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A note on the efficient evaluation of a modified Hilbert transformation

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Abstract

In this note we consider an efficient data–sparse approximation of a modified Hilbert type transformation as it is used for the space–time finite element discretization of parabolic evolution equations in the anisotropic Sobolev space $H^{1,1/2}(Q)$. The resulting bilinear form of the first order time derivative is symmetric and positive definite, and similar as the integration by parts formula for the Laplace hypersingular boundary integral operator in 2D. Hence we can apply hierarchical matrices for data–sparse representations and for acceleration of the computations. Numerical results show the efficieny in the approximation of the first order time derivative. These results are necessary for the use of general space–time finite element approximations of parabolic evolution equations.

1 Introduction

Space–time finite element methods for parabolic evolution equations in the anisotropic Sobolev space $H^{1,1/2}(Q)$ became quite attractive recently, see, e.g., [3, 4, 7, 11, 13]. While for an infinite time interval $(0,\infty)$ the classical Hilbert transformation can be used to define a symmetric and positive definite bilinear form for the first order time derivative, for a finite time horizon $T<\infty$ in [13] we have introduced a modified Hilbert type transformation by means of its Fourier series. Since the transformation of a piecewise polynomial local basis function is global, its Galerkin approximation results in dense stiffness matrices. Moreover, the definition of the series representation requires a tensor–product structure for the discretization which limits the applicability of the approach. In [13] we have already derived a closed form of the modified Hilbert transformation which is given as a Cauchy singular integral operator. In this note we will present an alternative representation which is similar to the integration by parts formula for the hypersingular boundary integral operator of the Laplacian in 2D. Hence, and as it is done in boundary element methods, we can use hierarchical matrices [1, 2, 6, 9] for data–sparse representations of the stiffness matrices to accelerate the computations. Moreover, this approach also allows the use of general space–time finite element meshes for the numerical solution of parabolic evolution equations.

As in [13], as a model problem we consider the initial value problem for a first–order linear ordinary differential equation,

$$\partial_t u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0, \tag{1.1}$$

where $\mu \geq 0$ and $T > 0$ are fixed. A related variational formulation is to find $u \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T v\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v\rangle_{(0,T)} \tag{1.2}$$

is satisfied for all $v \in H^{1/2}_{0,}(0,T)$. Note that

$$\begin{aligned} H^{1/2}_{0,}(0,T) &= \left\{ U_{|(0,T)} \colon\ U \in H^{1/2}(-\infty,T) \text{ with } U(t) = 0,\, t < 0 \right\}, \\ H^{1/2}_{,0}(0,T) &= \left\{ U_{|(0,T)} \colon\ U \in H^{1/2}(0,\infty) \text{ with } U(t) = 0,\, t > T \right\} \end{aligned}$$

with the Hilbertian norms

$$\begin{aligned} \|u\|_{H^{1/2}_{0,}(0,T)} &:= \sqrt{\|u\|^2_{H^{1/2}(0,T)} + \int_0^T \frac{|u(t)|^2}{t} \mathrm{d}t}, \\ \|u\|_{H^{1/2}_{,0}(0,T)} &:= \sqrt{\|u\|^2_{H^{1/2}(0,T)} + \int_0^T \frac{|u(t)|^2}{T-t} \mathrm{d}t}. \end{aligned}$$

In (1.2), $\langle \cdot, \cdot \rangle_{(0,T)}$ denotes the duality pairing in $[H^{1/2}_{,0}(0,T)]'$ and $H^{1/2}_{,0}(0,T)$ as extension of the inner product in $L^2(0,T)$.

The modified Hilbert transformation $\mathcal{H}_T \colon\ L^2(0,T) \to L^2(0,T)$ is defined as

$$(\mathcal{H}_T v)(t) = \sum_{k=0}^{\infty} v_k \cos\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad t \in (0,T), \tag{1.3}$$

where

$$v_k = \frac{2}{T} \int_0^T v(t) \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \mathrm{d}t \tag{1.4}$$

are the Fourier coefficients of the series representation

$$v(t) = \sum_{k=0}^{\infty} v_k \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \mathrm{d}t, \quad t \in (0,T),$$

when $v \in L^2(0,T)$ is given. The modified Hilbert transformation $\mathcal{H}_T$, as given in (1.3), is a mapping $\mathcal{H}_T\colon\ H^{1/2}_{0,}(0,T) \to H^{1/2}_{,0}(0,T)$ or a mapping $\mathcal{H}_T\colon\ H^1_{0,}(0,T) \to H^1_{,0}(0,T)$, where the latter spaces are defined accordingly. As shown in [13], we have

$$\langle \partial_t v, \mathcal{H}_T v\rangle_{(0,T)} = \|v\|^2_{H^{1/2}_{0,}(0,T)}, \quad \langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} \geq 0 \quad \text{for all } v \in H^{1/2}_{0,}(0,T).$$

Hence, we immediately conclude that the bilinear form

$$a(u,v) := \langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T v\rangle_{L^2(0,T)} \quad \text{for } u,v \in H^{1/2}_{0,}(0,T)$$

is continuous and $H^{1/2}_{0,}(0,T)$–elliptic, implying unique solvability of the variational problem (1.2). Moreover, using a conforming finite element space

$$S^p_{h,0,}(0,T) := S^p_h(0,T) \cap H^{1/2}_{0,}(0,T) = \text{span}\,\{\varphi^p_k\}_{k=1}^{pN}$$

of, e.g., piecewise polynomial and continuous basis functions φ^p_k of polynomial degree $p \geq 1$, which are defined with respect to a locally quasi–uniform decomposition

$$0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T, \quad h_k = t_k - t_{k-1} \quad \text{for } k = 1, \ldots, N, \tag{1.5}$$

the Galerkin finite element formulation of (1.2) is to find $u_h \in S^p_{h,0,}(0,T)$ such that

$$\langle \partial_t u_h, \mathcal{H}_T v_h\rangle_{(0,T)} + \mu \langle u_h, \mathcal{H}_T v_h\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_h\rangle_{(0,T)} \tag{1.6}$$

is satisfied for all $v_h \in S^p_{h,0,}(0,T)$. While the stability and error analysis of the Galerkin formulation (1.6) is standard, the practical realisation requires the computation of the matrix entries

$$A_h[k,\ell] = \langle \partial_t \varphi^p_\ell, \mathcal{H}_T \varphi^p_k\rangle_{L^2(0,T)}, \quad M_h[k,\ell] = \langle \varphi^p_\ell, \mathcal{H}_T \varphi^p_k\rangle_{L^2(0,T)} \quad \text{for } k,\ell = 1,\ldots,pN, \tag{1.7}$$

and of the vector entries

$$f_k = \langle f, \mathcal{H}_T \varphi^p_k\rangle_{(0,T)} \quad \text{for } k = 1, \ldots, pN. \tag{1.8}$$

While in our previous work [13], we have used the series representation (1.3) for the implementation of the modified Hilbert transformation $\mathcal{H}_T$, such an approach is most likely not efficient from a numerical point of view. Recall that the modified Hilbert transformation $\mathcal{H}_T \varphi^p_k$ of a basis function φ^p_k with local support is in general non–local, which results in dense stiffness and mass matrices A_h and M_h. This motivates to consider data–sparse approximations as known, e.g., from boundary element methods [1, 6, 9, 10]. The use of the series representation (1.3) in the numerical implementation requires the Fourier coefficients (1.4) of the basis function φ^p_k, i.e. the basis function φ^p_k is needed in a closed form in the whole time interval $(0,T)$. While this is not a restriction for the model problem (1.1), in the more general case of the heat equation this restricts to tensor–product meshes in space and time. Hence, to be able to use the modified Hilbert transformation within rather general space–time finite element meshes, we need to have a closed form of $\mathcal{H}_T$ which allows a direct evaluation, and the use of data–sparse approximations also in more general cases.

