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Technische Universität Graz
Institut für Numerische Mathematik
Steyrergasse 30
A 8010 Graz

WWW: http://www.numerik.math.tu-graz.at

On Existence analysis of steady flows of generalized Newtonian fluids with concentration dependent power–law index

M. Bulíček[1] P. Pustějovská[2]

[1] Charles University,
Faculty of Mathematics and Physics,
Mathematical Institute,
Sokolovská 83, 186 75 Prague 8, Czech Republic

[2] Institute of Computational Mathematics,
Graz University of Technology,
Steyrergaße 30,
A-8010 Graz, Austria

mbul8060@karlin.mff.cuni.cz, pustejovska@tugraz.at

Abstract

We study a system of partial differential equations describing a steady flow of an incompressible generalized Newtonian fluid, wherein the Cauchy stress is concentration dependent. Namely, we consider a coupled system of the generalized Navier–Stokes equations and convection–diffusion equation with non–linear diffusivity. We prove the existence of a weak solution for certain class of models by using a generalization of the monotone operator theory which fits into the framework of generalized Sobolev spaces with variable exponent. Such a framework is involved since the function spaces, where we look for the weak solution, are "dependent" of the solution itself, and thus, we a priori do not know them. This leads us to the principal a priori assumptions on the model parameters that ensure the Hölder continuity of the variable exponent. We present here a constructive proof based on the Galerkin method that allows us to obtain the result for very general class of models.

Keywords: generalized Navier–Stokes system, incompressible fluid, concentration dependent viscosity, shear–rate dependent viscosity, Sobolev spaces with variable exponent

1 Introduction

The mathematical study of non–Newtonian fluids became very popular in the recent decades. This is also due to the fact that many biological fluids (such as blood, mucus or polymeric solutions), fluids used in the engineering, food industry or agriculture can be described in such a framework. This of course brings importance of mathematical analysis and generally fundamental research in these medical, biological or engineering applications.

In this article, we shall study a rheological response of a synovial fluid (a biological fluid occurring in the mammalian movable joint cavities) during the shear experiment. From rheological point of view, synovial fluid is composed of an ultrafiltrated blood plasma (having a characteristic of a Newtonian fluid) diluting specific polysaccharide called hyaluronan. In one way, one can describe such a solution in a framework of a mixture theory, we, on the other hand, shall restrict ourselves to the case, that the fluid can be described as a single constituent fluid. This can be defended by the fact, that the physiological mass concentration of the hyaluronan is very low (usually around 0.5%), and even though a local accumulation occurs, the mass concentration does not exceed 2%. The solution remains in a practically homogeneous state[1]. Nevertheless, one still needs to include to the model the experimentally proven chemical influence on the fluid rheology. To be more specific, already in the early 50' it was shown in the work of Ogston and Stanier (1953) (see also (Fung, 1993, Section 6.7)), that the synovial fluid exhibit a strong shear–thinning behavior during a simple viscosimetric experiment, qualitatively and quantitatively depending on the concentration of hyaluronan in the solution. In other words, the apparent viscosity of the fluid is not just a function of shear–rate but also of the concentration. From mathematical modeling point of view, a class of power–law–like models with a concentration dependence seems to be then applicable. In that case, we need to additionally specify the concentration dependence.

By the experiment, it was shown that the hyaluronan concentration is not just a scaling factor of the viscosity (understand as $\nu(c,|D|^2) \sim f(c)\,\tilde{\nu}(|D|^2)$), but it influences the measure of how much the fluid thins the shear. For zero concentration, the viscosity remains constant for different shear–rates, reflecting the fact that that the fluid is consisting only from a blood plasma ultrafiltrate exhibiting a Newtonian character. With increasing concentration in the solution, the fluid exhibit higher apparent viscosity, nevertheless, it thins the shear more and more rapidly[2]. This led the authors of Hron et al. (2010) to propose a new power–law–like model for generalized viscosity of synovial fluid, where the shear–thinning index itself is a function of the concentration. Such a model is able to describe the synovial fluid viscous properties in a better way than the models used up to that time, and moreover, it naturally captures the character of decreasing concentration to zero for which the non–Newtonian effects become more and more diminishing. For detailed motivation and mathematical modeling of the synovial fluid we refer to the thesis

[1]Here, one needs to have in mind the biochemical constitution of synovial fluid, that the molecule chains are strongly hydrated, creating in the fluid "background" a quasi–continuous network.

[2]From biochemical point of view, this is a consequence of the chemical interactions between the hyaluronan molecules

Pustějovská (2012).

Based on the description above, we shall study a system of equations describing a flow of a shear–thinning/thickening fluid with a non–standard growth condition on the viscosity. Explicitly, we assume the generalized incompressible Navier–Stokes equations with a power–law–like viscosity wherein the power–law–like index is non–constant, depending on the concentration. It is clear, that additional governing equation for the concentration needs to be included to the equation system. For the balance of concentration we use the convection–diffusion equation[3], which results in a system that is fully coupled (see Bulíček et al. (2009)). Similar systems have been already studied for example by Růžička (2000), motivated by the electro–rheological fluid, or by Antontsev and Rodrigues (2006), describing a flow of thermo–mechanical fluids. In both cases, the power–law–like index was a function of another physical quantity, electric field and temperature, respectively. For closer comparison see following section of Historical remarks.

The system of equations, we are interested in, is described in the terms of the velocity field $\boldsymbol{v} : \Omega \to \mathbb{R}^d$, the pressure field $\pi : \Omega \to \mathbb{R}$ and the scalar field of concentration distribution $c : \Omega \to \mathbb{R}_+$, and consists of the generalized homogeneous incompressible Navier–Stokes equations coupled with the equation of the convection–diffusion for the concentration

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.1}$$

$$\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{S}(c, \boldsymbol{D}(\boldsymbol{v})) = -\nabla\pi + \boldsymbol{f}, \tag{1.2}$$

$$\operatorname{div}(c\boldsymbol{v}) - \operatorname{div} \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}(\boldsymbol{v})) = 0, \tag{1.3}$$

that is supposed to be satisfied in an open bounded domain $\Omega \subset \mathbb{R}^d$. Here $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ represents a given density of the bulk force, $\boldsymbol{D}(\boldsymbol{v})$ denote the symmetric part of the velocity gradient $\nabla\boldsymbol{v}$, it means $\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^{\mathrm{T}})$, and, $\boldsymbol{S}(c, \boldsymbol{D}(\boldsymbol{v}))$ and $\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}(\boldsymbol{v}))$ are the extra stress tensor of the Cauchy stress tensor and the concentration flux, respectively, given as

$$\boldsymbol{S}(c, \boldsymbol{D}(\boldsymbol{v})) = \nu(c, |\boldsymbol{D}(\boldsymbol{v})|^2)\boldsymbol{D}(\boldsymbol{v}), \qquad \boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}(\boldsymbol{v})) = \boldsymbol{K}(c, |\boldsymbol{D}(\boldsymbol{v})|)\nabla c. \tag{1.4}$$

