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Boundary element methods for exterior boundary control problems

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Abstract

As a model problem, we discuss the application of boundary element methods for the solution of exterior Dirichlet boundary control problems subject to the Laplace equation with box constraints on the control. The observation of L_2 tracking type is considered on some open or closed manifold in the exterior of a bounded Lipschitz domain, while the cost or regularisation term of the Dirichlet control is considered in the related energy norm. The solution of the exterior Dirichlet boundary value problem is given via a representation formula, which also covers the required far field behavior. It turns out that the optimality system is equivalent to a variational inequality on the control boundary. We provide a stability and error analysis for the approximate solution by using boundary element methods, and we present some numerical experiments which confirm the theoretical results.

1 Introduction

Optimal boundary control problems subject to elliptic or parabolic partial differential equations play an important role in many applications, e.g., [6]. The Dirichlet control problem with a boundary observation subject to some interior boundary value problem was already considered in [9]. Since the control was considered in $L_2(\Gamma)$, the observation of the Neumann datum was required to be considered in $H^{-1}(\Gamma)$, see [9, Remark 5.1]. Equivalently, an observation of the Neumann datum in $L_2(\Gamma)$ requires to consider the Dirichlet control in $H^1(\Gamma)$. However, the natural pairing seems to be $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ for the Dirichlet and Neumann data, respectively, see, e.g., [9, Sect. 2.4] for a Neumann boundary control problem, where later the control space $H^{-1/2}(\Gamma)$ was replaced by $L_2(\Gamma)$ to avoid fractional powers of the Laplace–Beltrami operator which was used to represent related Sobolev norms, and to simplify the complicated nature of the problem [9, Remark 2.4]. In any case, boundary integral operators such as the single layer boundary integral operator, or the hypersingular boundary integral operator, see, e.g., [7, 11, 15, 17], can be used to induce Sobolev norms of fractional order, in particular in $H^{\pm 1/2}(\Gamma)$. Moreover, the Steklov–Poincaré operator describing the Dirichlet to Neumann map subject to an elliptic partial differential equation of second order, or its inverse, induce equivalent norms in $H^{\pm 1/2}(\Gamma)$ which in fact describe the energy which is related to the partial differential equation.

Boundary integral equations and boundary element methods seem to be a natural choice when considering optimal control problems, where the control and the observation are defined on the boundary, and when the partial differential equation is considered in an unbounded exterior domain. But to our knowledge, there are only few results known on the use of boundary element methods to solve optimal boundary control problems, see, e.g., [3, 19] for problems with point observations. In [12] we have analysed boundary element methods to solve a Dirichlet boundary control problem with distributed observations subject to the Poisson equation, see [13] for a related finite element approach. The boundary element approach can also be used for parabolic problems such as the heat equation [14].

In particular when considering the solution of exterior boundary value problems it seems to be advantageous to use boundary integral equations. Applications in mind cover the control of acoustic and electromagnetic waves. But before tackling such much more challenging problems, we start with the simple model problem of the Laplace equation.

In Sect. 2 we introduce the model problem of an exterior Dirichlet boundary control problem, and we describe an equivalent formulation by using boundary integral equations. The optimality system and the related complementarity conditions are given in Sect. 3, while the boundary element discretization is described in Sect. 4. The related stability and error analysis is given in Sect. 5, where the final L_2 error estimate provides a convergence behavior of almost second order. Some numerical experiments in Sect. 6 confirm the theoretical results.

2 Exterior Dirichlet boundary control problem

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded and simply connected Lipschitz domain with boundary $\Gamma_C := \partial\Omega$, and let $\Omega^c := \mathbb{R}^n \backslash \overline{\Omega}$. By n_x we denote the normal vector for $x \in \Gamma_C$ exterior to Ω. Moreover, let $\Gamma_O \subset \Omega^c$ be an open or closed $(n-1)$–dimensional Lipschitz manifold, see Fig. 1. The boundary control will be considered on Γ_C while the observation is taken on Γ_O.

Before we state the boundary control problem, we first consider the exterior Dirichlet boundary value problem

$$-\Delta u(x) = 0 \text{ for } x \in \Omega^c, \quad u(x) = z(x) \text{ for } x \in \Gamma_C, \tag{2.1}$$

where u has to satisfy the radiation condition, for a given $u_0 \in \mathbb{R}$,

$$u(x) - u_0 = \mathcal{O}\Big(\frac{1}{|x|}\Big) \text{ as } |x| \to \infty. \tag{2.2}$$

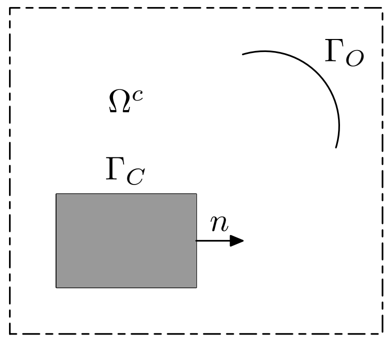

Figure 1: Configuration for $n = 2$.

The solution of the exterior Dirichlet boundary value problem (2.1) and (2.2) is given by the representation formula

$$u(x) = u_0 - \int_{\Gamma_C} U^*(x,y)\frac{\partial}{\partial n_y}u(y)ds_y + \int_{\Gamma_C} \frac{\partial}{\partial n_y}U^*(x,y)u(y)ds_y \quad \text{for } x \in \Omega^c \tag{2.3}$$

where

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } n=2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } n=3 \end{cases}$$

is the fundamental solution of the Laplacian. To satisfy the radiation condition (2.2), in the two–dimensional case $n = 2$ we have to ensure

$$\int_{\Gamma_C} \frac{\partial}{\partial n_y}u(y)\, ds_y = 0\,. \tag{2.4}$$

When considering the Dirichlet trace of the representation formula (2.3) we conclude the boundary integral equation

$$\int_{\Gamma_C} U^*(x,y)\frac{\partial}{\partial n_y}u(y)ds_y = u_0 - \frac{1}{2}z(x) + \int_{\Gamma_C} \frac{\partial}{\partial n_y}U^*(x,y)z(y)ds_y \quad \text{for almost all } x \in \Gamma_C.$$