2 Alternative representations of the modified Hilbert transformation

Instead of the series representation (1.3) of the modified Hilbert transformation $\mathcal{H}_T$, we aim to derive a closed form which is more suitable for a direct evaluation. We start to recall the proof of [13, Lemma 2.8]:

Lemma 2.1 *For $v \in L^2(0,T)$, the operator $\mathcal{H}_T$ as defined in (1.3) allows the integral representation*

$$(\mathcal{H}_T v)(t) = \text{v.p.} \int_0^T K(s,t)\, v(s)\, \mathrm{d}s, \quad t \in (0,T), \tag{2.1}$$

as a Cauchy principal value integral, where the kernel function is given as

$$K(s,t) = \frac{1}{2T}\left[\frac{1}{\sin\frac{\pi(s+t)}{2T}} + \frac{1}{\sin\frac{\pi(s-t)}{2T}}\right].$$

For $v \in H^1(0,T)$, the operator $\mathcal{H}_T$ as defined in (1.3) allows the integral representation

$$(\mathcal{H}_T v)(t) = -\frac{2}{\pi} v(0) \ln\tan\frac{\pi t}{4T} - \frac{1}{\pi}\int_0^T \partial_t v(s) \ln\left[\tan\frac{\pi(s+t)}{4T}\tan\frac{\pi\,|t-s|}{4T}\right]\mathrm{d}s \tag{2.2}$$

for $t \in (0,T)$ as a weakly singular integral.

Proof. First, we prove the lemma for functions in $v \in H^1(0,T)$. Let $t \in (0,T)$ be arbitrary but fixed. To show the representation (2.2), we have by definition

$$\begin{aligned}
(\mathcal{H}_T v)(t) &= \sum_{k=0}^{\infty} v_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \\
&= \sum_{k=0}^{\infty} \frac{2}{T}\int_0^T v(s)\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s}{T}\right)\mathrm{d}s\,\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{T}\int_0^T v(s)\left[\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s+t}{T}\right) + \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s-t}{T}\right)\right]\mathrm{d}s \\
&= \lim_{M\to\infty}\sum_{k=0}^{M}\frac{1}{T}\lim_{\varepsilon\to 0}\left\{\int_0^{t-\varepsilon} v(s)\left[\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s+t}{T}\right) + \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s-t}{T}\right)\right]\mathrm{d}s\right. \\
&\qquad\left. + \int_{t+\varepsilon}^T v(s)\left[\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s+t}{T}\right) + \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s-t}{T}\right)\right]\mathrm{d}s\right\}.
\end{aligned}$$

Since the double limit and the iterated limits exist, interchanging the order of the limit for the double sequence is justified and thus, with integration by parts follows

$$
\begin{aligned}
(\mathcal{H}_T v)(t) =& \lim_{\varepsilon\to 0}\lim_{M\to\infty}\sum_{k=0}^{M}\frac{1}{T}\Bigg\{-v(t-\varepsilon)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{2t-\varepsilon}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}+v(0)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\\
&+\int_0^{t-\varepsilon}\partial_t v(s)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s+t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\mathrm{d}s-v(t-\varepsilon)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{\varepsilon}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\\
&+v(0)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}+\int_0^{t-\varepsilon}\partial_t v(s)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s-t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\mathrm{d}s\\
&-v(T)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{T+t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}+v(t+\varepsilon)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{2t+\varepsilon}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\\
&+\int_{t+\varepsilon}^{T}\partial_t v(s)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s+t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\mathrm{d}s-v(T)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{T-t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\\
&+v(t+\varepsilon)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{\varepsilon}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}+\int_{t+\varepsilon}^{T}\partial_t v(s)\frac{\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s-t}{T}\right)}{\left(\frac{\pi}{2}+k\pi\right)\frac{1}{T}}\mathrm{d}s\Bigg\}.
\end{aligned}
$$

With this, it holds true that

$$
\begin{aligned}
(\mathcal{H}_T v)(t) \quad &= \lim_{\varepsilon\to 0}\frac{1}{\pi}\Bigg\{v(t-\varepsilon)\ln\tan\frac{\pi(2t-\varepsilon)}{4T}-2v(0)\ln\tan\frac{\pi t}{4T}\\
&-\int_0^{t-\varepsilon}\partial_t v(s)\ln\tan\frac{\pi(s+t)}{4T}\mathrm{d}s+v(t-\varepsilon)\ln\tan\frac{\pi\varepsilon}{4T}\\
&-\int_0^{t-\varepsilon}\partial_t v(s)\ln\tan\frac{\pi(t-s)}{4T}\mathrm{d}s+v(T)\ln\tan\frac{\pi(T+t)}{4T}\\
&-v(t+\varepsilon)\ln\tan\frac{\pi(2t+\varepsilon)}{4T}-\int_{t+\varepsilon}^{T}\partial_t v(s)\ln\tan\frac{\pi(s+t)}{4T}\mathrm{d}s\\
&+v(T)\ln\tan\frac{\pi(T-t)}{4T}-v(t+\varepsilon)\ln\tan\frac{\pi\varepsilon}{4T}-\int_{t+\varepsilon}^{T}\partial_t v(s)\ln\tan\frac{\pi(s-t)}{4T}\mathrm{d}s\Bigg\}, \quad (2.3)
\end{aligned}
$$

where the continuity of the inner products $\langle\cdot,\cdot\rangle_{L^2(0,t-\epsilon)}$, $\langle\cdot,\cdot\rangle_{L^2(t+\epsilon,T)}$, and the relation

$$
\sum_{k=0}^{\infty}\frac{\cos\left(\left(\frac{1}{2}+k\right)x\right)}{\frac{1}{2}+k}=-\ln\tan\frac{x}{4}\qquad \text{for } x\in(0,2\pi),
$$

see [5, 1.442], are used. Since v is Hölder continuous with exponent $\frac{1}{2}$, see [8, Chapitre 2, Théorème 3.8], we conclude further

$$\begin{aligned}(\mathcal{H}_T v)(t) =& \frac{1}{\pi}\lim_{\varepsilon\to 0}\left\{[v(t-\varepsilon)-v(t+\varepsilon)]\ln\tan\frac{\pi\varepsilon}{4T}\right\} - \frac{2}{\pi}v(0)\ln\tan\frac{\pi t}{4T} \\ &- \frac{1}{\pi}\int_0^T \partial_t v(s)\left[\ln\tan\frac{\pi(s+t)}{4T} + \ln\tan\frac{\pi\,|t-s|}{4T}\right]\mathrm{d}s \\ &+ \frac{1}{\pi}v(T)\underbrace{\left[\ln\tan\frac{\pi(T+t)}{4T} + \ln\tan\frac{\pi(T-t)}{4T}\right]}_{=0} \\ =& -\frac{2}{\pi}v(0)\ln\tan\frac{\pi t}{4T} - \frac{1}{\pi}\int_0^T \partial_t v(s)\ln\left[\tan\frac{\pi(s+t)}{4T}\tan\frac{\pi\,|t-s|}{4T}\right]\mathrm{d}s,\end{aligned}$$

where the integrals exist as weakly singular integrals. This shows the representation (2.2).