Here $\nu(c, |\boldsymbol{D}(\boldsymbol{v})|^2)$ denotes the generalized viscosity of the power–law type, dependent of the shear–rate and concentration

$$\nu(c, |\boldsymbol{D}(\boldsymbol{v})|^2) \sim \nu_0 \left(\kappa_1 + \kappa_2 |\boldsymbol{D}(\boldsymbol{v})|^2\right)^{\frac{p(c)-2}{2}}, \tag{1.5}$$

ν_0, κ_1, κ_2 stand for constants, $\boldsymbol{K}(c, |\boldsymbol{D}(\boldsymbol{v})|) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ denotes the diffusivity tensor function, at the moment not specified, and to $p(\cdot)$ we shall refer as to a variable

[3]Generally, one could even think of the extension of the equation upon the volumetric reaction term. This, however, is not proper for the application we have in mind. The hyaluronan molecules in the synovial fluid of a healthy person do not undergo covalent structural changes, and, they are created exclusively by the cells outside the joint cavity.

exponent. To complete the problem (1.1)–(1.3) we prescribe the Dirichlet boundary data, i.e., we assume that

$$\boldsymbol{v} = \boldsymbol{0}, \qquad c = c_d \qquad \text{on } \partial\Omega. \tag{1.6}$$

Finally, to end this introductory part, we formulate the main result of the paper. Please note that the theorem is stated very vaguely and in fact requires a precise definition of what we mean by a weak solution and consequently a proper definition of function spaces. All of those are provided in the next section.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $d \geqslant 2$. Assume that p is Hölder continuous function such that*

$$\max\left\{\frac{3d}{d+2}, \frac{d}{2}\right\} < p^- \leqslant p(c) \leqslant p^+ < \infty \qquad \text{for all } c \in \mathbb{R} \tag{1.7}$$

and let $\boldsymbol{K}$ be uniformly elliptic. In addition suppose that $c_d \in W^{1,q}(\Omega)$ for some $q > d$ and $\boldsymbol{f} \in (W_0^{1,p^-}(\Omega))^*$. Then there exists a weak solution to (1.1)–(1.6).

2 Historical remarks

The mathematical analysis of generalized Navier–Stokes equations for incompressible fluid with non–constant viscosity, particularly shear–rate dependent, started to be point of interest in late 60' due to fundamental works Ladyzhenskaya (1967) and Ladyzhenskaya (1969). There the author established the existence of a weak solution for $p \geqslant \frac{3d}{d+2}$, p being constant power–law index, using the theory of monotone operators. This result was then generalized by using the so–called L^∞ and Lipschitz approximation methods, obtaining the existence of a weak solution for constant[4] $p > \frac{2d}{d+2}$. We refer the interested reader to the series of papers Málek et al. (1993), Frehse et al. (1997), Růžička (1997), Málek et al. (2001), Frehse et al. (2000), Frehse et al. (2003) and Diening et al. (2010b), where a detailed description of the methods used for both, steady and evolutionary case can be found.

The coupled system of generalized Navier–Stokes equations with the convection–diffusion–reaction equation, considering the viscosity of the form $\nu(c, |\boldsymbol{D}(\boldsymbol{v})|^2) \sim \nu_1(c)\nu_2(|\boldsymbol{D}(\boldsymbol{v})|^2)^{\frac{p-2}{2}}$ was studied by Bulíček et al. (2009). There the authors treated the evolutionary case and established a long time and large data existence of a weak solution for constant $p > \frac{8}{5}$ with the help of L^∞ truncation method. Note here that the result presented in Bulíček et al. (2009) can be easily extended to the steady case to obtain the existence of a weak solution for constant $p > \frac{2d}{d+1}$.

The models with variable power–law index, developed for electrorheological fluids, are studied for instance in Růžička (2000) and Růžička (2004). For this kind of fluids the extra stress tensor is (non–trivially) dependent of electric field $\boldsymbol{E}$ and thus the Navier–Stokes

[4]Note that such a bound is in fact very natural since for lower p's we are not able to control the convective term.

equations have to be solved with the (quasi–static) Maxwell equations. Nevertheless, the governing equations are essentially uncoupled, hence the Maxwell equations can be solved first. The solution of electric field can be then considered as a known function, resulting that the problem reduces to the problem of incompressible Navier–Stokes problem with extra stress tensor having the growth property of $|\boldsymbol{S}| \leqslant C(1+|\boldsymbol{D}(\boldsymbol{v})|^2)^{\frac{p(x)-2}{2}}$, where $p(x) :=p(|\boldsymbol{E}(x)|^2)$ is a given function (under some assumption of Hölder continuity), satisfying $1 < p^- < p(x) < p^+ < \infty$. Using the theory of monotone operators, the author was able to prove the existence of a weak solution for the lower bound $p^- \geqslant \frac{3d}{d+2}$, and in the case of stationary problem the existence result was extended to $p^- \geqslant \frac{2d}{d+2}$ by Diening et al. (2008), by the means of the method of Lipschitz approximations.

The closest system to ours, (1.1)–(1.3), is studied in Antontsev and Rodrigues (2006). The authors considered the stationary system of generalized Navier–Stokes equations coupled with equation for thermal diffusion obtained as Oberbeck–Boussinesq approximation of the heat equation, under the consideration that the power–law index depends on the temperature θ. For Dirichlet boundary conditions, for both velocity and temperature, they prove the existence of a weak solution for the case of $\frac{3d}{d+2} \leqslant p^- < p(\theta) < p^+ < \infty$ for $d = 2, 3$ and sufficiently smooth data by using the fixed point argument. There, the important simplification is the assumption of the constant diffusion tensor (similar to our $\boldsymbol{K}$), which ensures the Hölder continuity of the temperature. On the other hand, in case that $\boldsymbol{K}$ depends on the concentration and the shear rate, such procedure is simply not possible due to the lack of the compactness of the corresponding operator required by the use of a fix–point theorem.