In fact, $t := \frac{\partial}{\partial n}u \in H^{-1/2}(\Gamma_C)$ is the unique solution of the boundary integral equation

$$(V_{CC}t)(x) = u_0 + (-\frac{1}{2}I + K_{CC})z(x) \quad \text{for } x \in \Gamma_C \tag{2.5}$$

where

$$(V_{CC}t)(x) = \int_{\Gamma_C} U^*(x,y)t(y)ds_y \quad \text{for } x \in \Gamma_C$$

is the single layer boundary integral operator $V_{CC} : H^{-1/2}(\Gamma_C) \to H^{1/2}(\Gamma_C)$, and

$$(K_{CC}z)(x) = \int_{\Gamma_C} \frac{\partial}{\partial n_y}U^*(x,y)z(y)ds_y \quad \text{for } x \in \Gamma_C$$

is the double layer boundary integral operator $K_{CC} : H^{1/2}(\Gamma_C) \to H^{1/2}(\Gamma_C)$. Note that the single layer boundary integral operator V_{CC} is $H^{-1/2}(\Gamma_C)$–elliptic, where in the two–dimensional case we assume $\operatorname{diam}\Omega < 1$, and therefore invertible. In the two–dimensional case $n = 2$ we then obtain from (2.4) that

$$\begin{aligned}0 = \langle t, 1\rangle_{\Gamma_C} &= \langle V_{CC}t, V_{CC}^{-1}1\rangle_{\Gamma_C}\\ &= \langle u_0 + (-\frac{1}{2}I + K_{CC})z, V_{CC}^{-1}1\rangle_{\Gamma_C}\\ &= \langle u_0, V_{CC}^{-1}1\rangle_{\Gamma_C} + \langle(\frac{1}{2}I + K_{CC})z, V_{CC}^{-1}1\rangle_{\Gamma_C} - \langle z, V_{CC}^{-1}1\rangle_{\Gamma_C}\\ &= \langle u_0, V_{CC}^{-1}1\rangle_{\Gamma_C} - \langle z, V_{CC}^{-1}1\rangle_{\Gamma_C}\end{aligned}$$

due to

$$\langle(\frac{1}{2}I + K_{CC})z, V_{CC}^{-1}1\rangle_{\Gamma_C} = \langle z, (\frac{1}{2}I + K'_{CC})V_{CC}^{-1}1\rangle_{\Gamma_C} = \langle z, V_{CC}^{-1}(\frac{1}{2}I + K_{CC})1\rangle_{\Gamma_C} = 0.$$

Hence we conclude that in the two–dimensional case $n = 2$ the given Dirichlet datum $z \in H^{1/2}(\Gamma_C)$ has to satisfy the compatibility condition

$$\langle z, V_{CC}^{-1}1\rangle_{\Gamma_C} = u_0\,\langle 1, V_{CC}^{-1}1\rangle_{\Gamma_C}. \tag{2.6}$$

In the general case, i.e. for $n = 2$ and $n = 3$, and without loss of generality it is sufficient to consider the case $u_0 = 0$ only. From the boundary integral equation (2.5) we then conclude the Dirichlet to Neumann map

$$t(x) = -V_{CC}^{-1}(\frac{1}{2}I - K_{CC})z(x) = -(S_{CC}^{\text{ext}}z)(x) \tag{2.7}$$

with the exterior Steklov–Poincaré operator

$$S_{CC}^{\text{ext}} := V_{CC}^{-1}(\frac{1}{2}I - K_{CC}) : H^{1/2}(\Gamma_C) \to H^{-1/2}(\Gamma_C). \tag{2.8}$$

Now we are in the position to state the optimal control problem. As a model problem we consider the exterior Dirichlet boundary control problem to minimize the cost functional

$$\mathcal{J}(u, z) := \frac{1}{2}\int_{\Gamma_O}[u(x) - \overline{u}(x)]^2\,ds_x + \frac{1}{2}\varrho\langle S_{CC}^{\text{ext}}z, z\rangle_{\Gamma_C} \tag{2.9}$$

subject to the exterior Dirichlet boundary value problem

$$-\Delta u(x) = 0 \text{ for } x \in \Omega^c, \quad u(x) = z(x) \text{ for } x \in \Gamma_C, \quad u(x) = \mathcal{O}\Big(\frac{1}{|x|}\Big) \text{ as } |x| \to \infty, \tag{2.10}$$

where the control z satisfies the box constraints

$$z \in \mathcal{U} = \left\{v \in H_*^{1/2}(\Gamma_C) : g_a(x) \le v(x) \le g_b(x) \text{ for } x \in \Gamma_C\right\}.$$

Note that we use $H_*^{1/2}(\Gamma_C) := H^{1/2}(\Gamma_C)$ in the case $n = 3$, while for $n = 2$ we define

$$H_*^{1/2}(\Gamma_C) := \left\{ v \in H^{1/2}(\Gamma_C) \;:\; \langle v, V_{CC}^{-1} 1 \rangle_{\Gamma_C} = 0 \right\}.$$

We assume $\overline{u} \in L_2(\Gamma_O)$, $\varrho \in \mathbb{R}_+$, $g_a, g_b \in H_*^{1/2}(\Gamma_C)$, and $g_a < g_b$ on Γ_C. Note that the exterior Steklov–Poincaré operator (2.8) is $H^{1/2}(\Gamma_C)$–elliptic, and hence, defines an equivalent norm in $H_*^{1/2}(\Gamma_C)$ which represents the exterior Dirichlet form for the solution of the exterior Dirichlet boundary value problem (2.10), i.e.

$$\langle S_{CC}^{\text{ext}} z, z \rangle_\Gamma = -\int_{\Gamma_C} \frac{\partial}{\partial n_x} u(x) u(x)\, ds_x = \int_{\Omega^c} |\nabla u(x)|^2\, dx.$$

Before we rewrite the cost functional (2.9) by using boundary integral equations we will derive a second representation of the exterior Steklov–Poincaré operator (2.8). For this we consider the exterior normal derivative of the representation formula (2.3), i.e. for $x \in \Gamma$ we obtain

$$\begin{aligned} t(x) = \frac{\partial}{\partial n_x} u(x) &= \frac{1}{2} t(x) - \int_{\Gamma_C} \frac{\partial}{\partial n_x} U^*(x,y) t(y) ds_y + \frac{\partial}{\partial n_x} \int_{\Gamma_C} \frac{\partial}{\partial n_y} U^*(x,y) z(y) ds_y \\ &= (\frac{1}{2} I - K_{CC}^*) t(x) - (D_{CC} z)(x), \end{aligned} \tag{2.11}$$

where

$$(K_{CC}^* t)(x) = \int_{\Gamma_C} \frac{\partial}{\partial n_x} U^*(x,y) t(y) ds_y \quad \text{for } x \in \Gamma_C$$

is the adjoint double layer boundary integral operator $K_{CC}^* : H^{-1/2}(\Gamma_C) \to H^{-1/2}(\Gamma_C)$, and

$$(D_{CC} z)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma_C} \frac{\partial}{\partial n_y} U^*(x,y) z(y) ds_y \quad \text{for } x \in \Gamma_C$$

is the hypersingular boundary integral operator $D_{CC} : H^{1/2}(\Gamma_C) \to H^{-1/2}(\Gamma_C)$. When inserting the Dirichlet to Neumann map (2.7) into the boundary integral equation (2.11) this gives

$$t = (\frac{1}{2} I - K_{CC}^*) t - D_{CC} z = -\Big[(\frac{1}{2} I - K_{CC}^*) V_{CC}^{-1} (\frac{1}{2} I - K_{CC}) + D_{CC} \Big] z = -S_{CC}^{\text{ext}} z$$

with the symmetric representation of the Steklov–Poincaré operator

$$S_{CC}^{\text{ext}} = (\frac{1}{2} I - K_{CC}^*) V_{CC}^{-1} (\frac{1}{2} I - K_{CC}) + D_{CC} : H^{1/2}(\Gamma_C) \to H^{-1/2}(\Gamma_C). \tag{2.12}$$

