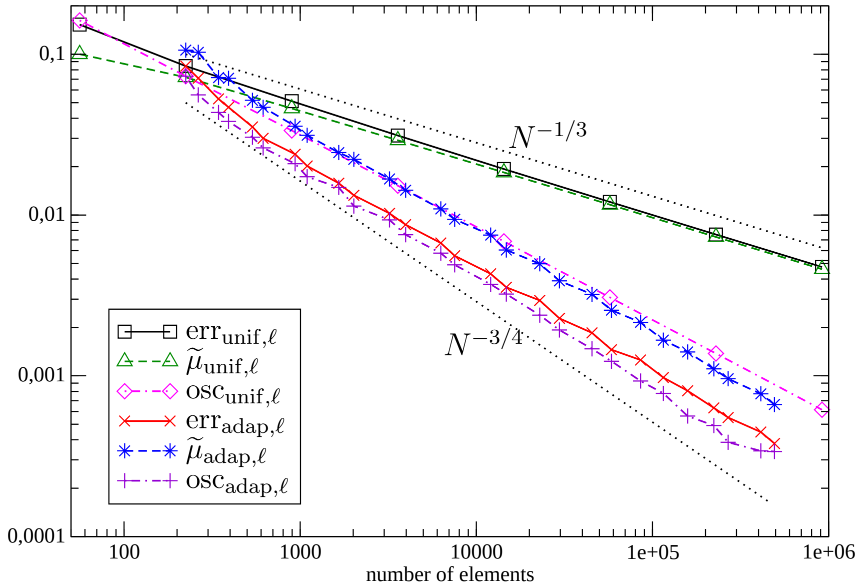

Figure 4: Error plots for uniform and the adaptive refinement with parameter $\theta = 0.5$.

For the uniform refinement with up to 917,504 triangles, we observe a convergence of the energy error err_ℓ of the Neumann data like $N^{-1/3}$, where N denotes the number of triangles as well as the number of degrees of freedom. This is in perfect agreement with the regularity of the predescribed solution $u \in H^{5/3}(\Omega)$, i.e., $\phi \in H^{1/6}(\Gamma)$, and the a priori error estimate for uniform refinement

$$\|\phi - \Phi_\ell\|_{H^{-1/2}} \lesssim h_\ell^{s+1/2} |\phi|_{H^s} \qquad \text{for all possible } s \in [0, 1].$$

For the data approximation error osc_ℓ, we observe a higher order of convergence, as the Dirichlet data g is zero at the reentrant edge and the adjacent faces.

For the adaptive refinement, we observe a significantly higher order of convergence than for the uniform refinement. As we use an isotropic refinement to resolve an edge singularity, we cannot expect to get the optimal convergence of $N^{-3/4}$. But we observe an order of convergence of approximately 0.7, i.e., close to the optimum. The error estimator $\widetilde{\mu}_{\mathrm{adap},\ell}$ proves to be a good estimator.

In Fig. 5, we compare the errors $\mathrm{err}_{\mathrm{unif},\ell}$ and $\mathrm{err}_{\mathrm{adap},\ell}$ to the related computational times. On a first glance, the definition of $t_{\mathrm{adap},\ell}$ and $t_{\mathrm{unif},\ell}$ seems to favor uniform mesh refinement. However, we observe that the adaptive computations outperform the uniform computations significantly. Only for small computational times and approximately 10,000 uniform elements, the uniform refinement gives slightly smaller errors. Moreover, the computational times are comparable for both values of θ.

Note that the computations in the adaptive algorithm do not take advantage of the fact that only parts of the geometry are refined in each step and thus significant parts of the matrices could be reused. Instead the matrices are generated from the scratch on each refinement level. Therefore, clever implementation would speed up the adaptive