To prove the representation (2.1), integration by parts in (2.3) yields

$$\begin{aligned}(\mathcal{H}_T v)(t) =& \lim_{\varepsilon\to 0}\left\{\int_0^{t-\varepsilon} v(s)\frac{1}{2T\sin\frac{\pi(s+t)}{2T}}\mathrm{d}s + \int_0^{t-\varepsilon} v(s)\frac{1}{2T\sin\frac{\pi(s-t)}{2T}}\mathrm{d}s\right. \\ &\left. + \int_{t+\varepsilon}^T v(s)\frac{1}{2T\sin\frac{\pi(s+t)}{2T}}\mathrm{d}s + \int_{t+\varepsilon}^T v(s)\frac{1}{2T\sin\frac{\pi(s-t)}{2T}}\mathrm{d}s\right\} \\ =& \text{v.p.}\int_0^T \frac{1}{2T}\left[\frac{1}{\sin\frac{\pi(s+t)}{2T}} + \frac{1}{\sin\frac{\pi(s-t)}{2T}}\right] v(s)\,\mathrm{d}s,\end{aligned}$$

where the last integral is a Cauchy principal value integral. Since $H^1(0,T)$ is dense in $L^2(0,T)$, the representation (2.1) for functions in $L^2(0,T)$ follows by a density argument. ∎

Remark 2.1 *In general, the representation* (2.1) *is only given as a Cauchy principal value integral, since for, e.g., $v(t) = 1$, the integral in* (2.1) *does not converge as a weakly singular integral. The representation* (2.2) *shows that for an arbitrary function $v \in H^1(0,T)$, the function $\mathcal{H}_T v \in L^2(0,T)$ is not a function in $H^s(0,T)$ for $s > \frac{1}{2}$. As an example, consider the function $v(t) = 1$ and its transformation*

$$(\mathcal{H}_T v)(t) = -\frac{2}{\pi}\ln\tan\frac{\pi t}{4T}, \quad t \in (0,T),$$

where the trace for $t = 0$ does not exist.

Corollary 2.2 *For $v \in H^1_{0,}(0,T)$, the operator $\mathcal{H}_T$ as defined in* (1.3) *allows the integral representation*

$$(\mathcal{H}_T v)(t) = -\frac{1}{\pi}\int_0^T \partial_t v(s)\ln\left[\tan\frac{\pi(s+t)}{4T}\tan\frac{\pi\,|t-s|}{4T}\right]\mathrm{d}s, \quad t \in (0,T), \tag{2.4}$$

as a weakly singular integral, and the traces are given as

$$(\mathcal{H}_T v)(0) = -\frac{2}{\pi} \int_0^T \partial_t v(s) \ln \tan \frac{\pi s}{4T} \mathrm{d}s, \qquad (\mathcal{H}_T v)(T) = 0,$$

where the integral is a weakly singular integral.

Proof. Let $v \in H^1_{0,}(0,T)$ be fixed. The representation (2.4) follows from (2.2). Since the Fourier series (1.3) converges also in $H^1_{,0}(0,T)$, we immediately obtain $(\mathcal{H}_T v)(T) = 0$. To compute the trace $(\mathcal{H}_T v)(0)$, we consider

$$(\mathcal{H}_T v)(0) = \sum_{k=0}^{\infty} v_k \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{0}{T}\right) = \sum_{k=0}^{\infty} v_k = \sum_{k=0}^{\infty} \frac{2}{T} \int_0^T v(s) \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{s}{T}\right) \mathrm{d}s,$$

where the same arguments as given in the proof of the representation (2.2) lead to the representation of the trace $(\mathcal{H}_T v)(0)$. ■

The integral representation (2.4) is used for the computation of the stiffness and mass matrices as given in (1.7). In particular for functions $u, v \in H^1_{0,}(0,T)$, we have

$$\langle \partial_t u, \mathcal{H}_T v \rangle_{L^2(0,T)} = -\frac{1}{\pi} \int_0^T \partial_t u(t) \int_0^T \ln\left[\tan \frac{\pi(s+t)}{4T} \tan \frac{\pi\,|t-s|}{4T}\right] \partial_t v(s)\, \mathrm{d}s\, \mathrm{d}t \qquad (2.5)$$

and

$$\langle u, \mathcal{H}_T v \rangle_{L^2(0,T)} = -\frac{1}{\pi} \int_0^T u(t) \int_0^T \ln\left[\tan \frac{\pi(s+t)}{4T} \tan \frac{\pi\,|t-s|}{4T}\right] \partial_t v(s)\, \mathrm{d}s\, \mathrm{d}t, \qquad (2.6)$$

which shows a relation with the integration by parts formula for the two–dimensional Laplace hypersingular boundary integral operator, see, e.g., [12].

3 Realisation of $\mathcal{H}_T$

In order to compute the stiffness matrix A_h and the mass matrix M_h in (1.7), we have to evaluate

$$\begin{aligned} A_h[k,\ell] &= \langle \partial_t \varphi^p_\ell, \mathcal{H}_T \varphi^p_k \rangle_{L^2(0,T)} \\ &= -\frac{1}{\pi} \int_0^T \partial_t \varphi^p_\ell(t) \int_0^T \ln\left[\tan \frac{\pi(s+t)}{4T} \tan \frac{\pi\,|t-s|}{4T}\right] \partial_t \varphi^p_k(s)\, \mathrm{d}s\, \mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} M_h[k,\ell] &= \langle \varphi^p_\ell, \mathcal{H}_T \varphi^p_k \rangle_{L^2(0,T)} \\ &= -\frac{1}{\pi} \int_0^T \varphi^p_\ell(t) \int_0^T \ln\left[\tan \frac{\pi(s+t)}{4T} \tan \frac{\pi\,|t-s|}{4T}\right] \partial_t \varphi^p_k(s)\, \mathrm{d}s\, \mathrm{d}t \end{aligned}$$

for $k, \ell = 1, \ldots, pN$. In the particular case $p = 1$ we have piecewise linear basis functions

$$\varphi_k^1(t) = \begin{cases} \dfrac{1}{h_k}(t - t_{k-1}) & \text{for } t \in (t_{k-1}, t_k], \\ \dfrac{1}{h_{k+1}}(t_{k+1} - t) & \text{for } t \in (t_k, t_{k+1}), \\ 0 & \text{else}, \end{cases}$$

for $k = 1, \ldots, N-1$ and

$$\varphi_N^1(t) = \begin{cases} \dfrac{1}{h_N}(t - t_{N-1}) & \text{for } t \in (t_{N-1}, t_N], \\ 0 & \text{else}. \end{cases}$$

Since the first–order derivative $\partial_t \varphi_k^1$ is piecewise constant, we need to compute the matrix $A_h^{0,0} \in \mathbb{R}^{N\times N}$ with the matrix entries of the form

$$A_h^{0,0}[i,j] := -\frac{1}{\pi} \int_{t_{j-1}}^{t_j} \int_{t_{i-1}}^{t_i} \ln\left[\tan\frac{\pi(s+t)}{4T} \tan\frac{\pi\,|t-s|}{4T}\right] \mathrm{d}s\,\mathrm{d}t \tag{3.1}$$

for $i, j = 1, \ldots, N$, to build the stiffness matrix

$$A_h = Z_{A_h}^{0,0} A_h^{0,0} (Z_{A_h}^{0,0})^\top \tag{3.2}$$

with the help of the assembling matrix

$$Z_{A_h}^{0,0} := \begin{pmatrix} \frac{1}{h_1} & -\frac{1}{h_2} & & & \\ & \frac{1}{h_2} & -\frac{1}{h_3} & & \\ & & \ddots & \ddots & \\ & & & \frac{1}{h_{N-1}} & -\frac{1}{h_N} \\ & & & & \frac{1}{h_N} \end{pmatrix} \in \mathbb{R}^{N\times N}. \tag{3.3}$$