By using the Dirichlet to Neumann map (2.7) we can write the evaluation of the representation formula (2.3) on the observation manifold as

$$\begin{aligned} u(x) &= \int_{\Gamma_C} U^*(x,y) (S_{CC}^{\text{ext}} z)(y) ds_y + \int_{\Gamma_C} \frac{\partial}{\partial n_y} U^*(x,y) z(y) ds_y \\ &= \Big[V_{CO} S_{CC}^{\text{ext}} + K_{CO} \Big] z(x) \quad \text{for } x \in \Gamma_O, \end{aligned}$$

where

$$(V_{CO}t)(x) = \int_{\Gamma_C} U^*(x,y)t(y)ds_y \quad \text{for } x \in \Gamma_O$$

describes the evaluation of the single layer potential $V_{CO} : H^{-1/2}(\Gamma_C) \to H^{1/2}(\Gamma_O)$ on Γ_O, and

$$(K_{CO}z)(x) = \int_{\Gamma_C} \frac{\partial}{\partial n_y} U^*(x,y)z(y)ds_y \quad \text{for } x \in \Gamma_O$$

is the double layer potential $K_{CO} : H^{1/2}(\Gamma_C) \to H^{1/2}(\Gamma_O)$ on Γ_O with its adjoint $K^*_{OC} : H^{-1/2}(\Gamma_O) \to H^{-1/2}(\Gamma_C)$. Hence we obtain the control to state operator

$$\mathcal{H} := V_{CO}S^{\text{ext}}_{CC} + K_{CO} : H^{1/2}(\Gamma_C) \to H^{1/2}(\Gamma_O) \subset L_2(\Gamma_O),$$

and we conclude the reduced cost functional

$$\widetilde{J}(z) := \frac{1}{2}\int_{\Gamma_O} [(\mathcal{H}z)(x) - \overline{u}(x)]^2\, ds_x + \frac{1}{2}\varrho\langle S^{\text{ext}}_{CC}z, z\rangle_{\Gamma_C} \to \min_{z\in\mathcal{U}} . \tag{2.13}$$

The solution of the minimization problem (2.13) is equivalent to find the solution $z \in \mathcal{U}$ of the variational inequality

$$\langle \mathcal{H}^*(\mathcal{H}z - \overline{u}) + \varrho S^{\text{ext}}_{CC}z, v - z\rangle_{\Gamma_C} \geq 0 \quad \text{for all } v \in \mathcal{U}, \tag{2.14}$$

where

$$\mathcal{H}^* := K^*_{OC} + S^{\text{ext}}_{CC}V_{OC} : L_2(\Gamma_O) \subset H^{-1/2}(\Gamma_O) \to H^{1/2}(\Gamma_C) \tag{2.15}$$

is the adjoint operator of $\mathcal{H}$.

Remark 2.1 *The adjoint operator $\mathcal{H}^*$ as given in* (2.15) *can be written as, by using that S^{ext}_{CC} is self–adjoint,*

$$\mathcal{H}^* = K^*_{OC} + (\frac{1}{2}I - K^*_{CC})V^{-1}_{CC}V_{OC}.$$

For a given $w \in L_2(\Gamma_O)$ the application $\mathcal{H}^*w$ is the Neumann trace

$$\frac{\partial}{\partial n_x}p(x) = \int_{\Gamma_O} \frac{\partial}{\partial n_x}U^*(x,y)w(y)ds_y + \frac{1}{2}(V^{-1}_{CC}V_{OC}w)(x) - \int_{\Gamma_C} \frac{\partial}{\partial n_x}U^*(x,y)(V^{-1}_{CC}V_{OC})w(y)ds_y$$

for $x \in \Gamma_C$ of the representation formula

$$p(x) = \int_{\Gamma_O} U^*(x,y)w(y)ds_y - \int_{\Gamma_C} U^*(x,y)q(y)ds_y \quad \textit{for } x \in \Omega^c,$$

where $q \in H^{-1/2}(\Gamma_C)$ solves the boundary integral equation

$$\int_{\Gamma_C} U^*(x,y)q(y)ds_y = \int_{\Gamma_O} U^*(x,y)w(y)ds_y \quad \textit{for } x \in \Gamma_C.$$

From this we conclude the adjoint boundary value problem

$$-\Delta p = 0 \quad \textit{in } \Omega^c\backslash\overline{\Gamma}_O, \quad p = 0 \quad \textit{on } \Gamma_C, \quad [p]_{|\Gamma_O} = 0, \quad [\partial_n p]_{|\Gamma_O} = w,$$

where p has to satsify the radiation condition $p(x) = \mathcal{O}(1/|x|)$ as $|x| \to \infty$. Note that $[\cdot]_{|\Gamma_O}$ denotes the jump across Γ_O.

The operator as used in the variational inequality (2.14),

$$\begin{aligned} T_\varrho &:= \mathcal{H}^*\mathcal{H} + \varrho S_{CC}^{ext} \\ &= \Big[(\frac{1}{2}I - K_{CC}^*)V_{CC}^{-1}V_{OC} + K_{OC}^*\Big]\Big[V_{CO}V_{CC}^{-1}(\frac{1}{2}I - K_{CC}) + K_{CO}\Big] \\ &\qquad + \varrho\Big[(\frac{1}{2}I - K_{CC}^*)V_{CC}^{-1}(\frac{1}{2}I - K_{CC}) + D_{CC}\Big] \end{aligned} \tag{2.16}$$

is bounded, i.e. $T_\varrho : H^{1/2}(\Gamma_C) \to H^{-1/2}(\Gamma_C)$, and $H^{1/2}(\Gamma_C)$–elliptic. Hence, (2.14) is an elliptic variational inequality of the first kind, and we can use standard arguments [4, 9] to establish unique solvability of the variational inequality (2.14), i.e. $z \in \mathcal{U}$ satisfies

$$\langle T_\varrho z, v - z\rangle_{\Gamma_C} \geq \langle f, v - z\rangle_{\Gamma_C} \quad \text{for all } v \in \mathcal{U}, \tag{2.17}$$

where

$$f := \mathcal{H}^*\overline{u} = \Big[K_{OC}^* + (\frac{1}{2}I - K_{CC}^*)V_{CC}^{-1}V_{OC}\Big]\overline{u} \in H^{-1/2}(\Gamma_C). \tag{2.18}$$

3 Optimality system and complementarity conditions

For the solution $z \in \mathcal{U}$ of the variational inequality (2.17) we introduce

$$\lambda := T_\varrho z - f \in H^{-1/2}(\Gamma_C),$$

and from the variational inequality

$$\langle \lambda, v - z\rangle_{\Gamma_C} \geq 0 \quad \text{for all } v \in \mathcal{U}$$

we conclude the complementarity conditions

$$\begin{aligned} \lambda &\leq 0 \qquad \text{for } z = g_b, \\ \lambda &= 0 \qquad \text{for } g_a < z < g_b, \\ \lambda &\geq 0 \qquad \text{for } z = g_a. \end{aligned} \tag{3.1}$$

