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A note on a modified Hilbert transform

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ABSTRACT
The Hilbert transform $\mathcal{H}$ is a useful tool in the mathematical analysis of time-dependent partial differential equations in order to prove coercivity estimates in anisotropic Sobolev spaces in case of a bounded spatial domain Ω, but an infinite time interval $(0, \infty)$. Instead, a modified Hilbert transform $\mathcal{H}_T$ can be used if we consider a finite time interval $(0, T)$. In this note we prove that the classical and the modified Hilbert transformations differ by a compact perturbation, when a suitable extension of a function defined on a bounded time interval $(0, T)$ onto $\mathbb{R}$ is used. This result is important when we deal with space-time variational formulations of time-dependent partial differential equations, and for the implementation of related space-time finite and boundary element methods for the numerical solution of parabolic and hyperbolic equations with the heat and wave equations as model problems, respectively.

KEYWORDS
Hilbert transform; modified Hilbert transform; compact perturbation

Dedicated to Wolfgang L. Wendland on the occasion of his 85th birthday.

1. Introduction

The Hilbert transform $\mathcal{H}$, see, e.g., [6] for an introduction, has many applications not only in signal processing [5], but also for the solution of singular integral equations and of Riemann–Hilbert problems [10]. There is also a strong relationship between the Hilbert transform and the definition of fractional derivatives, see [2] and the references given therein.

Despite of the above mentioned applications, more recently, the Hilbert transform became a useful tool in the analysis of time-dependent partial differential equations. In his doctoral thesis [4], M. Fontes used the Hilbert transform to analyse parabolic evolution equations, e.g., the heat equation $\partial_t u - \Delta_x u = f$ with zero initial and Dirichlet boundary conditions, in anisotropic Sobolev spaces with respect to a bounded spatial domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, but an infinite time interval $(0, \infty)$. The numerical analysis of a related discretization scheme using wavelets was then considered in [7], where an appropriate cut off of the infinite time interval has been included. In fact, using the Hilbert transform $\mathcal{H}$ one can prove boundedness and coercivity in anisotropic Sobolev spaces $H^{1,1/2}(\Omega \times \mathbb{R}_+)$, when considering the space-time bilinear form which is related to the transformed operator $(I + \delta\mathcal{H})(\partial_t - \Delta_x)$, $\delta > 0$. For a more general setting, see also [1].

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In [12], and for a finite time interval $(0,T)$, we have introduced a modified Hilbert transform $\mathcal{H}_T$. Its original definition was using the conjugate of a Fourier series with respect to the eigenfunctions of the one-dimensional Laplace operator with mixed Dirichlet and Neumann boundary conditions. The duality pairing of the first order time derivative with this modified Hilbert transform then results in a norm representation $\langle \partial_t v, \mathcal{H}_T v\rangle_{(0,T)} = \|v\|^2_{H^{1/2}(0,T)}$ for functions $v \in H^{1/2}(0,T)$ which are zero at the origin, $v(0) = 0$. With this we were able to analyse space-time finite element methods for the heat equation in anisotropic Sobolev spaces $H^{1,1/2}_{0;0,}(Q)$ in the case of a bounded space-time domain $Q := \Omega \times (0,T)$ with zero initial and Dirichlet boundary conditions. In particular, we have considered the transformed operator $\mathcal{H}_T^{-1}(\partial_t - \Delta_x)$ in the case of a finite time interval $(0,T)$. A comparison with the infinite time interval $(0,\infty)$ indicates a relation $\mathcal{H}_T^{-1} \sim I + \delta\mathcal{H}$ which should also include some suitable extension from $(0,T)$ onto $\mathbb{R}$.