A similar approach can be used for the mass matrix M_h, which leads to the representation

$$M_h = Z_{A_h}^{0,0} \left(A_h^{0,1} (Z_{A_h}^{0,0})^\top + A_h^{0,0} S_{M_h}^{0,0} \right) \tag{3.4}$$

with

$$A_h^{0,1}[i,j] := -\frac{1}{\pi} \int_{t_{j-1}}^{t_j} \int_{t_{i-1}}^{t_i} t \ln\left[\tan\frac{\pi(s+t)}{4T} \tan\frac{\pi\,|t-s|}{4T}\right] \mathrm{d}s\,\mathrm{d}t \tag{3.5}$$

for $i, j = 1, \ldots, N$, and

$$S_{M_h}^{0,0} := \begin{pmatrix} -\frac{t_0}{h_1} & & & & \\ \frac{t_2}{h_2} & -\frac{t_1}{h_2} & & & \\ & \ddots & \ddots & & \\ & & \frac{t_{N-1}}{h_{N-1}} & -\frac{t_{N-2}}{h_{N-1}} & \\ & & & \frac{t_N}{h_N} & -\frac{t_{N-1}}{h_N} \end{pmatrix} \in \mathbb{R}^{N\times N}.$$

More general, the computation of the stiffness and mass matrices in (1.7) is based on the calculation of the auxiliary matrix $B_h^{r,q} \in \mathbb{R}^{N\times N}$ with matrix entries

$$B_h^{r,q}[k,\ell] := -\frac{1}{\pi}\int_{t_{\ell-1}}^{t_\ell} \alpha_q(t) \int_{t_{k-1}}^{t_k} \ln\left[\tan\frac{\pi(s+t)}{4T}\tan\frac{\pi\,|t-s|}{4T}\right]\beta_r(s)\,\mathrm{d}s\,\mathrm{d}t, \tag{3.6}$$

for $k = 1,\dots,N$ and $\ell = 1,\dots,N$, where α_q is a fixed polynomial of degree $q \in \mathbb{N}_0$ and β_r is a fixed polynomial of degree $r \in \mathbb{N}_0$. With this notation, the matrix $A_h^{0,0}$ in (3.1) is equal to $B_h^{0,0}$ for the choice $\alpha_0(t) = 1$, $\beta_0(t) = 1$, and the matrix $A_h^{0,1}$ in (3.5) is equal to $B_h^{0,1}$ for the choice $\alpha_1(t) = t$, $\beta_0(t) = 1$. In addition, the auxiliary matrix $B_h^{r,q} \in \mathbb{R}^{N\times N}$ can be used for the realisation of the finite element method (1.6) with piecewise polynomial functions of arbitrary degree $p \geq 1$.

As a last point, the treatment of the right–hand side $\underline{f}$ of (1.8) is investigated for the special case of $p = 1$. We assume that $f \in L^2(0,T)$ and so, we compute the piecewise constant L^2 projection

$$Q_h^0 f = \sum_{\ell=1}^N a_\ell \varphi_\ell^0 \in S_h^0(0,T), \quad a_\ell = \frac{1}{h_\ell}\int_{t_{\ell-1}}^{t_\ell} f(t)\,\mathrm{d}t, \tag{3.7}$$

where we have the standard error estimate

$$\left\|f - Q_h^0 f\right\|_{[H_{,0}^{-\sigma}(0,T)]'} \leq c\,h^{s-\sigma}\|f\|_{H^s(0,T)}$$

for $s \in [0,1]$, $\sigma \in [-1,0]$ and a sufficiently regular right–hand side f. Note that the convergence for the approximate solution of (1.6) is not spoilt for $p = 1$. The resulting right–hand side $\underline{\hat{f}} \in \mathbb{R}^N$, approximating $\underline{f}$ in (1.8), is given by

$$\hat{f}_k = \langle Q_h^0 f, \mathcal{H}_T\varphi_k^1\rangle_{L^2(0,T)} = \sum_{\ell=1}^N a_\ell \langle \varphi_\ell^0, \mathcal{H}_T\varphi_k^1\rangle_{L^2(0,T)} = \left(Z_{A_h}^{0,0} A_h^{0,0}\underline{a}\right)_k \tag{3.8}$$

for $k = 1,\dots,N$, where the auxiliary matrix $A_h^{0,0}$ of (3.1) and the assembling matrix $Z_{A_h}^{0,0}$ of (3.3) are used. Note that for the general case of $p > 1$, the L^2 projection on discontinuous, piecewise polynomial functions of degree $p-1$, i.e. $Q_h^{p-1,-1}\colon\ L^2(0,T) \to S_h^{p-1,-1}(0,T)$, is used to define a computable approximation of the right–hand side $\underline{f} \in \mathbb{R}^{pN}$ in (1.8), which preserves the convergence for the approximate solution of (1.6).

3.1 Computation of the matrix entries of the auxiliary matrix

For a fixed polynomial α_q of degree $q \in \mathbb{N}_0$ and a fixed polynomial β_r of degree $r \in \mathbb{N}_0$, the matrix $B_h^{r,q} \in \mathbb{R}^{N\times N}$ with matrix entries (3.6) is considered. To compute the auxiliary matrix $B_h^{r,q}$, the integral in (3.6) is split into a regular part and into a singular part. For the regular part, a tensor Gauß quadrature

$$\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} \omega_{\nu_1,n_1}\omega_{\nu_2,n_2} G(\xi_{\nu_1,n_1}, \xi_{\nu_2,n_2}), \tag{3.9}$$

approximating

$$\int_0^1 \int_0^1 G(\xi_1, \xi_2)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2$$

for a function $G\colon\ [0,1] \times [0,1] \to \mathbb{R}$, is applied, where $\omega_{\nu_i,n_i} \in \mathbb{R}$ and $\xi_{\nu_i,n_i} \in [0,1]$ are the Gauß integration weights and Gauß integration nodes of order $n_i \in \mathbb{N}$ with respect to the coordinate direction $i \in \{1,2\}$. For the resulting singular part, an analytic integration or, alternatively, the usage of an adapted numerical integration is possible.

In particular, three different cases, which correspond to the singularities of the integral in (3.6), are investigated. First, for the indices $k = \ell = 1$, we consider

$$\begin{aligned}
B_h^{r,q}[1,1] &= -\frac{1}{\pi} \int_0^{t_1} \alpha_q(t) \int_0^{t_1} \ln\left[\tan\frac{\pi(s+t)}{4T} \tan\frac{\pi\,|t-s|}{4T}\right] \beta_r(s)\, \mathrm{d}s\, \mathrm{d}t \\
&= -\frac{1}{\pi} \int_0^{t_1} \int_0^{t_1} \underbrace{\alpha_q(t) \ln\left[\frac{\tan\frac{\pi(s+t)}{4T}}{s+t} \frac{\tan\frac{\pi|t-s|}{4T}}{|s-t|}\right] \beta_r(s)}_{=:F_0^{r,q}(s,t)}\, \mathrm{d}s\, \mathrm{d}t \\
&\quad \underbrace{-\frac{1}{\pi} \int_0^{t_1} \alpha_q(t) \int_0^{t_1} \ln\left[(s+t)\,|s-t|\right] \beta_r(s)\, \mathrm{d}s\, \mathrm{d}t}_{=:I^{r,q}[1,1]}
\end{aligned}$$