For the evaluation of λ we obtain

$$\begin{aligned} \lambda &= T_\varrho z - f \;=\; \mathcal{H}^*(\mathcal{H}z - \overline{u}) + \varrho S_{CC}^{\text{ext}} z \\ &= \Big[(\frac{1}{2}I - K_{CC}^*)V_{CC}^{-1}V_{OC} + K_{OC}^*\Big]\Big[\Big(V_{CO}V_{CC}^{-1}(\frac{1}{2}I - K_{CC}) + K_{CO}\Big)z - \overline{u}\Big] \\ &\qquad + \varrho\Big[(\frac{1}{2}I - K_{CC}^*)V_{CC}^{-1}(\frac{1}{2}I - K_{CC}) + D_{CC}\Big]z \\ &= (\frac{1}{2}I - K_{CC}^*)q + K_{OC}^*(u - \overline{u}) - \varrho(\frac{1}{2}I - K_{CC}^*)t + \varrho D_{CC} z \end{aligned}$$

where $t \in H^{-1/2}(\Gamma_C)$ is the unique solution of the boundary integral equation

$$V_{CC} t = (-\frac{1}{2}I + K_{CC})z \quad \text{on } \Gamma_C,$$

$q \in H^{-1/2}(\Gamma_C)$ solves

$$V_{CC} q = V_{OC}(u - \overline{u}) \quad \text{on } \Gamma_C,$$

and $u \in H^{1/2}(\Gamma_O) \subset L_2(\Gamma_O)$ is the point evaluation

$$u = -V_{CO} t + K_{CO} z \quad \text{on } \Gamma_O.$$

To conclude, we have to find $(z, t, q, u) \in \mathcal{U} \times H^{-1/2}(\Gamma_C) \times H^{-1/2}(\Gamma_C) \times L_2(\Gamma_O)$ such that

$$\langle (\frac{1}{2}I - K_{CC}^*) q + K_{OC}^*(u - \overline{u}) - \varrho(\frac{1}{2}I - K_{CC}^*) t + \varrho D_{CC} z, v - z\rangle_{\Gamma_C} \geq 0 \quad \text{for all } v \in \mathcal{U}, \tag{3.2}$$

$$\langle V_{CC} t + (\frac{1}{2}I - K_{CC}) z, \tau\rangle_{\Gamma_C} = 0 \quad \text{for all } \tau \in H^{-1/2}(\Gamma_C), \tag{3.3}$$

$$\langle V_{CC} q - V_{OC}(u - \overline{u}), r\rangle_{\Gamma_C} = 0 \quad \text{for all } r \in H^{-1/2}(\Gamma_C), \tag{3.4}$$

$$\langle u + V_{CO} t - K_{CO} z, w\rangle_{\Gamma_0} = 0 \quad \text{for all } w \in L_2(\Gamma_O). \tag{3.5}$$

Since the optimality system (3.2)–(3.5) is equivalent to the variational inequality (2.14), unique solvability follows. In the case of no box constraints, (3.2) becomes a variational equality, and we finally obtain a system of boundary integral equations to be solved:

$$\begin{pmatrix} & V_{CC} & & \frac{1}{2}I - K_{CC} \\ V_{CC} & & -V_{OC} & \\ & V_{CO} & I & -K_{CO} \\ \frac{1}{2}I - K_{CC}^* & -\varrho(\frac{1}{2}I - K_{CC}^*) & K_{OC}^* u & \varrho D_{CC} \end{pmatrix} \begin{pmatrix} q \\ t \\ u \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -V_{OC}\overline{u} \\ 0 \\ K_{OC}^*\overline{u} \end{pmatrix}. \tag{3.6}$$

By eliminating the first three equations we obtain the Schur complement in z, which in fact involves the operator T_ϱ as given in (2.17). To obtain a symmetric representaion also for the boundary integral equation system (3.6), the first equation is multiplied by $-\varrho$ and added to the second equation. Moreover, the third equation is multiplied by -1 to end up with a self–adjoint operator.

4 Boundary element discretization

For an approximate solution of the exterior Dirichlet boundary control problem (2.9) we consider a boundary element discretization of the optimality system (3.2)–(3.5). For this we introduce the ansatz space $S_h^1(\Gamma_C) = \text{span}\{\varphi_i\}_{i=1}^{M_C} \subset H^{1/2}(\Gamma_C)$ of piecewise linear and continuous basis functions φ_i which are defined with respect to a quasi–uniform boundary element mesh of mesh size h. In addition we define $S_h^0(\Gamma_C) = \text{span}\{\psi_k\}_{k=1}^{N_C} \subset H^{-1/2}(\Gamma_C)$ to be the ansatz space of piecewise constant basis functions ψ_k. Moreover, for the observation on Γ_O we use the ansatz space $S_h^0(\Gamma_O) = \text{span}\{\phi_k\}_{k=1}^{N_O} \subset L_2(\Gamma_O)$ of piecewise constant basis functions ϕ_k, where the underlying boundary element mesh is again of mesh size h.

In what follows we will only consider the three–dimensional case $n = 3$ where we have no additional constraints on the control z. In the two–dimensional case $n = 2$ we have to incorporate the constraint $\langle z, V_{CC}^{-1}1\rangle_{\Gamma_C} = 0$ which can be done by using a scalar Lagrange multiplier.

For continuous functions g_a und g_b we define the discrete convex set

$$\mathcal{U}_h := \Big\{ v_h \in S_h^1(\Gamma_C) : g_a(x_i) \le v_h(x_i) \le g_b(x_i) \quad \text{for all nodes } x_i \in \Gamma_C \Big\}.$$

The Galerkin boundary element discretization of the optimality system (3.2)–(3.5) is to find $(z_h, t_h, q_h, u_h) \in \mathcal{U}_h \times S_h^0(\Gamma_C) \times S_h^0(\Gamma_C) \times S_h^0(\Gamma_O)$ such that

$$\langle (\frac{1}{2}I - K_{CC}^*)q_h + K_{OC}^*(u_h - \overline{u}) - \varrho(\frac{1}{2}I - K_{CC}^*)t_h + \varrho D_{CC} z_h, v_h - z_h\rangle_{\Gamma_C} \ge 0 \quad \text{for all } v_h \in \mathcal{U}_h, \tag{4.1}$$

$$\langle V_{CC} t_h + (\frac{1}{2}I - K_{CC}) z_h, \tau_h\rangle_{\Gamma_C} = 0 \quad \text{for all } \tau_h \in S_h^0(\Gamma_C), \tag{4.2}$$

$$\langle V_{CC} q_h - V_{OC}(u_h - \overline{u}), r_h\rangle_{\Gamma_C} = 0 \quad \text{for all } r_h \in S_h^0(\Gamma_C), \tag{4.3}$$

$$\langle u_h + V_{CO} t_h - K_{CO} z_h, w_h\rangle_{\Gamma_0} = 0 \quad \text{for all } w_h \in S_h^0(\Gamma_O). \tag{4.4}$$