In view of this, it is of real interest to generalize the monotone operator theory and the Galerkin method in a proper way to prove the existence of a weak solution to (1.1)–(1.3) for general diffusion $\boldsymbol{K}$. Moreover, such a proof can then help to find a generalization of L^∞ and Lipschitz approximation method to overcome the bound $p > \frac{3d}{d+2}$ in Theorem 1.1. This we shall present in a forthcoming paper.

3 Notation and auxiliary results

It is obvious, that for the problem of non–constant viscosity with variable exponent of the power–law, the functional setting of classical Lebesgue/Sobolev spaces is not appropriate and more general approach needs to be introduced, in this case the setting of generalized Lebesgue/Sobolev spaces with variable exponent p. Such spaces have been studied by many authors, see for instance Orlicz (1931), Nakano (1950), Sharapudinov (1978), Musielak (1983), Kováčik and Rákosník (1991), Fan et al. (2001) and a compact and self–contained book Diening et al. (2011).

As in the above mentioned references, we shall use the following notation. We denote the set of all measurable functions $p : \Omega \to [1, \infty]$ by $\mathcal{P}(\Omega)$, and call the function $p \in \mathcal{P}(\Omega)$ a variable exponent. Moreover, we define $p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$. For simplicity we restrict ourselves only to the case when $1 < p^- \leqslant p^+ < \infty$. The dual variable exponent $p' \in \mathcal{P}(\Omega)$ is defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Next, for arbitrary variable exponent p, satisfying the above bounds, we introduce a generalized Lebesgue space

$$L^{p(\cdot)}(\Omega) := \{u \in L^1_{loc}(\Omega); \int_\Omega |u(x)|^{p(x)}\, dx < \infty\},$$

which is equipped with the Luxembourg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}.$$

In the same spirit we define the generalized Sobolev spaces

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in W^{1,1}(\Omega) \cap L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

with a norm

$$\|u\|_{1,p(\cdot),\Omega} = \|u\|_{1,p(\cdot)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} + \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}.$$

All above defined spaces are separable and reflexive Banach spaces (again, we use the simplification $1 < p^- \leqslant p^+ < \infty$). Moreover, $\left(L^{p(\cdot)}(\Omega)\right)^* = L^{p'(\cdot)}(\Omega)$ holds for any $p \in \mathcal{P}(\Omega)$, and the space $\mathcal{D}(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$. However, having such density (and further properties) also for generalized Sobolev spaces, an additional condition on the variable exponent is required (see e.g. Zhikov (1987), Fan et al. (2006), Hästö (2005) and Diening et al. (2005) for counter–examples). Therefore, we introduce a class of log–Hölder continuous exponents, i.e., we define $\mathcal{P}^{\log}(\Omega)$ as the set of all $p \in \mathcal{P}(\Omega)$ which satisfy

$$|p(x) - p(y)| \leqslant \frac{C_1}{-\ln|x-y|} \quad \forall\, x, y \in \Omega : \, 0 < |x-y| \leqslant \frac{1}{2}. \tag{3.1}$$

Under such additional regularity, we obtain for all $p \in \mathcal{P}^{\log}(\Omega)$ and arbitrary open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary

- The density of smooth functions, i.e.,

$$\overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\cdot\|_{1,p(\cdot)}} = W^{1,p(\cdot)}(\Omega). \tag{3.2}$$

- The embedding theorem, i.e., if $1 < p^- \leqslant p^+ < d$ then

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \qquad 1 \leqslant q(x) \leqslant \frac{dp(x)}{d - p(x)} =: p^*(x). \tag{3.3}$$

  Moreover, if $q(x) < p^*(x)$ for almost all $x \in \Omega$, the embedding is compact.

- Korn inequality

$$\|\nabla \boldsymbol{v}\|_{p(\cdot)} \leqslant C\|\boldsymbol{D}(\boldsymbol{v})\|_{p(\cdot)} \tag{3.4}$$

  for all $\boldsymbol{v} \in [W_0^{1,p(\cdot)}]^d$.

The proof of (3.4) can be found in Diening et al. (2010a) or Diening et al. (2010b), Section 14.3.

Additionally to the Lebesgue and Sobolev spaces and their generalizations, we introduce the function spaces relevant for the treatment of the problems of incompressible fluids. By $W_0^{1,r}(\Omega)$ and $W_{0,\mathrm{div}}^{1,r}(\Omega)$ we define the spaces

$$\begin{aligned} W_0^{1,r}(\Omega) &:= \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{1,r}}, \\ W_{0,\mathrm{div}}^{1,r}(\Omega) &:= \{\boldsymbol{\phi} \in [W_0^{1,r}(\Omega)]^d : \mathrm{div}\, \boldsymbol{\phi} = 0\}. \end{aligned}$$

Under the assumption of $\partial\Omega \in \mathcal{C}^{0,1}$, there holds $W_0^{1,r}(\Omega) \equiv \{\phi \in W^{1,r}(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. Similarly we have

$$\begin{aligned} W_0^{1,r(\cdot)}(\Omega) &:= \{\phi \in W^{1,r(\cdot)}(\Omega) : \phi = 0 \text{ on } \partial\Omega\}, \\ W_{0,\mathrm{div}}^{1,r(\cdot)}(\Omega) &:= \{\boldsymbol{\phi} \in [W_0^{1,r(\cdot)}(\Omega)]^d : \mathrm{div}\, \boldsymbol{\phi} = 0\}. \end{aligned}$$

For the notation of the duality pairing between $f \in X$ and $g \in X^\star$ we use symbol $\langle f, g\rangle_{X,X^\star}$, or, if it is obvious from the context, we skip for simplicity the indexes and write $\langle f, g\rangle$.