Surprisingly, the modified Hilbert transform $\mathcal{H}_T$ can also be used for the formulation of space-time finite element methods for the numerical solution of the wave equation with the D'Alembert operator $\Box := \partial_{tt} - \Delta_x$. It turns out that the related bilinear form

$$a(v,v) = \langle \mathcal{H}_T \partial_t v, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x v, \mathcal{H}_T \nabla_x v\rangle_{L^2(Q)}$$

is positive in $H^{1,1}_{0;0,}(Q)$, i.e., zero initial and Dirichlet boundary conditions; see [8] for first numerical results. In the case of the spatially one-dimensional wave equation, the composition of the modified Hilbert transform $\mathcal{H}_T$ and the wave single layer boundary integral operator becomes elliptic in the natural Sobolev trace space $H^{-1/2}(\Sigma)$, where Σ is the lateral boundary of the space-time domain Q, see [11]. On the other hand, recent work of M. Costabel and M. Zank shows coercivity of the wave single layer boundary integral operator on screens, composed with the classical Hilbert transform $I - \mathcal{H}$ in the infinite time interval, see [3].

The aim of this note is to provide a relation of the modified Hilbert transform $\mathcal{H}_T$ with the classical Hilbert transform $\mathcal{H}$ in order to be able to identify both in the mathematical and numerical analysis of time-dependent problems in bounded time intervals, where $\mathcal{H}_T$ seems to be more appropriate. However, for a practical realisation, the classical Hilbert transform $\mathcal{H}$ is probably simpler to implement. It turns out that the classical and the modified Hilbert transformations differ by a compact perturbation when extending a function defined on the bounded time interval $(0,T)$ onto $\mathbb{R}$ in a suitable way. Note that in [9] we already used, in a naive way, an intermediate result, namely $\mathcal{H}_\infty$ instead of $\mathcal{H}_T$, to solve the heat equation in a finite time interval.

The remainder of this note is organised as follows: the definitions of the classical Hilbert transform $\mathcal{H}$ and the modified Hilbert transform $\mathcal{H}_T$ and some of their properties are summarized in Sections 2 and 3, respectively. The main results are given in Section 4, where we first present an alternative representation of $\mathcal{H}_T$ combining $\mathcal{H}$ with a suitable extension from $(0,T)$ onto $\mathbb{R}$, and a perturbation which turns out to be compact. Finally we give some conclusions and comment on ongoing work.

2. Hilbert transform

For a sufficient regular density function φ, the Hilbert transform $\mathcal{H}$ is defined as Cauchy principal value integral

$$(\mathcal{H}\varphi)(t) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{\varphi(s)}{t-s}\,ds \quad \text{for } t \in \mathbb{R}. \tag{1}$$

It is well known that the Hilbert transform (1) commutes with the (fractional) derivative, i.e.,

$$D^\alpha(\mathcal{H}\varphi)(t) = \mathcal{H}(D^\alpha\varphi)(t), \tag{2}$$

see, e.g., [2, Lemma 2.4] in the case $\alpha = k \in \mathbb{N}$, and [2, Folgerung 5.21] in the case $\alpha \in [n, n+1), n \in \mathbb{N}$ for the Riemann–Liouville partial derivative [2, p. 26]

$$(D^\alpha\varphi)(t) = \frac{1}{\Gamma(n+1-\alpha)}\frac{d^{n+1}}{dt^{n+1}}\int_{-\infty}^{t}(t-s)^{n-\alpha}\varphi(s)\,ds\,.$$

In particular for $\varphi \in L^2(\mathbb{R})$ we also have [2, Lemma 2.3]

$$(\mathcal{H}(\mathcal{H}\varphi))(t) = -\varphi(t) \quad \text{for } t \in \mathbb{R} \text{ almost everywhere.}$$

When introducing the Fourier transform as

$$\mathcal{F}[\varphi](\eta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(t)\,e^{-i\eta t}\,dt,$$

there holds the signum rule

$$\mathcal{F}[\mathcal{H}\varphi](\eta) = (-i\,\operatorname{sgn}\eta)\,\mathcal{F}[\varphi](\eta)\,.$$

This relation is not only important to prove (2), but also in the proof of the coercivity estimate of the wave single layer boundary integral operator as discussed in [3].

While we can easily consider (1) for functions φ with $\varphi(s) = 0$ for $s < 0$, we still have to assume a suitable decay condition for $\varphi(s)$ as $s \to \infty$. From a computational point of view, one is therefore interested in the realisation of the Hilbert transform $\mathcal{H}$ in a finite time interval $(0, T)$ for a continuous function φ when assuming $\varphi(0) = 0$, but $\varphi(T) \neq 0$. While we can define a continuous zero extension of φ for $s < 0$, this is not obvious for $s > T$. However, in Section 4 we will describe a suitable extension of φ from $(0, T)$ onto $\mathbb{R}$, first by reflecting φ with respect to the final time T, and then skew-reflecting the previous result with respect to the origin.