By using the Galerkin boundary element stiffness matrices

$$\begin{aligned}
V_{CC,h}[\ell,k] &= \langle V_{CC}\psi_k, \psi_\ell\rangle_{\Gamma_C} \quad \text{for } k,\ell = 1,\dots,N_C,\\
V_{CO,h}[\ell,k] &= \langle V_{CO}\psi_k, \phi_\ell\rangle_{\Gamma_O} \quad \text{for } k = 1,\dots,N_C, \ell = 1,\dots,N_O,\\
K_{CC,h}[i,\ell] &= \langle K_{CC}\varphi_i, \psi_\ell\rangle_{\Gamma_C} \quad \text{for } i = 1,\dots,M_C, \ell = 1,\dots,N_C,\\
K_{CO,h}[i,\ell] &= \langle K_{CO}\varphi_i, \phi_\ell\rangle_{\Gamma_O} \quad \text{for } i = 1,\dots,M_C, \ell = 1,\dots,N_O,\\
D_{CC,h}[i,j] &= \langle D_{CC}\varphi_i, \varphi_j\rangle_{\Gamma_C} \quad \text{for } i,j = 1,\dots,M_C,\\
M_{CC,h}[i,\ell] &= \langle \varphi_i, \psi_\ell\rangle_{\Gamma_C} \quad \text{for } i = 1,\dots,M_C, \ell = 1,\dots,N_C,\\
\overline{M}_{OO}[\ell,k] &= \langle \phi_k, \phi_\ell\rangle_{L_2(\Gamma_O)} \quad \text{for } k,\ell = 1,\dots,N_O,
\end{aligned}$$

and the load vectors

$$\begin{aligned}
f_1[j] &= \langle K_{OC}^*\overline{u}, \varphi_j\rangle_{\Gamma_C} \quad \text{for } j = 1,\dots,M_C,\\
f_2[\ell] &= \langle V_{OC}\overline{u}, \psi_\ell\rangle_{\Gamma_C} \quad \text{for } \ell = 1,\dots,N_C,
\end{aligned}$$

we obtain

$$\begin{aligned}
((\frac{1}{2}M_{CC,h}^\top - K_{CC,h}^\top)\underline{q} + K_{CO,h}^\top\underline{u} - \underline{f}_1 - \varrho(\frac{1}{2}M_{CC,h}^\top - K_{CC,h}^\top)\underline{t} + \varrho D_{CC,h}\underline{z}, \underline{v} - \underline{z}) &\ge 0,\\
V_{CC,h}\underline{t} + (\frac{1}{2}M_{CC,h} - K_{CC,h})\underline{z} &= \underline{0},\\
V_{CC,h}\underline{q} - V_{CO,h}^\top\underline{u} + \underline{f}_2 &= \underline{0},\\
\overline{M}_{OO}\underline{u} + V_{CO,h}\underline{t} - K_{CO,h}\underline{z} &= \underline{0}.
\end{aligned}$$

With

$$\begin{aligned}
\underline{t} &= -V_{CC,h}^{-1}(\frac{1}{2}M_{CC,h} - K_{CC,h})\underline{z}, \\
\underline{u} &= \overline{M}_{OO}^{-1}\Big(V_{CO,h}V_{CC,h}^{-1}(\frac{1}{2}M_{CC,h} - K_{CC,h}) + K_{CO,h}\Big)\underline{z}, \\
\underline{q} &= V_{CC,h}^{-1}V_{CO,h}^{\top}\overline{M}_{OO}^{-1}\Big(V_{CO,h}V_{CC,h}^{-1}(\frac{1}{2}M_{CC,h} - K_{CC,h}) + K_{CO,h}\Big)\underline{z} - V_{CC,h}^{-1}\underline{f}_2
\end{aligned}$$

we conclude the discrete variational inequality to find $\underline{z} \in \mathbb{R}^{M_C} \leftrightarrow z_h \in \mathcal{U}_h$ such that

$$(\widetilde{T}_h\underline{z}, \underline{v} - \underline{z}) \geq (\underline{\widetilde{f}}, \underline{v} - \underline{z}) \quad \text{for all } \underline{v} \in \mathbb{R}^{M_C} \leftrightarrow v_h \in \mathcal{U}_h, \tag{4.5}$$

where

$$\begin{aligned}
\widetilde{T}_{\varrho,h} &:= \varrho\Big[(\frac{1}{2}M_{CC,h}^{\top} - K_{CC,h}^{\top})V_{CC,h}^{-1}(\frac{1}{2}M_{CC,h} - K_{CC,h}) + D_{CC,h}\Big] \qquad (4.6) \\
&+\Big[(\frac{1}{2}M_{CC,h}^{\top} - K_{CC,h}^{\top})V_{CC,h}^{-1}V_{CO,h}^{\top} + K_{CO,h}^{\top}\Big]\overline{M}_{OO}^{-1}\Big[V_{CO,h}V_{CC,h}^{-1}(\frac{1}{2}M_{CC,h} - K_{CC,h}) + K_{CO,h}\Big]
\end{aligned}$$

is a symmetric and positive definite boundary element approximation of $T_\varrho = \mathcal{H}^*\mathcal{H} + \varrho S_{CC}^{\mathrm{ext}}$, and

$$\underline{\widetilde{f}} := \underline{f}_1 + ((\frac{1}{2}M_{CC,h}^{\top} - K_{CC,h}^{\top})V_{CC,h}^{-1}\underline{f}_2 \tag{4.7}$$

is the related approximation of (2.18).

Since the discrete variational inequality (4.5) is again an elliptic variational inequality of the first kind, unique solvability follows as in the continuous case. By introducing

$$\begin{aligned}
\underline{\lambda} &:= \widetilde{T}_{\varrho,h}\underline{z} - \underline{\widetilde{f}} \\
&= (\frac{1}{2}M_{CC,h}^{\top} - K_{CC,h}^{\top})\underline{q} + K_{CO,h}^{\top}\underline{u} - \underline{f}_1 - \varrho(\frac{1}{2}M_{CC,h}^{\top} - K_{CC,h}^{\top})\underline{t} + \varrho D_{CC,h}\underline{z}
\end{aligned}$$

we conclude the discrete complementarity conditions

$$\begin{aligned}
\lambda_i &\leq 0 \quad \text{for } z_i = g_b(x_i), \\
\lambda_i &= 0 \quad \text{for } g_a(x_i) < z_i < g_b(x_i), \\
\lambda_i &\geq 0 \quad \text{for } z_i = g_a(x_i).
\end{aligned} \tag{4.8}$$

From this we find the equivalent characterizations, $c \in \mathbb{R}_+$,

$$\lambda_i^- = \min\Big\{0, \lambda_i^- + c[g_b(x_i) - z_i]\Big\}, \quad \lambda_i^+ = \max\Big\{0, \lambda_i^+ + c[g_a(x_i) - z_i]\Big\},$$

where $\lambda^\pm$ corresponds to the positive and negative contributions. Hence, for the solution of the discrete variational inequality (4.5) we can apply a semi–smooth Newton method, which turns out to be an active set strategy [5, 8].