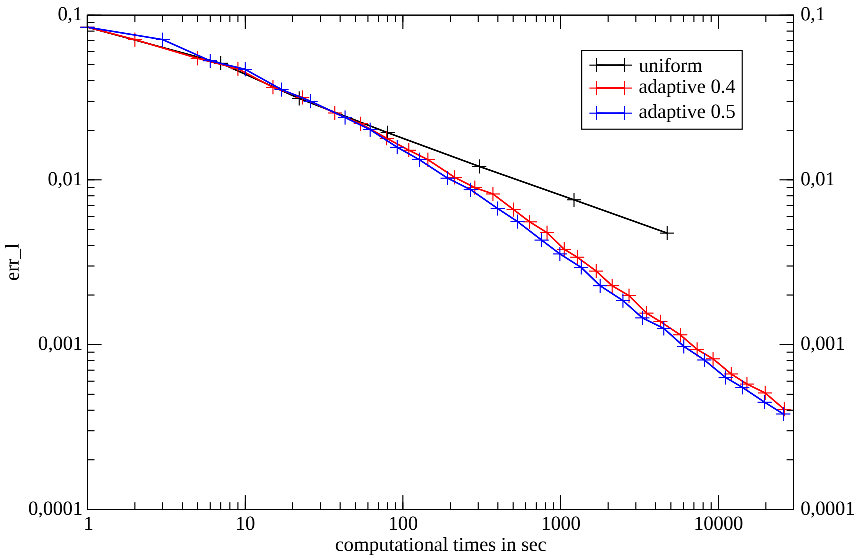

Figure 5: Errors $\mathrm{err}_{\mathrm{unif},\ell}$ and $\mathrm{err}_{\mathrm{adap},\ell}$ over computational times for uniform refinement and adaptive refinement with parameter $\theta \in \{0.4, 0.5\}$.

computations significantly. In addition, the parameters of the FMM are chosen to be fixed during the whole adaptive algorithm and are large enough to guarantee no loss of accuracy for the finest levels. If we adjusted the parameters in a suitable fashion to the actual error on each level, we would expect an additional speedup of the computations.

7 Conclusions

7.1. Analytical Results. We proposed and analyzed an adaptive mesh-refinement algorithm for the numerical solution of the Laplace equation by a lowest-order Galerkin-boundary element method. To enable the use of fast methods for boundary integral equations, we approximated the Dirichlet data by discrete functions by means of the L_2-projection onto piecewise linears. The resolution of the data approximation is included into the adaptive algorithm. This work transfers and extends the analysis of [AFGKMP] to three dimensions. In the latter work, nodal interpolation for the approximation of the Dirichlet data is used. However, in the three-dimensional case nodal interpolation is not feasible anymore for H^1-functions. We propose to use Scott-Zhang-type quasi-interpolation operators or L_2-orthogonal projections. To that end, (local) approximation estimates for quasi-interpolation operators in fractional-order Sobolev spaces were shown. We rigorously prove that the proposed adaptive algorithm drives the error estimator to zero. The convergence of the computed discrete solutions to the exact solution in the energy norm $|||\phi - \Phi_\ell|||$ was shown under the so-called *saturation assumption*.

7.2. Numerical Results. In the numerical experiment, the adaptive algorithm shows a significantly higher order of convergence than the computations based on uniform mesh-refinement. It almost regains the optimal order even with the restriction to isotropic mesh-refinement, which cannot be optimal for problems with generic edges singularities; see [CMPS, Section 7.3]. Regarding the computational error, the adaptive algorithm is faster than the computations with uniformly refined meshes, even though our BEM implementation is not adapted to the adaptive algorithm yet.

7.3. Future Work. For an adaptive algorithm driven by the weighted-residual error estimator from [CMS], even quasi-optimality could be proved recently in [FKMP]. We aim to combine the ideas presented in the work at hand with those of [FKMP] to prove quasi-optimal convergence rates for adaptive algorithms driven by $h - h/2$-based estimators. Furthermore, as already mentioned, the implementation that was used for the experiments in this work was not fitted to the adaptive approach. Including the effective update of the system matrix into the adaptive algorithm, as was done e.g. in [DJ], will certainly enhance computational times. In addition, we aim to include the choice of the parameters used for the Fast-Multipole method into the adaptive algorithm, thereby raising the accuracy that can be achieved by coupling of the Fast-Multipole method and adaptivity with lowest expenses.

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References

[A] M. Ainsworth, J.T. Oden: *A posteriori error estimation in finite element analysis*, Wiley-Interscience [John Wiley & Sons], New York, 2000.

[AFGKMP] M. Aurada, S. Ferraz-Leite, P. Goldenits, M. Karkulik, M. Mayr, D. Praetorius: *Convergence of adaptive BEM for some mixed boundary value problem*, Appl. Numer. Math **62** (2012), 226–245.

[AFP] M. Aurada, M. Feischl, D. Praetorius: *Convergence of some adatpvie FEM-BEM coupling for elliptic but possibly nonlinear interface problems* ESAIM: M2AN **46** (2012), 1147–1173.

[AFP+] M. Aurada, S. Ferraz-Leite, D. Praetorius: *Estimator reduction and convergence of adaptive BEM*, Appl. Numer. Math. **62** (2012), 787–801.

[AKP] M. Aurada, M. Karkulik, D. Praetorius: *Simple error estimators for hypersingular integral equations in adaptive 3D-BEM*, work in progress, 2011.

[B] R. Bank: *Hierarchical bases and the finite element method*, Acta Numerica **5** (1996), 1–45.

[BPS] J. H. Bramble, J. E. Pasciak, O. Steinbach: *On the stability of the L_2 projection in $H^1(\Omega)$*, Math. Comp. **71** (2001), 147–156.