and further

$$\begin{aligned}
B_h^{r,q}[1,1] &= -\frac{1}{\pi} \int_0^1 \int_0^1 \underbrace{F_0^{r,q}((1-\xi)\eta h_1, \eta h_1)\, h_1^2\, \eta}_{=:G_{0,1}^{r,q}(\xi,\eta)}\, \mathrm{d}\xi\, \mathrm{d}\eta \\
&\quad -\frac{1}{\pi} \int_0^1 \int_0^1 \underbrace{F_0^{r,q}(\xi h_1, (1-\eta)\xi h_1)\, h_1^2\, \xi}_{=:G_{0,2}^{r,q}(\xi,\eta)}\, \mathrm{d}\eta\, \mathrm{d}\xi - I^{r,q}[1,1] \\
&\approx -\frac{1}{\pi} \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} \omega_{\nu_1,n_1} \omega_{\nu_2,n_2} G_{0,1}^{r,q}(\xi_{\nu_1,n_1}, \xi_{\nu_2,n_2}) \\
&\quad -\frac{1}{\pi} \sum_{\nu_1=1}^{n_1} \sum_{\nu_2=1}^{n_2} \omega_{\nu_1,n_1} \omega_{\nu_2,n_2} G_{0,2}^{r,q}(\xi_{\nu_1,n_1}, \xi_{\nu_2,n_2}) - I^{r,q}[1,1] =: \widetilde{B}_h^{r,q}[1,1],
\end{aligned} \tag{3.10}$$

where the integration domain is split and integration by substitution is applied.

Second, we examine the indices $k = \ell = N$, i.e.

$$\begin{aligned} B_h^{r,q}[N,N] &= -\frac{1}{\pi}\int_{t_{N-1}}^{T}\alpha_q(t)\int_{t_{N-1}}^{T}\ln\left[\tan\frac{\pi(s+t)}{4T}\tan\frac{\pi\,|t-s|}{4T}\right]\beta_r(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= -\frac{1}{\pi}\int_{t_{N-1}}^{T}\int_{t_{N-1}}^{T}\underbrace{\alpha_q(t)\ln\left[\tan\frac{\pi(s+t)}{4T}(2T-s-t)\frac{\tan\frac{\pi|t-s|}{4T}}{|s-t|}\right]\beta_r(s)}_{=:F_N^{r,q}(s,t)}\,\mathrm{d}s\,\mathrm{d}t \\ &\quad \underbrace{-\frac{1}{\pi}\int_{t_{N-1}}^{T}\alpha_q(t)\int_{t_{N-1}}^{T}\ln\frac{|s-t|}{2T-s-t}\beta_r(s)\,\mathrm{d}s\,\mathrm{d}t}_{=:I^{r,q}[N,N]}\,. \end{aligned}$$

Again, splitting the integration domain and integration by substitution yield

$$\begin{aligned} B_h^{r,q}[N,N] &= -\frac{1}{\pi}\int_0^1\int_0^1\underbrace{F_N^{r,q}(t_{N-1}+(1-\xi)\eta h_N, t_{N-1}+\eta h_N)\,h_N^2\,\eta}_{=:G_{N,1}^{r,q}(\xi,\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &\quad -\frac{1}{\pi}\int_0^1\int_0^1\underbrace{F_N^{r,q}(t_{N-1}+\xi h_N, t_{N-1}+(1-\eta)\xi h_N)\,h_N^2\,\xi}_{=:G_{N,2}^{r,q}(\xi,\eta)}\,\mathrm{d}\eta\,\mathrm{d}\xi - I^{r,q}[N,N] \\ &\approx -\frac{1}{\pi}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\omega_{\nu_1,n_1}\omega_{\nu_2,n_2}G_{N,1}^{r,q}(\xi_{\nu_1,n_1},\xi_{\nu_2,n_2}) \\ &\quad -\frac{1}{\pi}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2}\omega_{\nu_1,n_1}\omega_{\nu_2,n_2}G_{N,2}^{r,q}(\xi_{\nu_1,n_1},\xi_{\nu_2,n_2}) - I^{r,q}[N,N] =: \widetilde{B}_h^{r,q}[N,N]. \end{aligned} \tag{3.11}$$

Third, for the indices $k,\ell = 1,\ldots,N$, excluding the cases $k=\ell=1$ and $k=\ell=N$, it holds true that

$$\begin{aligned} B_h^{r,q}[k,\ell] &= -\frac{1}{\pi}\int_{t_{\ell-1}}^{t_\ell}\alpha_q(t)\int_{t_{k-1}}^{t_k}\ln\left[\tan\frac{\pi(s+t)}{4T}\tan\frac{\pi\,|t-s|}{4T}\right]\beta_r(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= -\frac{1}{\pi}\int_{t_{\ell-1}}^{t_\ell}\int_{t_{k-1}}^{t_k}\underbrace{\alpha_q(t)\ln\left[\tan\frac{\pi(s+t)}{4T}\frac{\tan\frac{\pi|t-s|}{4T}}{|s-t|}\right]\beta_r(s)}_{=:F^{r,q}(s,t)}\,\mathrm{d}s\,\mathrm{d}t \\ &\quad \underbrace{-\frac{1}{\pi}\int_{t_{\ell-1}}^{t_\ell}\alpha_q(t)\int_{t_{k-1}}^{t_k}\ln|s-t|\,\beta_r(s)\,\mathrm{d}s\,\mathrm{d}t}_{=:I^{r,q}[k,\ell]} \end{aligned}$$

and further

$$
\begin{aligned}
B_h^{r,q}[k,\ell] &= -\frac{1}{\pi}\int_0^1\int_0^1 \underbrace{F^{r,q}(t_{k-1}+(1-\xi)\eta h_k, t_{\ell-1}+\eta h_\ell)\, h_k h_\ell\, \eta}_{=:G_{k,\ell,1}^{r,q}(\xi,\eta)} \,\mathrm{d}\xi\,\mathrm{d}\eta \\
&\quad -\frac{1}{\pi}\int_0^1\int_0^1 \underbrace{F^{r,q}(t_{k-1}+\xi h_k, t_{\ell-1}+(1-\eta)\xi h_\ell)\, h_k h_\ell\, \xi}_{=:G_{k,\ell,2}^{r,q}(\xi,\eta)} \,\mathrm{d}\eta\,\mathrm{d}\xi - I^{r,q}[k,\ell] \\
&\approx -\frac{1}{\pi}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} \omega_{\nu_1,n_1}\omega_{\nu_2,n_2} G_{k,\ell,1}^{r,q}(\xi_{\nu_1,n_1},\xi_{\nu_2,n_2}) \\
&\quad -\frac{1}{\pi}\sum_{\nu_1=1}^{n_1}\sum_{\nu_2=1}^{n_2} \omega_{\nu_1,n_1}\omega_{\nu_2,n_2} G_{k,\ell,2}^{r,q}(\xi_{\nu_1,n_1},\xi_{\nu_2,n_2}) - I^{r,q}[k,\ell] =: \widetilde{B}_h^{r,q}[k,\ell].
\end{aligned} \tag{3.12}
$$

In (3.10), (3.11) and (3.12), a tensor Gauß quadrature (3.9) is applied for the regular part, whereas the singular parts $I^{r,q}[1,1]$, $I^{r,q}[N,N]$ and $I^{r,q}[k,\ell]$ are calculated analytically or, alternatively, with an adapted numerical integration. Hence, this leads to the approximation

$$\widetilde{B}_h^{r,q} \approx B_h^{r,q}. \tag{3.13}$$

As a special case of (3.13), we consider the approximations

$$\widetilde{A}_h^{0,0} \approx A_h^{0,0} \quad \text{and} \quad \widetilde{A}_h^{0,1} \approx A_h^{0,1} \tag{3.14}$$

of the matrices given in (3.1) and (3.5). Thus, with the representations (3.2), (3.4), (3.8), we have the computable approximations

$$\widetilde{A_h} := Z_{A_h}^{0,0}\widetilde{A}_h^{0,0}(Z_{A_h}^{0,0})^\top \approx A_h, \tag{3.15}$$

$$\widetilde{M_h} := Z_{A_h}^{0,0}\left(\widetilde{A}_h^{0,1}(Z_{A_h}^{0,0})^\top + \widetilde{A}_h^{0,0}S_{M_h}^{0,0}\right) \approx M_h \tag{3.16}$$

and

$$\underline{\widetilde{f}} := Z_{A_h}^{0,0}\widetilde{A}_h^{0,0}\underline{a} \approx \underline{\hat{f}} \tag{3.17}$$

for the special case of piecewise linear ansatz and test functions, i.e. $p = 1$, in (1.6). The treatment of the case of higher polynomial degrees, i.e. $p > 1$, in (1.6) is straightforward.