Finally, we recall the famous result of De Giorgi (1957) and Nash (1958), see also Bensoussan and Frehse (2002), Chapter 2, applied to our problem as following theorem

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded set with Lipschitz boundary and let $q > d$ be given. Assume that that there are $C_1, C_2 > 0$ such that*

$$K_{ij} \in L^\infty, \qquad |K_{ij}| \leqslant C_1, \qquad K_{ij} b_i b_j \geqslant C_2 |\boldsymbol{b}|^2 \text{ for all } \boldsymbol{b} \in \mathbb{R}^d. \tag{3.5}$$

Then there exists $\alpha_0 > 0$ depending only on Ω, C_1, C_2 and q, such that for any $\boldsymbol{g} \in [L^q(\Omega)]^d$ and any $c_d \in W^{1,q}(\Omega)$ there exists unique $c \in W^{1,2}(\Omega)$ such that $c - c_d \in W_0^{1,2}(\Omega) \cap \mathcal{C}^{0,\alpha_0}(\Omega)$ solving

$$\int_\Omega K_{ij} \frac{\partial c}{\partial x_j} \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega \boldsymbol{g} \cdot \nabla \varphi \, dx \qquad \forall \varphi \in W_0^{1,2}(\Omega), \tag{3.6}$$

and fulfilling the uniform estimate

$$\|c\|_{W^{1,2} \cap \mathcal{C}^{0,\alpha_0}} \leqslant C(\Omega, C_1, C_2, q, \|\boldsymbol{g}\|_q, \|c_d\|_{1,q}). \tag{3.7}$$

4 The precise statement of the result

Having already introduced the function spaces, we can precisely reformulate Theorem 1.1. First, recall that $\boldsymbol{q}_c$ is given by

$$\boldsymbol{q}_c(c, \nabla c, \boldsymbol{D}(\boldsymbol{v})) = \boldsymbol{K}(c, \boldsymbol{D}(\boldsymbol{v})) \nabla c. \tag{4.1}$$

We assume that $\boldsymbol{S}:\mathbb{R}_0^+\times\mathbb{R}^{d\times d}_{sym}\to\mathbb{R}^{d\times d}_{sym}$ fulfills following growth, strict monotonicity and coercivity conditions for all $c\in\langle\min_{x\in\partial\Omega}c_d,\max_{x\in\partial\Omega}c_d\rangle$ and all $\boldsymbol{D}$, $\boldsymbol{D}_1$, $\boldsymbol{D}_2\in\mathbb{R}^{d\times d}_{sym}$

$$|\boldsymbol{S}(c,\boldsymbol{D})|\leqslant C_1(|\boldsymbol{D}|^{p(c)-1}+1), \tag{4.2}$$

$$(\boldsymbol{S}(c,\boldsymbol{D}_1)-\boldsymbol{S}(c,\boldsymbol{D}_2))\cdot(\boldsymbol{D}_1-\boldsymbol{D}_2)>0 \quad \boldsymbol{D}_1\neq\boldsymbol{D}_2, \tag{4.3}$$

$$\boldsymbol{S}(c,\boldsymbol{D})\cdot\boldsymbol{D}\geqslant C_2(|\boldsymbol{D}|^{p(c)}+|\boldsymbol{S}(c,\boldsymbol{D})|^{p'(c)}-1), \tag{4.4}$$

where $p(\cdot)$ is Hölder continuous function such that $1<p^-\leqslant p(\cdot)\leqslant p^+<\infty$, and the concentration flux vector $\boldsymbol{q}_c$ satisfies (4.1), where $\boldsymbol{K}(c,|\boldsymbol{D}(\boldsymbol{v})|):\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}^{d\times d}$ is continuous mapping with $K_{i,j}\in L^\infty(\Omega)$ such that the flux vector fulfills following conditions

$$|\boldsymbol{q}_c(c,\xi,\boldsymbol{D})|\leqslant K_1|\xi|, \tag{4.5}$$

$$\boldsymbol{q}_c(c,\xi,\boldsymbol{D})\cdot\xi\geqslant K_2\,|\xi|^2\,. \tag{4.6}$$

Above $C_1,C_2,K_1,K_2\in(0,\infty)$ are constants and $\boldsymbol{A}\cdot\boldsymbol{B}$ is notation for the scalar product between two tensors.

Now, we are ready to formulate the main theorem.

Theorem 4.1. *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let p be a Hölder continuous function such that $p^-\leqslant p(c)\leqslant p^+<\infty$ for all c, where $p^-\geqslant\frac{3d}{d+2}$ and $p^->\frac{d}{2}$. Assume that $\boldsymbol{f}\in(W^{1,p^-}_{0,\mathrm{div}}(\Omega))^*$, and $\boldsymbol{S}$ and $\boldsymbol{q}_c$ satisfy conditions (4.2)–(4.6). Moreover, let $c_d\in W^{1,q}(\Omega)$ for some $q>d$. Then there exists $(\boldsymbol{v},c)$ such that for some $\alpha>0$*

$$\boldsymbol{v}\in W^{1,p(c)}_{0,\mathrm{div}}(\Omega), \tag{4.7}$$

$$(c-c_d)\in\mathcal{C}^{0,\alpha}(\Omega)\cap W^{1,2}_0(\Omega), \tag{4.8}$$

fulfilling

$$-\int_\Omega\boldsymbol{v}\otimes\boldsymbol{v}\cdot\nabla\boldsymbol{\psi}\,dx+\int_\Omega\boldsymbol{S}(c,\boldsymbol{D}(\boldsymbol{v}))\cdot\boldsymbol{D}(\boldsymbol{\psi})\,dx=\langle\boldsymbol{f},\boldsymbol{\psi}\rangle \qquad \forall\boldsymbol{\psi}\in W^{1,p(c)}_{0,\mathrm{div}}(\Omega), \tag{4.9}$$

$$-\int_\Omega\boldsymbol{v}c\cdot\nabla\varphi\,dx+\int_\Omega\boldsymbol{q}_c(c,\nabla c,\boldsymbol{D}(\boldsymbol{v}))\cdot\nabla\varphi\,dx=0 \qquad \forall\varphi\in W^{1,2}_0(\Omega). \tag{4.10}$$

5 Proof of the main theorem

We present the proof based on Galerkin method together with the application of the generalized theory of monotone operators. From that, we are restricted to the assumption of $p^-\geqslant\frac{3d}{d+2}$ (in 3D setting $p^-\geqslant\frac{9}{5}$) which is required for the convective term $(\boldsymbol{v}\otimes\boldsymbol{v})$ to be well defined for the test functions from $W^{1,p(c)}_{0,\mathrm{div}}(\Omega)$. The second restriction on the minimal value of p in Theorem 4.1, explicitly $p^->\frac{d}{2}$, comes from the necessity of c being Hölder continuous, ensuring the Hölder continuity of the variable exponent which is a crucial assumption for the density of smooth functions in generalized Sobolev spaces, embedding theorems and Korn inequality. For that we use Theorem 3.1, where, nevertheless, it is required that $v\in L^q(\Omega)$ for $q>d$, and therefore, using the Sobolev embedding, the second bound on p arises.