3. A modified Hilbert transform

In [12], and for a finite time interval $(0,T)$, we have introduced a modified Hilbert transform $\mathcal{H}_T$ for $\varphi \in L^2(0,T)$ as conjugate of the Fourier series

$$\varphi(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad u_k = \frac{2}{T}\int_0^T \varphi(s)\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) ds, \quad (3)$$

i.e.,

$$(\mathcal{H}_T\varphi)(t) := \sum_{k=0}^{\infty} u_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad t \in (0,T). \quad (4)$$

Note that

$$\varphi_k(t) = \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)$$

are the eigenfunctions of the one-dimensional Laplacian eigenvalue problem

$$-\partial_{tt}\varphi(t) = \lambda\varphi(t) \quad \text{for } t \in (0,T), \quad \varphi(0) = 0, \quad \partial_t\varphi(t)_{|t=0} = 0.$$

For $\varphi \in H^{1/2}_{0,}(0,T) := \left\{v \in H^{1/2}(0,T), v(0) = 0\right\}$ we now conclude [12, Lemma 2.3]

$$\partial_t \mathcal{H}_T \varphi = -\mathcal{H}_T^{-1}\partial_t\varphi, \quad (5)$$

which can be seen as the counterpart of (2). Moreover, the operator

$$-\partial_t\mathcal{H}_T : H^{1/2}_{0,}(0,T) \to [H^{1/2}_{,0}(0,T)]'$$

is self-adjoint and elliptic, inducing an equivalent norm in $H^{1/2}_{0,}(0,T)$.

As discussed in [12], see also [13], we can conclude from (4) the closed representation

$$(\mathcal{H}_T\varphi)(t) = \frac{1}{2T}\,\text{p.v.}\int_0^T \left[\frac{1}{\sin\left(\frac{\pi}{2}\frac{s-t}{T}\right)} + \frac{1}{\sin\left(\frac{\pi}{2}\frac{s+t}{T}\right)}\right]\varphi(s)\,ds \quad \text{for } t \in (0,T). \quad (6)$$

In particular for $T \to \infty$, and assuming $\widetilde{\varphi} \in L^2(\mathbb{R}_+)$, this gives

$$(\mathcal{H}_\infty\widetilde{\varphi})(t) = \frac{1}{\pi}\,\text{p.v.}\int_0^\infty \frac{\widetilde{\varphi}(s)}{s-t}\,\frac{2s}{s+t}\,ds \quad \text{for } t \in (0,\infty), \quad (7)$$

which obviously shows a connection with the Hilbert transform (1). This relation will be discussed in more detail in the next section.

4. A relation between modified and classical Hilbert transformations

For a given $\varphi \in L^2(0,T)$, let $\widetilde{\varphi}$ be some extension on $(0,\infty)$ to be specified. Using

$$\frac{2s}{(s-t)(s+t)} = \frac{1}{s-t} + \frac{1}{s+t}$$

we can write (7) as

$$(\mathcal{H}_\infty \widetilde{\varphi})(t) = \frac{1}{\pi}\,\text{p.v.}\int_0^\infty \frac{\widetilde{\varphi}(s)}{s-t}\,ds + \frac{1}{\pi}\,\text{p.v.}\int_0^\infty \frac{\widetilde{\varphi}(s)}{s+t}\,ds \quad \text{for } t \in (0,\infty).$$