5 Error analysis

In this section we provide error estimates for the approximate solution $z_h \in \mathcal{U}_h \leftrightarrow \underline{z} \in \mathbb{R}^{M_C}$ of the discrete variational inequality (4.5). Since the variational inequality (4.5) involves Galerkin approximations of both the operator T_ϱ and the right hand side f, we have to apply Strang type error estimates. First, we consider the Galerkin variational formulation of the variational inequality (2.17) to find $\overline{z}_h \in \mathcal{U}_h$ such that

$$\langle T_\varrho \overline{z}_h, v_h - \overline{z}_h \rangle_{\Gamma_C} \geq \langle f, v_h - \overline{z}_h \rangle_{\Gamma_C} \quad \text{for all } v_h \in \mathcal{U}_h. \tag{5.1}$$

Using [18, Theorem 3.4] we first obtain the energy error estimate

$$\|z - \overline{z}_h\|_{H^{1/2}(\Gamma_C)} \leq c\, h^{\sigma - 1/2} \left(|z|^2_{H^\sigma_{\text{pw}}} + |g_a|^2_{H^\sigma_{\text{pw}}(\Gamma_C)} + |g_b|^2_{H^\sigma_{\text{pw}}(\Gamma_C)} \right)^{1/2} \tag{5.2}$$

when assuming $z, g_a, g_b \in H^\sigma_{\text{pw}}(\Gamma_C)$ for some $\sigma \in (\frac{n-1}{2}, 2]$.

For the solution $\underline{z} \in \mathbb{R}^{M_C} \leftrightarrow z_h \in \mathcal{U}_h$ of the discrete variational inequality (4.5) we can use [13, Theorem 3.2]. Let $\widetilde{T}_\varrho : H^{1/2}(\Gamma_C) \to H^{-1/2}(\Gamma_C)$ be a bounded and $S^1_h(\Gamma_C)$–elliptic approximation of T_ϱ satisfying

$$\langle \widetilde{T}_\varrho v_h, v_h \rangle_{\Gamma_C} \geq c_1^{\widetilde{T}_\varrho} \|v_h\|^2_{H^{1/2}(\Gamma_C)} \quad \text{for all } S^1_h(\Gamma_C)$$

and

$$\|\widetilde{T}_\varrho v\|_{H^{-1/2}(\Gamma)} \leq c_2^{\widetilde{T}_\varrho} \|v\|^2_{H^{1/2}(\Gamma_C)} \quad \text{for all } v \in H^{1/2}(\Gamma_C).$$

Let $\widetilde{f} \in H^{-1/2}(\Gamma_C)$ be some approximation of f. Then there holds the error estimate

$$\|z - z_h\|_{H^{1/2}(\Gamma_C)} \leq c \left[\|z - \overline{z}_h\|_{H^{1/2}(\Gamma_C)} + \|(T_\varrho - \widetilde{T}_\varrho) z\|_{H^{-1/2}(\Gamma_C)} + \|f - \widetilde{f}\|_{H^{-1/2}(\Gamma_C)} \right]. \tag{5.3}$$

While the first part of the above error estimate corresponds to the Galerkin error (5.2), it remains to estimate the approximation errors of $\widetilde{T}_\varrho$ and $\widetilde{f}$, respectively.

Lemma 5.1 *Let f be given as defined in* (2.18), *and let $\widetilde{f}$ be the approximation which is related to the load vector* (4.7). *Assume $V_{CC}^{-1} V_{OC} \overline{u} \in H^s_{pw}(\Gamma_C)$ for some $s \in [0, 1]$. Then there holds the error estimate*

$$\|f - \widetilde{f}\|_{H^{-1/2}(\Gamma_C)} \leq c\, h^{s + \frac{1}{2}} \|V_{CC}^{-1} V_{OC} \overline{u}\|_{H^s_{pw}(\Gamma_C)}. \tag{5.4}$$

Proof. The Galerkin approximation of

$$f = K^*_{OC} \overline{u} + (\frac{1}{2} I - K^*_{CC}) V_{CC}^{-1} V_{OC} \overline{u}$$

is given by

$$\widetilde{f} = K^*_{OC} \overline{u} + (\frac{1}{2} I - K^*_{CC}) q_h,$$

where $q_h \in S_h^0(\Gamma_C)$ is the unique solution of the Galerkin variational formulation

$$\langle V_{CC} q_h, r_h \rangle_{\Gamma_C} = \langle V_{OC} \overline{u}, r_h \rangle_{\Gamma_C} \quad \text{for all } r_h \in S_h^0(\Gamma_C).$$

The assertion then follows from, by using standard boundary element error estimates, see, e.g., [15, 17],

$$\begin{aligned} \| f - \widetilde{f} \|_{H^{-1/2}(\Gamma_C)} &= \| (\frac{1}{2} I - K_{CC}^*)(q - q_h) \|_{H^{-1/2}(\Gamma_C)} \\ &\leq c \, \| q - q_h \|_{H^{-1/2}(\Gamma_C)} \leq c \, h^{s+\frac{1}{2}} \, \| q \|_{H^s_{\text{pw}}(\Gamma_C)} \end{aligned}$$

when assuming $q = V_{CC}^{-1} V_{CO} \overline{u} \in H^s_{\text{pw}}(\Gamma_C)$ for some $s \in [0, 1]$. ■

Lemma 5.2 *Let $T_\varrho : H^{1/2}(\Gamma_C) \to H^{-1/2}(\Gamma)$ as defined in* (2.17)*, and let $\widetilde{T}_\varrho$ be the approximate operator which is related to the approximate Galerkin discretization* (4.6)*. Assume*

$$t = V_{CC}^{-1}(\frac{1}{2} I - K_{CC}) z \in H^s_{pw}(\Gamma_C), \; u = V_{CO} t + K_{CO} z \in H^s_{pw}(\Gamma_O), \; q = V_{CC}^{-1} V_{OC} u \in H^s_{pw}(\Gamma_C)$$

for some $s \in [0, 1]$. Then there holds the error estimate

$$\| (T_\varrho - \widetilde{T}_\varrho) z \|_{H^{-1/2}(\Gamma_C)} \leq c \, h^{s+1/2} \Big[\| q \|_{H^s_{pw}(\Gamma_C)} + \| u \|_{H^s_{pw}(\Gamma_O)} + \| t \|_{H^s_{pw}(\Gamma_C)} \Big]. \tag{5.5}$$

Proof. For $z \in H^{1/2}(\Gamma_C)$ the application of T_ϱ is given by

$$\begin{aligned} T_\varrho z &= \Big((\frac{1}{2} I - K_{CC}^*) V_{CC}^{-1} V_{OC} + K_{OC}^* \Big) \Big(V_{CO} V_{CC}^{-1} (\frac{1}{2} I - K_{CC}) + K_{CO} \Big) z \\ &\qquad + \varrho \Big[(\frac{1}{2} I - K_{CC}^*) V_{CC}^{-1} (\frac{1}{2} I - K_{CC}) + D_{CC} \Big] z \\ &= (\frac{1}{2} I - K_{CC}^*) q + K_{OC}^* u + \varrho (\frac{1}{2} I - K_{CC}^*) t + \varrho D_{CC} z, \end{aligned}$$

where we have used

$$t = V_{CC}^{-1}(\frac{1}{2} I - K_{CC}) z, \quad u = V_{CO} t + K_{CO} z, \quad q = V_{CC}^{-1} V_{OC} u.$$