$(\boldsymbol{n},\, \boldsymbol{m})$–approximate problem and uniform estimates

Let $\{\boldsymbol{w}_i\}_{i=1}^{\infty}$ be a basis of $W^{1,p^+}_{0,\mathrm{div}}(\Omega)$ such that $\int_\Omega \boldsymbol{w}_i \boldsymbol{w}_j\, dx = \delta_{ij}$ and $\{z_j\}_{j=1}^{\infty}$ be a basis of $W_0^{1,2}(\Omega)$, again $\int_\Omega z_i z_j\, dx = \delta_{ij}$. Then for positive, fixed n, $m \in \mathbb{N}$ we define the Galerkin approximations

$$\boldsymbol{v}^{n,m} := \sum_{i=1}^{n} \alpha_i^{n,m} \boldsymbol{w}_i, \qquad\qquad c^{n,m} := \sum_{i=1}^{m} \beta_i^{n,m} z_i + c_d, \tag{5.1}$$

for which $\boldsymbol{\alpha}^{n,m}$ and $\boldsymbol{\beta}^{n,m}$ solve the approximate system

$$-\int_\Omega (\boldsymbol{v}^{n,m} \otimes \boldsymbol{v}^{n,m}) \cdot \nabla \boldsymbol{w}_i\, dx + \int_\Omega \boldsymbol{S}(c^{n,m}, \boldsymbol{D}(\boldsymbol{v}^{n,m})) \cdot \boldsymbol{D}(\boldsymbol{w}_i)\, dx = \langle \boldsymbol{f}, \boldsymbol{w}_i \rangle \tag{5.2}$$
$$\forall i = 1, \ldots, n,$$

$$-\int_\Omega \boldsymbol{v}^{n,m} c^{n,m} \cdot \nabla z_j\, dx + \int_\Omega \boldsymbol{q}_c(c^{n,m}, \nabla c^{n,m}, \boldsymbol{D}(\boldsymbol{v}^{n,m})) \cdot \nabla z_j\, dx = 0 \tag{5.3}$$
$$\forall j = 1, \ldots, m.$$

The straightforward application of the fixed point theorem, provided the later derived uniform estimates, then ensures the existence of the solution $(\boldsymbol{v}^{n,m}, c^{n,m})$ of the approximate system (5.2)–(5.3).

Now, we shall derive the uniform estimates that are independent of n, m in the corresponding function spaces. For simplicity, we shall use the notation of $\boldsymbol{S}^{n,m} := \boldsymbol{S}(c^{n,m}, \boldsymbol{D}(\boldsymbol{v}^{n,m}))$ and $\boldsymbol{q}_c^{n,m} := \boldsymbol{q}_c(c^{n,m}, \nabla c^{n,m}, \boldsymbol{D}(\boldsymbol{v}^{n,m}))$. Multiplying the i–th equation in (5.2) by $\alpha_i^{n,m}$ and taking the sum over $i = 1, ..., n$, we obtain

$$-\int_\Omega (\boldsymbol{v}^{n,m} \otimes \boldsymbol{v}^{n,m}) \cdot \nabla \boldsymbol{v}^{n,m}\, dx + \int_\Omega \boldsymbol{S}^{n} \cdot \boldsymbol{D}(\boldsymbol{v}^{n,m})\, dx = \langle \boldsymbol{f}, \boldsymbol{v}^{n,m} \rangle, \tag{5.4}$$

where the convective term vanishes after integration by parts since $\operatorname{div} \boldsymbol{v}^{n,m} = 0$ in Ω and $\boldsymbol{v}^{n,m} = \boldsymbol{0}$ on $\partial\Omega$. The equation (5.4) thus reduces to

$$\int_\Omega \boldsymbol{S}^{n,m} \cdot \boldsymbol{D}(\boldsymbol{v}^{n,m})\, dx = \langle \boldsymbol{f}, \boldsymbol{v}^{n,m} \rangle, \tag{5.5}$$

and further, using the assumptions (4.4), (4.2) and standard duality estimates with Young and Korn inequalities on the second term, we obtain that

$$\int_\Omega |\boldsymbol{D}(\boldsymbol{v}^{n,m})|^{p(c^{n,m})}\, dx < C. \tag{5.6}$$

Next, multiplying the j–th equation in (5.3) by $\beta_j^{n,m}$ and taking the sum over $j = 1, ..., m$, we arrive at

$$-\int_\Omega \boldsymbol{v}^{n,m} c^{n,m} \cdot \nabla (c^{n,m} - c_d)\, dx + \int_\Omega \boldsymbol{q}_c^{n,m} \cdot \nabla (c^{n,m} - c_d)\, dx = 0. \tag{5.7}$$

Again, the integration by parts of the first term and the assumptions of $\operatorname{div} \boldsymbol{v}^{n,m} = 0$, $\boldsymbol{v}^{n,m} = \mathbf{0}$ on $\partial\Omega$ reduce the equation (5.7) to

$$\int_\Omega \boldsymbol{q}_c^{n,m} \cdot \nabla c^{n,m} \, dx = \int_\Omega \boldsymbol{q}_c^{n,m} \cdot \nabla c_d \, dx + \int_\Omega c_d \nabla c^{n,m} \cdot \boldsymbol{v}^{n,m} \, dx. \tag{5.8}$$

After the use of assumption (4.5), the first therm on right hand side is estimated by the use of the Hölder inequality followed by standard use of Young inequality. The second term is estimated by the fact that c_d is bounded and thus we can use the Hölder inequality and consequently the Young inequality. All together with assumption (4.6), we arrive at

$$\int_\Omega |\nabla c^{n,m}|^2 \, dx \leqslant C(1 + \|\boldsymbol{v}^{n,m}\|_2^2) \leqslant C, \tag{5.9}$$

where we estimated the last term by the use of the Korn inequality on (5.6) together with the embedding $W^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega)$ for $p^- \geqslant \frac{2d}{d+2}$.