In the second integral we use the transformation $s = -\sigma$ to obtain

$$\frac{1}{\pi}\,\text{p.v.}\int_0^\infty \frac{\widetilde{\varphi}(s)}{s+t}\,ds = -\frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^0 \frac{\widetilde{\varphi}(-\sigma)}{\sigma-t}\,d\sigma\,,$$

and hence,

$$(\mathcal{H}_\infty \widetilde{\varphi})(t) = \frac{1}{\pi}\,\text{p.v.}\int_0^\infty \frac{\widetilde{\varphi}(s)}{s-t}\,ds + \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^0 \frac{-\widetilde{\varphi}(-s)}{s-t}\,ds = -(\mathcal{H}\overline{\varphi})(t) \tag{8}$$

follows, where we have used (1) for

$$\overline{\varphi}(s) := \begin{cases} \widetilde{\varphi}(s) & \text{for } s \in [0,\infty), \\ -\widetilde{\varphi}(-s) & \text{for } s \in (-\infty,0). \end{cases} \tag{9}$$

For $\varphi \in L^2(0,T)$ we now define

$$\widetilde{\varphi}(s) := \begin{cases} \varphi(s) & \text{for } s \in (0,T), \\ \varphi(2T-s) & \text{for } s \in (T,2T), \\ 0 & \text{for } s \in (2T,\infty), \end{cases} \tag{10}$$

and we consider

$$\begin{aligned} (\mathcal{H}_\infty \widetilde{\varphi})(t) &= \frac{1}{\pi}\,\text{p.v.}\int_0^\infty \frac{\widetilde{\varphi}(s)}{s-t}\,\frac{2s}{s+t}\,ds \\ &= \frac{1}{\pi}\,\text{p.v.}\int_0^T \frac{\varphi(s)}{s-t}\,\frac{2s}{s+t}\,ds + \frac{1}{\pi}\,\text{p.v.}\int_T^{2T} \frac{\varphi(2T-s)}{s-t}\,\frac{2s}{s+t}\,ds\,. \end{aligned}$$

With the transformation $\sigma = 2T - s$, i.e., $s = 2T - \sigma$, we obtain

$$\frac{1}{\pi}\,\text{p.v.}\int_T^{2T} \frac{\varphi(2T-s)}{s-t}\,\frac{2s}{s+t}\,ds = \frac{1}{\pi}\,\text{p.v.}\int_0^T \frac{\varphi(\sigma)}{2T-\sigma-t}\,\frac{2(2T-\sigma)}{2T-\sigma+t}\,d\sigma,$$

and therefore,

$$\begin{aligned}(\mathcal{H}_\infty\widetilde{\varphi})(t) &= \frac{1}{\pi}\,\text{p.v.}\int_0^T \varphi(s)\left[\frac{1}{s-t}\,\frac{2s}{s+t} + \frac{1}{2T-s-t}\,\frac{2(2T-s)}{2T-s+t}\right] ds \\ &= \frac{1}{\pi}\,\text{p.v.}\int_0^T \varphi(s)\left[\frac{1}{s-t} + \frac{1}{s+t} + \frac{1}{2T-s-t} + \frac{1}{2T-s+t}\right] ds \\ &= \frac{1}{\pi}\,\text{p.v.}\int_0^T \varphi(s)\left[\frac{2T}{(s-t)(2T-(s-t))} + \frac{2T}{(s+t)(2T-(s+t))}\right] ds\end{aligned}$$

follows. Hence we obtain

$$(B\varphi)(t) := (\mathcal{H}_T\varphi)(t) - (\mathcal{H}_\infty\widetilde{\varphi})(t) = \int_0^T \Big[k_1(s,t) + k_2(s,t)\Big]\,\varphi(s)\,ds \tag{11}$$

with the kernel functions, for $s,t \in (0,T)$,

$$\begin{aligned}k_1(s,t) &:= \frac{1}{2T}\frac{1}{\sin\left(\frac{\pi}{2}\frac{s-t}{T}\right)} - \frac{1}{\pi}\frac{2T}{(s-t)(2T-(s-t))}, \\ k_2(s,t) &:= \frac{1}{2T}\frac{1}{\sin\left(\frac{\pi}{2}\frac{s+t}{T}\right)} - \frac{1}{\pi}\frac{2T}{(s+t)(2T-(s+t))}\,.\end{aligned}$$

Summarizing the above, we have shown the following result.