If we define the Galerkin approximations $(t_h, u_h, q_h) \in S_h^0(\Gamma_C) \times S_h^0(\Gamma_O) \times S_h^0(\Gamma_C)$ as unique solutions of the variational formulations

$$\langle V_{CC} t_h, \tau_h \rangle_{\Gamma_C} = \langle (\frac{1}{2} I - K_{CC}) z, \tau_h \rangle_{\Gamma_C} \quad \text{for all } \tau_h \in S_h^0(\Gamma_C),$$

$$\langle u_h, w_h \rangle_{L_2(\Gamma_O)} = \langle V_{CO} t_h + K_{CO} z, w_h \rangle_{L_2(\Gamma_O)} \quad \text{for all } w_h \in S_h^0(\Gamma_O),$$

and

$$\langle V_{CC} q_h, r_h \rangle_{\Gamma_C} = \langle V_{OC} u_h, r_h \rangle_{\Gamma_C} \quad \text{for all } r_h \in S_h^0(\Gamma_C),$$

we obtain for the application of the approximate operator $\widetilde{T}_\varrho$

$$\widetilde{T}_\varrho z = (\frac{1}{2}I - K^*_{CC})q_h + K^*_{OC}u_h + \varrho(\frac{1}{2}I - K^*_{CC})t_h + \varrho D_{CC}z.$$

Hence we conclude

$$\begin{aligned} \|(T_\varrho - \widetilde{T}_\varrho)z\|_{H^{-1/2}(\Gamma)} &= \|(\frac{1}{2}I - K^*_{CC})(q-q_h) + K^*_{OC}(u-u_h) + \varrho(\frac{1}{2}I - K^*_{CC})(t-t_h)\|_{H^{-1/2}(\Gamma_C)} \\ &\leq c_1 \|q-q_h\|_{H^{-1/2}(\Gamma_C)} + c_2 \|u-u_h\|_{H^{-1/2}(\Gamma_O)} + c_3\varrho \|t-t_h\|_{H^{-1/2}(\Gamma_C)}. \end{aligned}$$

Using standard arguments we first find

$$\|t-t_h\|_{H^{-1/2}(\Gamma_C)} \leq c\, h^{s+1/2} \|t\|_{H^s_{\rm pw}(\Gamma_C)}$$

if we assume $t \in H^s_{\rm pw}(\Gamma)$ for some $s \in [0,1]$, and by using the Aubin–Nitsche trick we conclude

$$\|t-t_h\|_{H^{-1}(\Gamma_C)} \leq ch^{s+1} \|t\|_{H^s_{\rm pw}(\Gamma_C)}.$$

Since $u_h \in S^0_h(\Gamma_O)$ is itself the Galerkin solution of a perturbed variational problem, we find, if we assume $u \in H^s_{\rm pw}(\Gamma_O)$,

$$\begin{aligned} \|u-u_h\|_{L_2(\Gamma_O)} &\leq \inf_{w_h\in S^0_h(\Gamma_O)} \|u-w_h\|_{L_2(\Gamma_O)} + \|V_{CO}(t-t_h)\|_{L_2(\Gamma_O)} \\ &\leq \inf_{w_h\in S^0_h(\Gamma_O)} \|u-w_h\|_{L_2(\Gamma_O)} + c\,\|t-t_h\|_{H^{-1}(\Gamma_C)} \\ &\leq c_1\, h^s \|u\|_{H^s_{\rm pw}(\Gamma_O)} + c_2 h^{s+1} \|t\|_{H^s_{\rm pw}(\Gamma_C)}. \end{aligned}$$

By using the Aubin–Nitsche trick we further obtain

$$\|u-u_h\|_{H^{-1/2}(\Gamma_O)} \leq c_1\, h^{s+1/2} \|u\|_{H^s_{\rm pw}(\Gamma_O)} + c_2 h^{s+1} \|t\|_{H^s_{\rm pw}(\Gamma_C)}.$$

In the same way we finally conclude

$$\begin{aligned} \|q-q_h\|_{H^{-1/2}(\Gamma_C)} &\leq c_1 \inf_{r_h\in S^0_h(\Gamma_C)} \|q-r_h\|_{H^{-1/2}(\Gamma_C)} + c_2\|u-u_h\|_{H^{-1/2}(\Gamma_O)} \\ &\leq c_1\, h^{s+1/2} \|q\|_{H^s_{\rm pw}(\Gamma_C)} + c_2\, h^{s+1/2} \|u\|_{H^s_{\rm pw}(\Gamma_O)} + c_3 h^{s+1} \|t\|_{H^s_{\rm pw}(\Gamma_C)}. \end{aligned}$$

■

By combing the error estimate (5.3) with the error estimates (5.2), (5.4) and (5.5) we are now in the position to formulate the main result of this paper.

Theorem 5.3 *Let $z \in \mathcal{U}$ be the unique solution of* (2.14), *and let $z_h \in \mathcal{U}_h \leftrightarrow \underline{z} \in \mathbb{R}^{M_C}$ be the unique solution of the discrete variational inequality* (4.5). *We assume $z, g_a, g_b \in H^\sigma_{pw}(\Gamma_C)$ for some $\sigma \in (\frac{n-1}{2}, 2]$ and the regularity result*

$$\|t\|_{H^{\sigma-1}_{pw}(\Gamma_C)} = \|S^{ext}_{CC} z\|_{H^{\sigma-1}(\Gamma_C)} \leq c\,\|z\|_{H^\sigma_{pw}(\Gamma_C)}.$$

Then there holds the error estimate

$$\|z-z_h\|_{H^{1/2}(\Gamma_C)} \leq c\, h^{\sigma-1/2} \left(\|z\|^2_{H^\sigma_{pw}} + |g_a|^2_{H^\sigma_{pw}(\Gamma_C)} + |g_b|^2_{H^\sigma_{pw}(\Gamma_C)}\right)^{1/2}.$$

Finally, when applying the Aubin–Nitsche trick for variational inequalities [18] we conclude the error estimate

$$\|z - z_h\|_{L_2(\Gamma_C)} \le c\, h^{\sigma} \left(\|z\|^2_{H^{\sigma}_{\mathrm{pw}}} + |g_a|^2_{H^{\sigma}_{\mathrm{pw}}(\Gamma_C)} + |g_b|^2_{H^{\sigma}_{\mathrm{pw}}(\Gamma_C)} \right)^{1/2}.$$

In fact, as for the solution of boundary value problems with Signorini boundary conditions we find for the state $u \in H^{5/2-\varepsilon}(\Omega)$, $\varepsilon > 0$, i.e. $z \in H^{2-\varepsilon}(\Gamma)$. Hence we conclude

$$\|z - z_h\|_{L_2(\Gamma_C)} \le c\, h^{2-\varepsilon} \left(\|z\|^2_{H^{2-\varepsilon}_{\mathrm{pw}}} + |g_a|^2_{H^{2}_{\mathrm{pw}}(\Gamma_C)} + |g_b|^2_{H^{2}_{\mathrm{pw}}(\Gamma_C)} \right)^{1/2} \tag{5.6}$$

when assuming $g_a, g_b \in H^2_{\mathrm{pw}}(\Gamma_C)$, i.e. an almost quadratic convergence in h.