Moreover, it is easy to show as a direct consequence of assumption (4.2) and estimates (5.6), (5.9), that

$$\int_\Omega \left(|\boldsymbol{S}^{n,m}|^{p^{+'}} + |\boldsymbol{q}_c^{n,m}|^2 \right) \, dx \leqslant C. \tag{5.10}$$

Limit $m \to \infty$

Having the uniform estimate (5.6), the equivalence of norms in the finite dimensional spaces leads to $|\boldsymbol{\alpha}^{n,m}| \leqslant C(n)$. Then, together with estimate (5.9), we can establish the following $m \to \infty$ convergence results for a suitable sub–sequences (for simplification not relabeled)

$$\begin{aligned}
\boldsymbol{\alpha}^{n,m} &\to \boldsymbol{\alpha}^n && \text{strongly in } \mathbb{R}^n, && (5.11)\\
c^{n,m} &\rightharpoonup c^n && \text{weakly in } W^{1,2}(\Omega), && (5.12)\\
\boldsymbol{S}^{n,m} &\rightharpoonup \bar{\boldsymbol{S}}^n && \text{weakly in } [L^{p^{+'}}(\Omega)]^{d\times d}, && (5.13)\\
\boldsymbol{q}_c^{n,m} &\rightharpoonup \bar{\boldsymbol{q}}_c^n && \text{weakly in } [L^2(\Omega)]^d, && (5.14)
\end{aligned}$$

and thus directly from (5.11) and the compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ we obtain the convergences results

$$\begin{aligned}
\boldsymbol{v}^{n,m} &\to \boldsymbol{v}^n && \text{strongly in } W^{1,p^+}_{0,\mathrm{div}}(\Omega), && (5.15)\\
c^{n,m} &\to c^n && \text{strongly in } L^2(\Omega). && (5.16)
\end{aligned}$$

Since $\boldsymbol{S}$ and $\boldsymbol{q}_c$ are continuous with respect to their unknowns, (5.15)–(5.16), growth condition on $\boldsymbol{S}$ and $\boldsymbol{q}_c$ and the linearity of $\boldsymbol{q}_c$ with respect to the gradient of c imply that

$$\bar{\boldsymbol{S}}^n = \boldsymbol{S}^n := \boldsymbol{S}(c^n, \boldsymbol{D}(\boldsymbol{v}^n)), \tag{5.17}$$

$$\bar{\boldsymbol{q}}_c^n = \boldsymbol{q}_c^n := \boldsymbol{q}_c(c^n, \nabla c^n, \boldsymbol{D}(\boldsymbol{v}^n)). \tag{5.18}$$

The convergence results allow us to make the limit passage in the equation set (5.2)–(5.3) and thus we obtain following system

$$-\int_\Omega (\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \cdot \nabla \boldsymbol{w}_i \, dx + \int_\Omega \boldsymbol{S}^n \cdot \boldsymbol{D}(\boldsymbol{w}_i) \, dx = \langle \boldsymbol{f}, \boldsymbol{w}_i \rangle \qquad \forall i = 1, \dots, n, \tag{5.19}$$

$$-\int_\Omega \boldsymbol{v}^n c^n \cdot \nabla \varphi \, dx + \int_\Omega \boldsymbol{q}_c^n \cdot \nabla \varphi \, dx = 0 \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{5.20}$$

Minimum and maximum principle and further a priori estimates

Define $z_1^n := (c^n - \min_{x\in\partial\Omega} c_d)_-$ and $z_2^n := (c^n - \max_{x\in\partial\Omega} c_d)_+$, where $(a)_-$ and $(a)_+$ denote the negative and positive parts of a, respectively. It is clear, that functions $z_1^n, z_2^n \in W_0^{1,2}(\Omega)$ since $c^n = c_d$ on $\partial\Omega$, and thus, from (5.20), we get

$$-\int_\Omega \boldsymbol{v}^n c^n \cdot \nabla z_1^n \, dx + \int_\Omega \boldsymbol{q}_c^n \cdot \nabla z_1^n \, dx = 0, \tag{5.21}$$

and

$$-\int_\Omega \boldsymbol{v}^n c^n \cdot \nabla z_2^n \, dx + \int_\Omega \boldsymbol{q}_c^n \cdot \nabla z_2^n \, dx = 0. \tag{5.22}$$

First, let us consider equation (5.21). Using $\operatorname{div} \boldsymbol{v}^n = 0$ in Ω and $\boldsymbol{v}^n = \boldsymbol{0}$ on $\partial\Omega$ on the first term, (4.6) and the property of negative part on the second term, we arrive at

$$\int_{\Omega^-} \boldsymbol{v}^n \cdot \nabla c^n z_1^n \, dx + \int_{\Omega^-} K_2 |\nabla c^n|^2 \, dx \leqslant 0, \tag{5.23}$$

where Ω^- is the part of the domain on which $z_1^n < 0$. Next, using that $\nabla c^n = \nabla z_1^n$ on Ω^- and again the extension of ∇c^n from Ω^- on the whole domain Ω by using the negative part, we obtain

$$\int_\Omega \boldsymbol{v}^n \cdot \nabla z_1^n z_1^n \, dx + \int_\Omega K_2 |\nabla z_1^n|^2 \, dx = 0. \tag{5.24}$$

Application of the chain rule on the first term and the integration by parts, we find that the first term vanishes, and thus

$$z_1^n = (c^n - \min_{x\in\partial\Omega} c_d)_- = \text{const.} \quad \text{a. e. in } \Omega. \tag{5.25}$$

Analogous treatment can be used on the equation (5.22) which consequently leads to $z_2^n = (c^n - \max_{x\in\partial\Omega} c_d)_+ = \text{const.}$ a. e. in Ω. Combining both results, we obtain the boundedness of the concentration c^n

$$\min_{x\in\partial\Omega} c_d \leqslant c^n \leqslant \max_{x\in\partial\Omega} c_d \quad \text{a. e. in } \Omega. \tag{5.26}$$

Similar to the previous subsection, we can establish further uniform estimates. Multiplying the i–th equation in (5.19) by α_i^n and taking the sum over $i = 1, ..., n$, we obtain, after integration by parts together with condition (4.4) and using the same arguments on the term with force as above,

$$\int_\Omega (|\boldsymbol{D}(\boldsymbol{v}^n)|^{p(c^n)} + |\boldsymbol{S}^n|^{p'(c^n)})\, dx \leqslant C, \tag{5.27}$$

and thus, using (4.2),

$$\int_\Omega |\boldsymbol{D}(\boldsymbol{v}^n)|^{p^-}\, dx \leqslant C. \tag{5.28}$$

In equation (5.20) we can already use the approximation c^n as a test function and thus, similar as above, obtain

$$\int_\Omega |\nabla c^n|^2\, dx \leqslant C. \tag{5.29}$$

Additionally, it is easy to show from above estimates and (4.5) that

$$\int_\Omega \left(|\boldsymbol{S}^n|^{p^{+'}} + |\boldsymbol{q}_c^n|^2 \right) dx \leqslant C. \tag{5.30}$$

Since c^n is bounded, see (5.26), we can apply Theorem 3.1, and thus

$$\|c^n\|_{0,\alpha_0} \leqslant C \qquad \text{for some } \alpha_0 > 0. \tag{5.31}$$

However, here we need to assume $\boldsymbol{v}^n \in [L^q(\Omega)]^d$, where $q > d$. For that we use the embedding $W^{1,p^-}_{0,\mathrm{div}}(\Omega) \hookrightarrow L^q(\Omega)$, which raise the second condition on the lower bound of the variable index, $p^- > \frac{d}{2}$.