Lemma 4.1. *For $\varphi \in L^2(0,T)$ we find the alternative representation*

$$(\mathcal{H}_T\varphi)(t) = -(\mathcal{H}\overline{\varphi})(t) + (B\varphi)(t), \quad t \in (0,T), \tag{12}$$

of the modified Hilbert transform (6) *when using* (1) *and* (11), *as well as*

$$\overline{\varphi}(s) := \begin{cases} \varphi(s) & \text{for } s \in (0,T), \\ \varphi(2T-s) & \text{for } s \in (T,2T), \\ -\varphi(-s) & \text{for } s \in (-T,0), \\ -\varphi(2T+s) & \text{for } s \in (-2T,-T), \\ 0 & \text{else}\,. \end{cases} \tag{13}$$

It remains to state some mapping properties of B when assuming $\varphi \in L^2(0,T)$.

Lemma 4.2. *For the operator B as defined in* (11) *there hold the bounds*

$$\|B\varphi\|_{L^2(0,T)} \le \frac{1}{2}\,\|\varphi\|_{L^2(0,T)}, \quad \|\partial_t B\varphi\|_{L^2(0,T)} \le \frac{1}{T}\left[\frac{23}{36}\frac{1}{\pi} + \frac{\pi}{24}\right]\|\varphi\|_{L^2(0,T)} \tag{14}$$

for all $\varphi \in L^2(0,T)$.

Proof. Using the transformation $x := (s-t)/T \in (-1,1)$ for $s,t \in (0,T)$ we obtain

$$k_1(s,t) = \frac{1}{\pi T}\left[\frac{\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}x\right)} - \frac{2}{x(2-x)}\right] =: \frac{1}{\pi T}\,f(x), \; x \in (-1.1).$$

With the transformation $y := (s+t)/T \in (0,2)$ for $s,t \in (0,T)$ we also conclude

$$k_2(s,t) := \frac{1}{\pi T}\left[\frac{\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}y\right)} - \frac{2}{y(2-y)}\right] = \frac{1}{\pi T} f(y)\,.$$

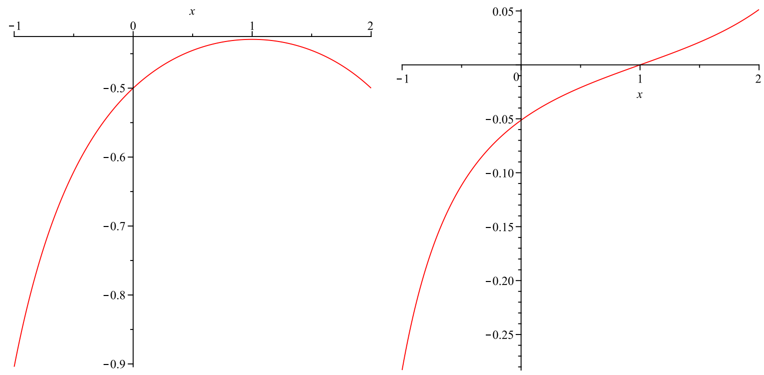

Figure 1. Graphs of $f(x) = \frac{\frac{\pi}{2}}{\sin(\frac{\pi}{2}x)} - \frac{2}{x(2-x)}$ (left) and $g(x) = \frac{\pi}{4}\frac{\cos(\frac{\pi}{2}x)}{\sin^2(\frac{\pi}{2}x)} + \frac{4}{\pi}\frac{x-1}{x^2(2-x)^2}$ (right).

From the behavior of the function $f(x)$ for $x \in (-1,2)$, see the left graph in Fig. 1, we then obtain

$$|k_1(s,t)| \le \frac{1}{\pi T}\left(\frac{\pi}{2} - \frac{2}{3}\right), \quad |k_2(s,t)| \le \frac{1}{2\pi T} \quad \text{for } (s,t) \in (0,T),$$

i.e.,

$$|k_1(s,t) + k_2(s,t)| \le \frac{1}{\pi T}\left[\frac{\pi}{2} - \frac{1}{6}\right] \le \frac{1}{2T} \quad \text{for } (s,t) \in (0,T).$$