3.2 Exponential convergence of the tensor Gauß quadrature

In this subsection, we show that the proposed tensor Gauß quadrature (3.9) applied in (3.10), (3.11) and (3.12) converges exponentially. To prove that, the integrands of (3.10), (3.11) and (3.12) should have an analytical extension with respect to every variable on an ellipse $\mathcal{E}^\rho \subset \mathbb{C}$ with focal points 0, 1 and the sum of the half-axes $\rho > 1/2$, see [10, p. 330]. In particular, the following theorem states that these integrands can be analytically extended.

Theorem 3.1 *Let the mesh* (1.5) *fulfil the assumption* $\max_\ell h_\ell \leq \frac{T}{2}$ *and let the function* $G\colon [0,1]\times[0,1]\to\mathbb{R}$ *be one of the functions* $G^{r,q}_{0,1}$, $G^{r,q}_{0,2}$, $G^{r,q}_{N,1}$, $G^{r,q}_{N,2}$, $G^{r,q}_{k,\ell,1}$ *or* $G^{r,q}_{k,\ell,2}$, *which are the integrands of* (3.10), (3.11) *and* (3.12). *Then, the function* $G\colon [0,1]\times[0,1]\to\mathbb{R}$ *can be componentwise analytically extended, i.e.* $\rho > 1/2$ *exists such that for all* $\xi\in[0,1]$, *the function*

$$G_\xi\colon [0,1]\to\mathbb{R},\quad G_\xi(\eta):=G(\xi,\eta),$$

can be extended to an analytic function $\widetilde{G}_\xi\colon \mathcal{E}^\rho\to\mathbb{C}$, *and for all* $\eta\in[0,1]$, *the function*

$$G_\eta\colon [0,1]\to\mathbb{R},\quad G_\eta(\xi):=G(\xi,\eta),$$

can be extended to an analytic function $\widetilde{G}_\eta\colon \mathcal{E}^\rho\to\mathbb{C}$.

Proof. We prove only the case $G=G^{r,q}_{0,1}$ since the other cases are analogous. With the tanc function

$$\operatorname{tanc}(z)=\begin{cases}\frac{\tan(z)}{z}, & z\in\left\{w\in\mathbb{C}\colon \operatorname{Re}(w)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right\},\\ 1, & z=0,\end{cases}$$

we have the representation

$$\begin{aligned}G^{r,q}_{0,1}(\xi,\eta)&=F^{r,q}_0((1-\xi)\eta h_1,\eta h_1)\,h_1^2\,\eta\\ &=\alpha_q(\eta h_1)\ln\left[\frac{\tan\frac{\pi(2-\xi)\eta h_1}{4T}}{\frac{\pi(2-\xi)\eta h_1}{4T}}\frac{\tan\frac{\pi\xi\eta h_1}{4T}}{\frac{\pi\xi\eta h_1}{4T}}\frac{\pi^2}{16T^2}\right]\beta_r((1-\xi)\eta h_1)\,h_1^2\,\eta\\ &=\alpha_q(\eta h_1)\ln\left[\operatorname{tanc}\left(\frac{\pi(2-\xi)\eta h_1}{4T}\right)\operatorname{tanc}\left(\frac{\pi\xi\eta h_1}{4T}\right)\frac{\pi^2}{16T^2}\right]\beta_r((1-\xi)\eta h_1)\,h_1^2\,\eta\end{aligned}$$

for ξ, $\eta\in[0,1]$. Since the tanc function is analytic and different from zero on the strip $\left\{w\in\mathbb{C}\colon \operatorname{Re}(w)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right\}$, the logarithm $\ln(\cdot)$ can be extended to the complex logarithm. As all involved functions have an analytic extension, for all $\eta\in[0,1]$, the function $G^{r,q}_{0,1,\eta}\colon [0,1]\to\mathbb{R}$, $G^{r,q}_{0,1,\eta}(\xi):=G^{r,q}_{0,1}(\xi,\eta)$, can be extended to an analytic function on

$$\{z\in\mathbb{C}\colon \operatorname{Re}(z)\in(-2,2)\},$$

since for $\xi,\eta\in[0,1]$, it holds true that

$$\frac{\pi\xi\eta h_1}{4T}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\Longleftrightarrow \underbrace{\xi\eta}_{\in[0,1]}\in\left(-\frac{2T}{h_1},\frac{2T}{h_1}\right)\supset(-4,4)$$

and

$$\frac{\pi(2-\xi)\eta h_1}{4T}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\Longleftrightarrow \underbrace{(2-\xi)\eta}_{\in[0,2]}\in\left(-\frac{2T}{h_1},\frac{2T}{h_1}\right)\supset(-4,4),$$

where the assumption $\max_\ell h_\ell\leq\frac{T}{2}$ enters. For all $\xi\in[0,1]$, the analytic extension of $G^{r,q}_{0,1,\xi}\colon [0,1]\to\mathbb{R}$, $G^{r,q}_{0,1,\xi}(\eta):=G^{r,q}_{0,1}(\xi,\eta)$, follows in the same manner. ■

Corollary 3.2 *Let the mesh* (1.5) *fulfil the assumption* $\max_\ell h_\ell \leq \frac{T}{2}$. *The tensor Gauß quadrature* (3.9) *applied in* (3.10), (3.11) *and* (3.12) *converges exponentially with respect to the number of Gauß integration nodes.*

Proof. This follows immediately from Theorem 3.1 with [10, Theorem 5.3.15, p. 330]. ∎

3.3 Approximation of the stiffness and mass matrices with hierarchical matrices

As the stiffness matrix A_h and mass matrix M_h in (1.7) are dense for piecewise polynomial functions, the memory consumption of storing these matrices is $\mathcal{O}(N^2)$. A way out is the usage of hierarchical matrices ($\mathcal{H}$ matrices) with a memory consumption of $\mathcal{O}(N \log N)$, see, e.g., [1, 2, 6, 9]. In this subsection, we consider the special case of piecewise linear functions, i.e. $p = 1$, in (1.6). As a first step towards $\mathcal{H}$ matrices, a geometrical clustering with respect to the elements

$$\tau_\ell = (t_{\ell-1}, t_\ell), \quad \ell = 1, \ldots, N,$$

is used. For the resulting block cluster tree, the admissibility condition

$$\max\{\operatorname{diam}(C_1), \operatorname{diam}(C_2)\} \leq \eta^{\mathrm{ACA}} \cdot \operatorname{dist}(C_1, C_2) \tag{3.18}$$

for clusters C_1, C_2 with a admissibility parameter $\eta^{\mathrm{ACA}} > 1$ identifies admissible blocks. Here, $\operatorname{diam}(\cdot)$ denotes the diameter of the cluster and $\operatorname{dist}(\cdot,\cdot)$ denotes the distance of two clusters. Next, the construction of an approximation for the matrix $\widetilde{B}_h^{r,q}$, given in (3.13), via $\mathcal{H}$ matrices is realised with the help of the Adaptive Cross Approximation (ACA), where the parameter $\varepsilon^{\mathrm{ACA}} > 0$ controls the approximation accuracy of the low–rank matrices, see, e.g., [1, 9]. Hence, this procedure gives low–rank approximations

$$\widetilde{B}_h^{r,q,\mathrm{ACA}} \approx \widetilde{B}_h^{r,q} \approx B_h^{r,q} \tag{3.19}$$

of (3.13) as well as

$$\widetilde{A}_h^{0,0,\mathrm{ACA}} \approx \widetilde{A}_h^{0,0} \approx A_h^{0,0} \quad \text{and} \quad \widetilde{A}_h^{0,1,\mathrm{ACA}} \approx \widetilde{A}_h^{0,1} \approx A_h^{0,1} \tag{3.20}$$

of (3.14).