Limit $n \to \infty$

It follows from the estimates (5.26)–(5.30) that there exist $\boldsymbol{v}$ and c such that for some (again not relabeled) subsequences

$$c^n \rightharpoonup c \qquad \text{weakly in } W^{1,2}(\Omega), \tag{5.32}$$

$$c^n \overset{*}{\rightharpoonup} c \qquad *\text{–weakly in } \mathcal{C}^{0,\alpha_0}(\Omega), \tag{5.33}$$

$$\boldsymbol{v}^n \rightharpoonup \boldsymbol{v} \qquad \text{weakly in } W^{1,p^-}_{0,\mathrm{div}}(\Omega), \tag{5.34}$$

$$\boldsymbol{S}^n \rightharpoonup \overline{\boldsymbol{S}} \qquad \text{weakly in } [L^{p^{+'}}(\Omega)]^{d\times d}, \tag{5.35}$$

$$\boldsymbol{q}_c^n \rightharpoonup \overline{\boldsymbol{q}}_c \qquad \text{weakly in } [L^2(\Omega)]^d, \tag{5.36}$$

and thus, from the compact embedding $W^{1,p^-}_{0,\mathrm{div}}(\Omega) \hookrightarrow\hookrightarrow L^2_{\mathrm{div}}(\Omega)$ and above convergence results

$$c^n \to c \qquad \text{strongly in } \mathcal{C}^{0,\alpha}(\Omega), \quad 0 < \alpha < \alpha_0, \tag{5.37}$$

$$\boldsymbol{v}^n \to \boldsymbol{v} \qquad \text{strongly in } L^2_{\mathrm{div}}(\Omega). \tag{5.38}$$

If we identify the limits $\overline{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}(\boldsymbol{v}))$ and $\overline{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \boldsymbol{D}(\boldsymbol{v}), \nabla c)$, these convergence results on the second level of the Galerkin approximations, together with the density of smooth functions argument, lead us to the full weak formulation

$$-\int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\psi}\, dx + \int_\Omega \boldsymbol{S}(c, \boldsymbol{D}(v)) \cdot \boldsymbol{D}(\boldsymbol{\psi})\, dx = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \qquad \forall \boldsymbol{\psi} \in W^{1,p(c)}_{0,\mathrm{div}}(\Omega), \tag{5.39}$$

$$-\int_\Omega \boldsymbol{v} c \cdot \nabla \varphi\, dx + \int_\Omega \boldsymbol{q}_c(c, \boldsymbol{D}(\boldsymbol{v}), \nabla c) \cdot \nabla \varphi\, dx = 0 \qquad \forall \varphi \in W^{1,2}_0(\Omega). \tag{5.40}$$

Identification of the limit $\overline{\boldsymbol{S}}$ and $\overline{\boldsymbol{q}}_c$

First, we need to show that $\overline{\boldsymbol{S}}$ and the convective term $(\boldsymbol{v} \otimes \boldsymbol{v})$ belong to the right dual space $L^{p'(c)}(\Omega)$. Then we will be able to test the equation for velocity with the functions from energy space corresponding to velocity. After, we shall use the theory of monotone operators to identify the non–linear term $\overline{\boldsymbol{S}}$ and at the end, we unify the limit of the concentration flux $\overline{\boldsymbol{q}}_c$.

We start with the limit equation which can be obtained from (5.19) by letting $n \to \infty$ together with (5.35) and (5.38)

$$-\int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \nabla \boldsymbol{\psi}\, dx + \int_\Omega \overline{\boldsymbol{S}} \cdot \boldsymbol{D}(\boldsymbol{\psi})\, dx = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \qquad \forall \boldsymbol{\psi} \in W^{1,p^+}_{0,\mathrm{div}}(\Omega). \tag{5.41}$$

Let us divide the domain Ω on parts where $|\overline{\boldsymbol{S}}| < 1$ and $|\overline{\boldsymbol{S}}| \geqslant 1$, and denote them in the same manner, then

$$\int_\Omega |\overline{\boldsymbol{S}}|^{p'(c)}\, dx \leqslant \int_{|\overline{\boldsymbol{S}}|<1} |\overline{\boldsymbol{S}}|^{p'(c)}\, dx + \int_{|\overline{\boldsymbol{S}}|\geqslant 1} |\overline{\boldsymbol{S}}|^{p'(c)}\, dx. \tag{5.42}$$

It is obvious, that first term is bounded by a constant due to the boundedness of the variable exponent itself. Next, using (5.37) and the continuity of p, it holds that

$$\forall \varepsilon > 0\ \exists N > 0 : \quad \forall n \geqslant N \quad |p'(c^n) - p'(c)| \leqslant \frac{\varepsilon}{\Theta}, \tag{5.43}$$

where $\Theta > 1$, large enough to satisfy $p'(c) - \frac{\Theta-1}{\Theta}\varepsilon > 1$. We can then deduce from (5.27) that

$$\begin{aligned} C &\geqslant \int_\Omega |\boldsymbol{S}^n|^{p'(c^n)}\, dx \geqslant \int_{|\boldsymbol{S}^n|\geqslant 1} |\boldsymbol{S}^n|^{p'(c^n)}\, dx \\ &\geqslant \int_{|\boldsymbol{S}^n|\geqslant 1} |\boldsymbol{S}^n|^{p'(c^n)-p'(c)+p'(c)-\varepsilon}\, dx \geqslant \int_{|\boldsymbol{S}^n|\geqslant 1} |\boldsymbol{S}^n|^{p'(c)-\frac{\Theta-1}{\Theta}\varepsilon}\, dx. \end{aligned} \tag{5.44}$$