6 Numerical results

In this section we will consider some numerical examples to test our theoretical results. For all the simulations the software `BEM++` [16] is used. As this software provides the `AHMED` library [1], all discrete boundary integral operators are assembled with the Adaptive Cross Aproximation (ACA).

First we consider a test case where the exact solution is known. It turns out that the construction of an exact solution is easier, when we do not claim homogenous Dirichlet conditions for the adjoint boundary value problem. Hence we solve a modified minimization problem, where the term $\langle S^{\mathrm{ext}}_{CC} z, g\rangle_{\Gamma_C}$ is added to the cost functional (2.9). In this case it turns out that the Dirichlet datum of the adjoint problem is equal to g. The used geometry is given by, see Fig. 2,

$$\Gamma_C := \Big\{ x \in \mathbb{R}^3 \colon |x| = 1 \Big\}, \quad \Gamma_O := \Big\{ x \in \mathbb{R}^3 \colon |x - (0.5, 0, 0)^{\top}| = 2 \Big\}.$$

The cost coefficient ϱ, the desired state $\overline{u}$, and the Dirichlet boundary condition of the adjoint problem are given by

$$\varrho = 10^{-6}, \quad \overline{u} = \frac{1+\varrho}{2}, \quad g = \varrho r^{-1} - \varrho \text{ with } r = \sqrt{(x_1 - \frac{1}{2})^2 + x_2^2 + x_3^2}.$$

Using these data it is easy to verify that, in the case of no box constraints, the primal variable u is given by $u = r^{-1}$, see Fig. 3, while the dual variable p is determined piecewise, i.e.

$$p = \begin{cases} \varrho r^{-1} - \varrho & \text{for } 1 < r < 2, \\ -\varrho r^{-1} & \text{for } r > 2. \end{cases}$$

Hence we find for the boundary conditions of the adjoint problem

$$[p]_{|\Gamma_O} = 0, \quad [\partial_n p]_{|\Gamma_O} = -\frac{\varrho}{2}.$$

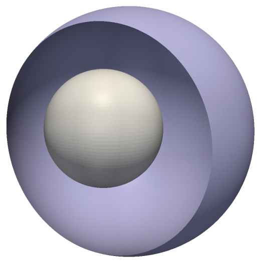

Figure 2: Geometric configuration.

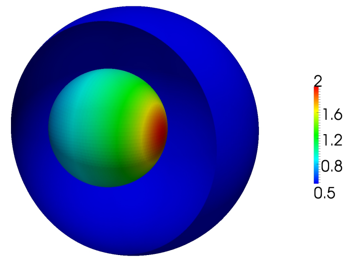

Figure 3: Primal solution u without box constraints.

L	N_C	N_O	M_C	M_O	$\|z - z_h\|_{L_2(\Gamma_C)}$	eoc
0	128	920	66	46	4.7 –0	
1	512	3680	258	1842	3.2 –1	3.9
2	2048	14464	1026	7234	1.5 –2	4.36
3	8192	57856	4098	28930	3.6 –3	2.09
4	32768	231424	16386	115714	8.8 –4	2.04

Table 1: L_2–error of the control $\|z - z_h\|_{L_2(\Gamma_C)}$.

As the exact solution is known, we can compute the L_2–error $\|z - z_h\|_{L_2(\Gamma_C)}$ for a sequence of uniform boundary element meshes of level L, see Table 1, where we observe a quadratic order of convergence (eoc), as predicted in (5.6).

Next we consider the case of box constraints, i.e. $g_b = 1.8$, where the exact solution is unknown. The primal solution u is depicted in Fig. 4, and in Fig. 5 we give a comparison of the constrained and unconstrained primal solutions on Γ_C along the line $x_3 = 0$. We observe that the constraints are active in a neighborhood of $(\frac{1}{2}, 0, 0)^\top$.

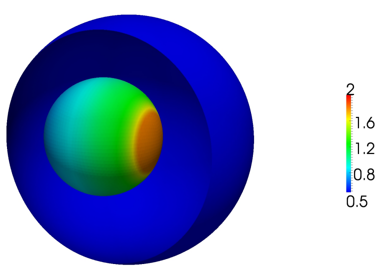

Figure 4: Primal solution u with box constraints.

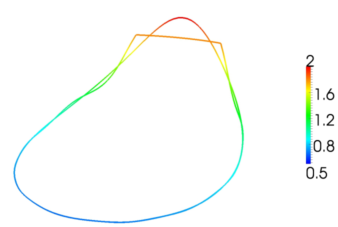

Figure 5: Primal solutions u with and without box constraints.

As a second example we consider the control boundary $\Gamma_C = \partial\Omega$ of the cube $\Omega = (-1, 1)^3$, where the observation boundary is given by the plane manifold

$$\Gamma_O := \Big\{ x \in \mathbb{R}^3 \colon x_1 = 3, x_2, x_3 \in (-2, 2) \Big\}.$$

The cost coefficient ϱ and the desired state $\overline{u}$ are given by

$$\varrho = 10^{-6}, \quad \overline{u}(x) = \frac{1}{|x|}\,.$$

The primal solution u of the optimal control problem without constraints is given in Fig. 6 (left), and with the constraint $g_b = 0.9$ in Fig. 6 (right). We observe that the constraints are active in a neighborhood of $(1,0,0)^\top$. In Fig. 7 we give a comparison of the constrained and unconstrained primal solutions on Γ_C along the line $x_3 = 0$.

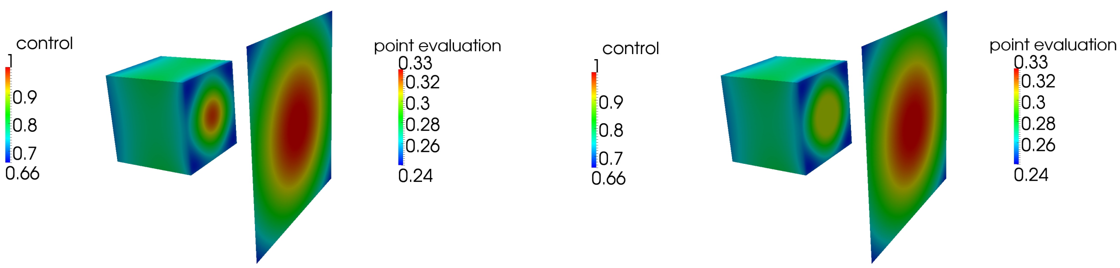

Figure 6: Primal solution u without (left) and with (right) constraints.

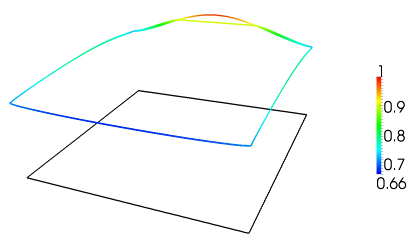

Figure 7: Primal solutions u with and without box constraints.

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