3.4 Realisation of the linear system for piecewise linear functions

The linear system corresponding to (1.6),

$$(A_h + \mu M_h)\underline{u} = \underline{\hat{f}},$$

with the approximation $\underline{\hat{f}} \approx \underline{f}$ of the right–hand side given in (3.8) is equivalent to the linear system

$$\left[A_h^{0,0}(Z_{A_h}^{0,0})^\top + \mu\left(A_h^{0,1}(Z_{A_h}^{0,0})^\top + A_h^{0,0}S_{M_h}^{0,0}\right)\right]\underline{u} = A_h^{0,0}\underline{a}, \tag{3.21}$$

where the representations (3.2), (3.4), (3.8) are used. First, replacing the matrices $A_h^{0,0}$ and $A_h^{0,1}$ in (3.21) with their approximations (3.14), i.e. using $\widetilde{A_h} \approx A_h$ in (3.15), $\widetilde{M_h} \approx M_h$ in (3.16) and $\underline{\widetilde{f}} \approx \underline{\hat{f}}$ in (3.17), gives the linear system

$$\left[\widetilde{A}_h^{0,0}(Z_{A_h}^{0,0})^\top + \mu\left(\widetilde{A}_h^{0,1}(Z_{A_h}^{0,0})^\top + \widetilde{A}_h^{0,0}S_{M_h}^{0,0}\right)\right]\underline{\widetilde{u}} = \widetilde{A}_h^{0,0}\underline{a} \tag{3.22}$$

with the approximation $\underline{\widetilde{u}} \approx \underline{u}$. Second, replacing the matrices $A_h^{0,0}$ and $A_h^{0,1}$ in (3.21) with their low–rank approximations (3.20) gives the linear system

$$\left[\widetilde{A}_h^{0,0,\mathrm{ACA}}(Z_{A_h}^{0,0})^\top + \mu\left(\widetilde{A}_h^{0,1,\mathrm{ACA}}(Z_{A_h}^{0,0})^\top + \widetilde{A}_h^{0,0,\mathrm{ACA}}S_{M_h}^{0,0}\right)\right]\underline{\widetilde{u}}^{\mathrm{ACA}} = \widetilde{A}_h^{0,0,\mathrm{ACA}}\underline{a} \tag{3.23}$$

with the approximation $\underline{\widetilde{u}}^{\mathrm{ACA}} \approx \underline{u}$. Note that at every iteration of an iterative solver, e.g., GMRES, for the linear system (3.23), the matrix–vector product of the system matrix in (3.23) with a vector $\underline{v}$ is realised as

$$\widetilde{A}_h^{0,0,\mathrm{ACA}}\left[((Z_{A_h}^{0,0})^\top + \mu S_{M_h}^{0,0})\underline{v}\right] + \widetilde{A}_h^{0,1,\mathrm{ACA}}\left[\mu(Z_{A_h}^{0,0})^\top\underline{v}\right],$$

where the matrix–vector product for $\mathcal{H}$ matrices occurs.

4 Numerical results for piecewise linear functions

In this section, numerical experiments show the exponential convergence of the numerical integration of Subsection 3.1 and the advantages of the usage of $\mathcal{H}$ matrices as described in Subsection 3.3, where only the special case of piecewise linear functions, i.e. $p = 1$, in (1.6) is examined.

4.1 Numerical integration

As an example for the quality of the numerical integration of Subsection 3.1, consider the non–uniform time mesh

$$0.0 = t_0 < t_1 = 0.625 < t_2 = 1.25 < t_3 = 1.875 < t_4 = 2.5 < t_5 = 6.25 < t_6 = T = 10.0$$

with $n = n_1 = n_2$ Gauß–Legendre points per coordinate direction and analytic integration of $I^{r,q}$ in (3.10), (3.11) and (3.12) for the calculation of the auxiliary matrices $\widetilde{A}_h^{0,0}$, $\widetilde{A}_h^{0,1}$ in (3.14). After assembling with the representations (3.2) and (3.4), the results for the stiffness matrix A_h and mass matrix M_h in (1.7) are given in Table 1, where we observe exponential convergence with respect to the number of Gauß integration nodes, as stated in Corollary 3.2. Different numerical experiments, i.e. more degrees of freedom N, show the same behaviour of the proposed numerical quadrature.

n	$\max_{k,\ell} \lvert A_h[k,\ell] - \widetilde{A}_h[k,\ell]\rvert$	$\max_{k,\ell} \lvert M_h[k,\ell] - \widetilde{M}_h[k,\ell]\rvert$
1	5.0287e-02	3.2988e-01
2	7.3350e-04	4.7820e-03
4	7.6937e-07	4.6192e-06
8	4.6935e-12	2.7540e-11
16	4.9741e-13	7.4433e-12
32	4.9730e-13	7.4435e-12
64	4.9716e-13	7.4424e-12

Table 1: Numerical results of the numerical integration with n Gauß–Legendre points per coordinate direction with a non–uniform time mesh for the stiffness matrix A_h and the mass matrix M_h for piecewise linear functions.

4.2 Approximations with hierarchical matrices

Consider the ordinary differential equation (1.1) with $\mu = 10$, $T = 10$ for the exact solutions

$$u_1(t) = \mathrm{e}^{-\frac{t}{5}} \sin(10\pi t), \quad u_2(t) = t^{3/4},$$

where u_1 is smooth and $u_2 \in H^{5/4-\delta}(0,T)$, $\delta > 0$, has less regularity. The time mesh is given by

$$t_\ell = \ell \frac{T}{N}, \quad \ell = 0, \ldots, N,$$

for $N = 4 \cdot 2^L$ with the levels $L = 0, \ldots, 11$, i.e. a uniform mesh with a uniform refinement strategy is considered. The numerical integration is realised with $n = n_1 = n_2 = 10$ Gauß–Legendre points per coordinate direction and analytic integration of $I^{r,q}$ in (3.10), (3.11) and (3.12) for the calculation of the auxiliary matrices $\widetilde{A}_h^{0,0}$, $\widetilde{A}_h^{0,1}$ in (3.14). The appearing integrals for the L^2 projection (3.7) of the related right–hand side are computed by the usage of high–order integration rules. In Table 2 the L^2 and H^1 errors of the Galerkin finite element formulation (1.6), realised via solving the approximated linear system (3.22), for the exact solution u_1 are given, where the optimal convergence rates are achieved. In Table 3 the convergence rates for the singular solution u_2 are presented, which are reduced due to the less regularity of the solution u_2.