Then, after adding to the inequality the term $\int_{|\boldsymbol{S}^n|<1} |\boldsymbol{S}^n|^{p'(c)-\frac{\Theta-1}{\Theta}\varepsilon}\,dx$, which is bounded by some constant $\bar{C}$, we obtain

$$C+\bar{C} \geqslant \int_{|\boldsymbol{S}^n|\geqslant 1} |\boldsymbol{S}^n|^{p'(c^n)}\,dx + \bar{C} \geqslant \int_\Omega |\boldsymbol{S}^n|^{p'(c)-\frac{\Theta-1}{\Theta}\varepsilon}\,dx. \tag{5.45}$$

Using the weak lower semicontinuity, we see that

$$\int_\Omega |\overline{\boldsymbol{S}}|^{p'(c)-\frac{\Theta-1}{\Theta}\varepsilon}\,dx \leqslant C, \tag{5.46}$$

and consequently the Fatou lemma with $\varepsilon \to 0$ leads to

$$\int_\Omega |\overline{\boldsymbol{S}}|^{p'(c)} \leqslant C. \tag{5.47}$$

Next, to have the convective term from admissible space $L^{p'(c)}(\Omega)$, we have to set up a constrain on the lower bound of the variable index p such that $W^{1,p(c)}(\Omega) \hookrightarrow L^{2p'(c)}(\Omega)$. This is exactly the constrain $p(c) \geqslant \frac{3d}{d+2}$ from Theorem 4.1. Having this, we obtain

$$(\boldsymbol{v}\otimes\boldsymbol{v}) \in L^{p'(c)}(\Omega). \tag{5.48}$$

Now, since $p(\cdot)$ is Hölder continuous, smooth functions $C^\infty_{0,\mathrm{div}}$ are dense[5] in $W^{1,p(c)}_{0,\mathrm{div}}$, we can rewrite the weak formulation of equation for velocity as

$$-\int_\Omega (\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{D}(\boldsymbol{\psi})\,dx + \int_\Omega \overline{\boldsymbol{S}}\cdot\boldsymbol{D}(\boldsymbol{\psi})\,dx = \langle \boldsymbol{f},\boldsymbol{\psi}\rangle \qquad \forall \boldsymbol{\psi}\in W^{1,p(c)}_{0,\mathrm{div}}, \tag{5.49}$$

and consequently take the velocity $\boldsymbol{v}$ as a test function. Moreover, if we take $\boldsymbol{v}^n$ as a test function in (5.19), the convective terms vanish for both cases, and since

$$\lim_{n\to\infty} \langle \boldsymbol{f},\boldsymbol{v}^n\rangle = \langle \boldsymbol{f},\boldsymbol{v}\rangle, \tag{5.50}$$

by comparison of the right hand sides we obtain

$$\lim_{n\to\infty}\int_\Omega \boldsymbol{S}(c^n,\boldsymbol{D}(\boldsymbol{v}^n))\cdot\boldsymbol{D}(\boldsymbol{v}^n)\,dx = \int_\Omega \overline{\boldsymbol{S}}\cdot\boldsymbol{D}(\boldsymbol{v})\,dx. \tag{5.51}$$

To finish the proof, we use Minty method. Indeed, for all $\boldsymbol{\phi}\in C^\infty_{0,\mathrm{div}}(\Omega)$ it is valid that

$$0 \leqslant \int_\Omega \left(\boldsymbol{S}(c^n,\boldsymbol{D}(\boldsymbol{v}^n)) - \boldsymbol{S}(c^n,\boldsymbol{D}(\boldsymbol{\phi}))\cdot(\boldsymbol{D}(\boldsymbol{v}^n)-\boldsymbol{D}(\boldsymbol{\phi}))\right)\,dx. \tag{5.52}$$

[5]In fact, we know that smooth functions are dense in $W^{1,p(\cdot)}$. To prove also the density of smooth solenoidal functions, it is enough to apply the Bogovskii lemma in spaces with variable exponents, which holds also for log–continuous exponents as is shown in Diening et al. (2011).

Using the strong convergence (5.37) and Lebesgue dominated convergence theorem, we obtain that

$$\boldsymbol{S}(c^n, \boldsymbol{D}(\boldsymbol{\phi})) \to \boldsymbol{S}(c, \boldsymbol{D}(\boldsymbol{\phi})) \text{ strongly in } [L^q(\Omega)]^{d\times d},\ q < \infty. \tag{5.53}$$

Now, we can take the limit $n \to \infty$ because of the (5.51), the growth condition (4.2) together with the weak convergence of $\boldsymbol{S}(c^n, \boldsymbol{D}(\boldsymbol{v}^n))$ and $\boldsymbol{D}(\boldsymbol{v}^n)$ in the admissible spaces. Thus, we obtain

$$0 \leqslant \int_\Omega (\overline{\boldsymbol{S}} - \boldsymbol{S}(c, \boldsymbol{D}(\boldsymbol{\phi}))) \cdot (\boldsymbol{D}(\boldsymbol{v}) - \boldsymbol{D}(\boldsymbol{\phi}))\, dx \quad \forall \boldsymbol{\phi} \in W^{1,p(c)}_{0,\mathrm{div}}(\Omega), \tag{5.54}$$

which is already written for all $\boldsymbol{\phi} \in W^{1,p(c)}_{0,\mathrm{div}}(\Omega)$ since of the density of smooth functions. Then, the Minty trick with test functions $\boldsymbol{\phi} = \boldsymbol{v} \pm \lambda \boldsymbol{w}$, $\lambda > 0$, implies the desired identification $\overline{\boldsymbol{S}} = \boldsymbol{S}(c, \boldsymbol{D}(\boldsymbol{v}))$.

Finally, if $\boldsymbol{S}$ is strictly monotone, it follows from convergences results (5.35) and (5.37) that also $\boldsymbol{D}(\boldsymbol{v}^n) \to \boldsymbol{D}(\boldsymbol{v})$ strongly in $[L^1(\Omega)]^{d\times d}$. Hence, we can use the Lebesgue dominated convergence theorem to identify the limit of $\boldsymbol{K}(c^n, \boldsymbol{D}(\boldsymbol{v}^n))$ and obtain

$$\overline{\boldsymbol{q}}_c = \boldsymbol{q}_c(c, \boldsymbol{D}(\boldsymbol{v}), \nabla c), \tag{5.55}$$

which finishes the proof.

□

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