[BV] I. Babuška, M. Vogelius: *Feedback and Adaptive Finite Element Solution of One-Dimensional Boundary Value Problems*, Numer. Math. **44** (1984), 75–102.

[BPX] J. H. Bramble, J. E. Pasciak, and J. Xu: *Parallel multilevel preconditioners*, Math. Comp. **55** (1990), 1–22.

[BX] J. H. Bramble, J. Xu: *Some estimates for a weighted L_2-projection*, Math. Comp. **56** (1991), 463–476.

[C96] C. Carstensen: *Efficiency of a posteriori BEM-error estimates for first-kind integral equations on quasi-uniform meshes*, Math. Comp. **65** (1996), 69–84.

[C97] C. Carstensen: *An a posteriori error estimate for a first-kind integral equation*, Math. Comp. **66** (1997), 139–155.

[C01] C. Carstensen: *Merging the Bramble-Pasciak-Steinbach and the Crouzeix-Thomée criterion for H^1-stability of the L_2-projection onto finite element spaces*, Math. Comp. **71** (2001), 157–163.

[C02] C. Carstensen: *An adaptive mesh-refining algorithm allowing for an H^1-stable L_2-projection onto Courant finite element spaces*, Constructive Approximation **20** (2004), 549–564

[CF] C. Carstensen, B. Faermann: *Mathematical foundation of a posteriori error estimates and adaptive mesh-refining algorithms for boundary integral equations of the first kind*, Eng. Anal. Bound. Elem. **25** (2001), 497–509.

[CFS] C. Carstensen, S. Funken, E. Stephan: *A posteriori error estimates for hp-boundary element methods*, Appl. Anal. **61** (1996), 233–253.

[CMPS] C. Carstensen, M. Maischak, D. Praetorius, E. Stephan: *Residual-based a posteriori error estimate for hypersingular equation on surfaces*, Numer. Math. **97** (2004), 397–425.

[CMS] C. Carstensen, M. Maischak, E. Stephan: *A posteriori error estimate and h-adaptive algorithm on surfaces for Symm's integral equation*, Numer. Math, **90** (2001), 197–213.

[CP06] C. Carstensen, D. Praetorius: *Averaging techniques for the effective numerical solution of Symm's integral equation of the first kind*, SIAM J. Sci. Comput. **27** (2006), 1226–1260.

[CS95] C. Carstensen, E. P. Stephan: *A posteriori error estimates for boundary element methods*, Math. Comp. 64 (1995), 483–500.

[CS96] C. Carstensen, E. P. Stephan: *Adaptive boundary element methods for some first kind integral equations*, SIAM J. Numer. Anal. 33 (1996), 2166–2183.

[CPS] A. Chernov, T. von Petersdorff, C. Schwab: *Exponential convergence of hp quadrature for integral operators with gevrey kernels*, ESAIM Math. Model. Numer. Anal. **45** (2011), 387–422.

[Cl] Ph. Clemént: *Approximation by finite element functions using local regularization*, RAIRO Analyse Numérique **9** (1975), 77–84.

[DJ] J. Djokić: *Efficient update of hierarchical matrices in the case of adaptive discretisation schemes*, Dissertation, Universität Leipzig, 2006.

[DLY] P. Deuflhard, P. Leinen, H. Yserentant: *Concepts of an adaptive hierarchical finite element code*, IMPACT Comput. in. Sci. and Eng. **1** (1989), 3–35.

[D] W. Dörfler: *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. **33** (1996) 1106–1124.

[DN] W. Dörfler, R. H. Nochetto: *Small data oscillation implies the saturation assumption*, Numer. Math. **91** (2002), 1–12.

[EFFP] C. Erath, S. Ferraz-Leite, S. Funken, D. Praetorius: *Energy norm based a posteriori error estimation for boundary element methods in two dimensions*, Appl. Numer. Math. **59** (2009), 2713–2734.

[EFGP] C. Erath, S. Funken, P. Goldenits, D. Praetorius: *Simple error estimators for the Galerkin BEM for some hypersingular integral equation in 2D*, accepted for publication in Appl. Anal. (2012).

[F98] B. Faermann: *Local a posteriori error indicators for the Galerkin discretisation of boundary integral equations*, Numer. Math. **79** (1998), 43–76.

[F00] B. Faermann: *Localization of the Aronszajn-Slobodeckij norm and applications to adaptive boundary element methods. Part I. The two-dimensional case.* IMA J. Numer. Anal. **20** (2000), 203–234.

[F02] B. Faermann: *Localization of the Aronszajn-Slobodeckij norm and applications to adaptive boundary element methods. Part II. The three-dimensional case.* Numer. Math. **92** (2002), 467–499.