In the last part of this section, numerical results for the linear system (3.23) are given in Table 4 and Table 5, where the experiments are the same as in Table 2 and Table 3. Here, the admissibility parameter is $\eta^{\mathrm{ACA}} = 2.0$ in (3.18) and the accuracy parameter $\varepsilon^{\mathrm{ACA}} > 0$ is chosen such that the additional approximations via $\mathcal{H}$ matrices are smaller than the approximation error of the finite element method. Table 4 and Table 5 show that the memory consumption can be significantly decreased via $\mathcal{H}$ matrices, including the same range for the L^2 and H^1 error, where the numbers of the errors are the same as in Table 2 and Table 3. Note that the memory consumptions $\mathrm{mem}(\widetilde{A}_h^{0,0,\mathrm{ACA}})$ and $\mathrm{mem}(\widetilde{A}_h^{0,1,\mathrm{ACA}})$ in Table 4 and Table 5 contain also the memory consumption for the hierarchical clustering, i.e. the cluster trees. The last column of Table 4 and Table 5 gives the relative error in the

L	N	$\frac{\left\|u_1-u_{1,h}\right\|_{L^2(0,T)}}{\|u_1\|_{L^2(0,T)}}$	eoc	$\frac{\left\|u_1-u_{1,h}\right\|_{H^1(0,T)}}{\|u_1\|_{H^1(0,T)}}$	eoc
0	4	1.01e+00	0.00	9.83e-01	0.00
1	8	9.89e-01	0.03	1.01e+00	-0.04
2	16	9.65e-01	0.04	1.00e+00	0.01
3	32	1.13e+00	-0.22	9.94e-01	0.01
4	64	1.14e+00	-0.02	9.96e-01	-0.00
5	128	4.77e-01	1.26	6.48e-01	0.62
6	256	1.31e-01	1.87	3.46e-01	0.91
7	512	3.37e-02	1.96	1.76e-01	0.97
8	1024	8.51e-03	1.99	8.84e-02	0.99
9	2048	2.13e-03	2.00	4.43e-02	1.00
10	4096	5.34e-04	2.00	2.21e-02	1.00
11	8192	1.34e-04	2.00	1.11e-02	1.00

Table 2: Numerical results of (3.22) with uniform meshes for $T = 10, \quad \mu = 10$ and the function u_1 for piecewise linear functions.

L	N	$\frac{\left\|u_2-u_{2,h}\right\|_{L^2(0,T)}}{\|u_2\|_{L^2(0,T)}}$	eoc	$\frac{\left\|u_2-u_{2,h}\right\|_{H^1(0,T)}}{\|u_2\|_{H^1(0,T)}}$	eoc
0	4	7.48e-02	0.00	6.06e-01	0.00
1	8	2.83e-02	1.40	4.62e-01	0.39
2	16	8.48e-03	1.74	2.75e-01	0.75
3	32	2.52e-03	1.75	1.65e-01	0.74
4	64	7.92e-04	1.67	1.05e-01	0.64
5	128	2.81e-04	1.50	7.55e-02	0.48
6	256	1.12e-04	1.32	5.92e-02	0.35
7	512	4.77e-05	1.24	4.85e-02	0.29
8	1024	2.05e-05	1.22	4.04e-02	0.26
9	2048	8.79e-06	1.22	3.39e-02	0.26
10	4096	3.74e-06	1.23	2.84e-02	0.25
11	8192	1.59e-06	1.24	2.39e-02	0.25

Table 3: Numerical results of (3.22) with uniform meshes for $T = 10, \quad \mu = 10$ and the function u_2 for piecewise linear functions.

Euclidean norm between the solution of the linear system (3.22) with a direct solver and the solution of the linear system (3.23) with the GMRES method with GMRES accuracy 10^{-10}. As implementation of the $\mathcal{H}$ matrices, the GYPSILAB software [14] is used.

L	N	$\frac{\mathrm{mem}(\widetilde{A}_h^{0,0,\mathrm{ACA}})}{\mathrm{mem}(\widetilde{A}_h^{0,0})}$	$\frac{\mathrm{mem}(\widetilde{A}_h^{0,1,\mathrm{ACA}})}{\mathrm{mem}(\widetilde{A}_h^{0,1})}$	$\varepsilon^{\mathrm{ACA}}$	$\frac{\left\|\underline{\widetilde{u}}_1-\underline{\widetilde{u}}_1^{\mathrm{ACA}}\right\|_2}{\|\underline{\widetilde{u}}_1\|_2}$
0	4	5.12	5.12	5.0e-05	3.1e-15
1	8	2.41	2.41	2.5e-05	6.6e-15
2	16	1.54	1.54	1.3e-05	3.3e-15
3	32	1.23	1.23	6.3e-06	4.7e-14
4	64	1.10	1.10	3.1e-06	7.6e-11
5	128	0.83	0.82	1.6e-06	6.4e-07
6	256	0.56	0.55	7.8e-07	7.5e-06
7	512	0.36	0.36	3.9e-07	3.3e-06
8	1024	0.22	0.22	2.0e-07	4.0e-06
9	2048	0.13	0.13	9.8e-08	3.8e-06
10	4096	0.08	0.08	4.9e-08	5.7e-06
11	8192	0.06	0.05	2.4e-08	4.1e-06

Table 4: Numerical results of (3.23) with uniform meshes for $T = 10$, $\mu = 10$ and the function u_1 for piecewise linear functions.

L	N	$\frac{\mathrm{mem}(\widetilde{A}_h^{0,0,\mathrm{ACA}})}{\mathrm{mem}(\widetilde{A}_h^{0,0})}$	$\frac{\mathrm{mem}(\widetilde{A}_h^{0,1,\mathrm{ACA}})}{\mathrm{mem}(\widetilde{A}_h^{0,1})}$	$\varepsilon^{\mathrm{ACA}}$	$\frac{\left\|\underline{\widetilde{u}}_2-\underline{\widetilde{u}}_2^{\mathrm{ACA}}\right\|_2}{\|\underline{\widetilde{u}}_2\|_2}$
0	4	5.12	5.12	5.0e-05	2.4e-15
1	8	2.41	2.41	2.5e-05	1.9e-15
2	16	1.54	1.54	1.3e-05	1.3e-14
3	32	1.23	1.23	6.3e-06	3.0e-09
4	64	1.10	1.10	3.1e-06	4.9e-09
5	128	0.82	0.83	1.6e-06	2.6e-08
6	256	0.56	0.56	7.8e-07	1.4e-07
7	512	0.36	0.36	3.9e-07	2.4e-07
8	1024	0.22	0.22	2.0e-07	1.1e-07
9	2048	0.13	0.13	9.8e-08	4.3e-08
10	4096	0.08	0.08	4.9e-08	4.7e-08
11	8192	0.06	0.05	2.4e-08	5.6e-08

Table 5: Numerical results of (3.23) with uniform meshes for $T = 10$, $\mu = 10$ and the function u_2 for piecewise linear functions.

5 Conclusions

We have proposed a realisation of the modified Hilbert transform $\mathcal{H}_T$, which is based on its integral representation. This appearing integral is split into a regular part and into a singular part. For the first, a numerical quadrature is applied, where an exponential convergence of the quadrature with respect to the number of Gauß integration nodes is achieved, and for the latter, the integration is done analytically. Furthermore, a fast realisation of $\mathcal{H}_T$ is derived via $\mathcal{H}$ matrices, where numerical examples show a significant reduction of the memory consumption.

The proposed realisation of $\mathcal{H}_T$ is directly applicable for the numerical solution of parabolic evolution equations in anisotropic Sobolev spaces, see [13], for a tensor–product space–time finite element space, where a realisation for higher polynomial degrees is straightforward. Moreover, the integral representation of $\mathcal{H}_T$ gives the possibility for a realisation of a space–time finite element method with unstructured and adaptive space–time meshes, which is a topic for future work.

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