[FKMP] M. Feischl, M. Karkulik, J. M. Melenk, D. Praetorius: *Quasi-optimal convergence rate for an adaptive boundary element method*, ASC Report **28/2011**, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien 2011.

[FOP] S. Ferraz-Leite, C. Ortner, D. Praetorius: *Convergence of simple adaptive Galerkin schemes based on $h - h/2$ error estimators*, Numer. Math. **116** (2010), 291–316.

[FP] S. Ferraz-Leite, D. Praetorius: *Simple a posteriori error estimators for the h-version of the boundary element method*, Computing **83** (2008), 135–162.

[FS] S. A. Funken, E. P. Stephan: *The BPX preconditioner for the single layer potential operator*, Appl. Anal. **67** (1997), 327–340.

[GHS] I. Graham, W. Hackbusch, S. Sauter: *Finite elements on degenerate meshes: inverse-type inequalities and applications*, IMA J. Numer. Anal. **25** (2005), 379–407.

[GR] L. Greengard, V. Rokhlin: *A fast algorithm for particle simulations*, J. Comput. Phys. **73** (1987), 325–348.

[HNW] E. Hairer, S. Nørsett, G. Wanner: *Solving ordinary differential equations I. Nonstiff problems*, Springer, New York, 1987.

[H02] N. Heuer: *An hp-adaptive refinement strategy for hypersingular operators on surfaces*, Numer. Methods Partial Differential Equations **18** (2002), 396–419.

[HMS01] N. Heuer, M. E. Mellado, E. P. Stephan: *hp-adaptive two-level methods for boundary integral equations on curves*, Computing **67** (2001), 305–334.

[HMS02] N. Heuer, M. E. Mellado, E. P. Stephan: *A p-adaptive algorithm for the BEM with the hypersingular operator on the plane screen*, Internat. J. Numer. Methods Engrg. **53** (2002), 85–104.

[HW] G, C. Hsiao, W. L. Wendland: *Boundary integral equations*, Applied Mathematical Sciences **164**, Springer, Berlin, 2008

[KPP] M. Karkulik, D. Pavlicek, D. Praetorius: *On 2D newest vertex bisection: Optimality of mesh-closure and H^1-stability of L_2-projection*, ASC Report **10/2012**, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien 2012.

[ML] W. McLean: *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.

[MS] P. Mund, E. P. Stephan: *An adaptive two-level method for hypersingular integral equations in* $\mathbb{R}^3$, ANZIAM J. **42** (2000), 1019–1033.

[MSW] P. Mund, E. P. Stephan, J. Weisse: *Two-Level Methods for the Single Layer Potential in* $\mathbb{R}^3$, Computing **60** (1998), 243–266.

[O06] G. Of: *BETI domain decomposition methods with fast boundary element methods and applications*, PhD thesis, University of Stuttgart, 2006, `http://elib.uni-stuttgart.de/opus/volltexte/2006/2543/`.

[OSW] G. Of, O. Steinbach, W. L. Wendland: *The fast multipole method for the symmetric boundary integral formulation*, IMA J. Numer. Anal. **26** (2006), 272–296.

[O98] P. Oswald: *Multilevel norms for $H^{-1/2}$*, Computing **61** (1998) 235–255.

[RS] S. Rjasanow, O. Steinbach: *The Fast Solution of Boundary Integral Equations*, Springer, New York, 2007.

[SaS] S. Sauter, C. Schwab: *Boundary element methods*, Translated and expanded from the 2004 German original. Springer Series in Computational Mathematics **39**, Springer-Verlag, Berlin, 2011.

[SSt] H. Schulz, O. Steinbach: *A new a posteriori error estimator in adaptive direct boundary element methods. The Dirichlet problem*, Calcolo **37** (2000), 79–96.

[S] C. Schwab: *Variable order composite quadrature of singular and nearly singular integrals* Computing **53** (1994), 173–194.

[SZ] L. Scott, S. Zhang: *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), 483–493.

[S00] O. Steinbach: *Adaptive boundary element methods based on computational schemes for Sobolev norms*, SIAM J. Sci. Comput. **22** (2000), 604–616.

[S01] O. Steinbach: *On the stability of the L_2 projection in fractional Sobolev spaces*, Numer. Math. **88** (2001), 367–379.

[S03] O. Steinbach: *Artificial multilevel boundary element preconditioners*, Proc. Appl. Math. Mech. **3** (2003), 539–542.

[S08] O. Steinbach: *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements*, Springer, New York , 2008.

[St] O. Steinbach: *Numerical approximation methods for elliptic boundary value problems*, Finite and boundary elements. Translated from the 2003 German original. Springer, New York, 2008.

[T] L. Tartar: *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer, Berlin-Heidelberg, 2007.

[V] R. Verfuerth: *A review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques*, Teubner, Stuttgart, 1996

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