With this,

$$|(B\varphi)(t)| \le \frac{1}{2T}\int_0^T |\varphi(s)|\,ds \le \frac{1}{2\sqrt{T}}\,\|\varphi\|_{L^2(0,T)},$$

i.e.,

$$|(B\varphi)(t)|^2 \le \frac{1}{4T}\,\|\varphi\|^2_{L^2(0,T)}$$

follows. Integration over $t \in (0,T)$ finally gives

$$\|B\varphi\|^2_{L^2(0,T)} \le \frac{1}{4}\,\|\varphi\|^2_{L^2(0,T)}.$$

Moreover, we also consider

$$\begin{aligned}
\frac{\partial}{\partial t} k_1(s,t) &= \frac{\pi}{4T^2} \frac{\cos\left(\frac{\pi}{2}\frac{s-t}{T}\right)}{\sin^2\left(\frac{\pi}{2}\frac{s-t}{T}\right)} + \frac{2T}{\pi} \frac{2(s-t)-2T}{(s-t)^2(2T-(s-t))^2} \\
&= \frac{1}{T^2}\left[\frac{\pi}{4}\frac{\cos\left(\frac{\pi}{2}x\right)}{\sin^2\left(\frac{\pi}{2}x\right)} + \frac{4}{\pi}\frac{x-1}{x^2(2-x)^2}\right] =: \frac{1}{T^2}\, g(x), \quad x \in (-1,1),
\end{aligned}$$

as well as

$$\begin{aligned}
\frac{\partial}{\partial t} k_2(s,t) &= -\frac{\pi}{4T^2} \frac{\cos\left(\frac{\pi}{2}\frac{s+t}{T}\right)}{\sin^2\left(\frac{\pi}{2}\frac{s+t}{T}\right)} - \frac{2T}{\pi} \frac{2(s+t)-2T}{(s+t)^2(2T-(s+t))^2} \\
&= -\frac{1}{T^2}\left[\frac{\pi}{4}\frac{\cos\left(\frac{\pi}{2}y\right)}{\sin^2\left(\frac{\pi}{2}y\right)} + \frac{4}{\pi}\frac{y-1}{y^2(2-y)^2}\right] = -\frac{1}{T^2}\, g(y), \quad y \in (0,2).
\end{aligned}$$

From the behavior of the function $g(x)$ for $x \in (-1,2)$, see the right plot in Fig. 1, we then obtain

$$|\partial_t k_1(s,t)| \leq \frac{8}{9\pi T^2}, \quad |\partial_t k_2(s,t)| \leq \frac{1}{T^2}\left(\frac{\pi}{24} - \frac{1}{4\pi}\right), \quad s,t \in (0,T),$$

i.e.,

$$|\partial_t k_1(s,t) + \partial_t k_2(s,t)| \leq \frac{1}{T^2}\left[\frac{23}{36}\frac{1}{\pi} + \frac{\pi}{24}\right], \quad (s,t) \in (0,T).$$

With this,

$$|\partial_t (B\varphi)(t)| \leq \frac{1}{T^2}\left[\frac{23}{36}\frac{1}{\pi} + \frac{\pi}{24}\right] \int_0^T |\varphi(s)|\, ds \leq \frac{1}{T^{3/2}}\left[\frac{23}{36}\frac{1}{\pi} + \frac{\pi}{24}\right] \|\varphi\|_{L^2(0,T)},$$

i.e.,

$$|\partial_t (B\varphi)(t)|^2 \leq \frac{1}{T^3}\left[\frac{23}{36}\frac{1}{\pi} + \frac{\pi}{24}\right]^2 \|\varphi\|^2_{L^2(0,T)}$$

follows. Integration over $t \in (0,T)$ finally gives

$$\|\partial_t B\varphi\|^2_{L^2(0,T)} \leq \frac{1}{T^2}\left[\frac{23}{36}\frac{1}{\pi} + \frac{\pi}{24}\right]^2 \|\varphi\|^2_{L^2(0,T)}.$$

$\square$

From (14) we immediately obtain that $B : L^2(0,T) \to H^1(0,T)$ is bounded, and since $H^1(0,T)$ is compactly embedded in $L^2(0,T)$ we conclude that $B : L^2(0,T) \to L^2(0,T)$ is compact. With this we are now in a position to formulate the main result of this note.

Theorem 4.3. *The modified Hilbert transform $\mathcal{H}_T$ as given in* (6) *is a compact perturbation of the Hilbert transform* (1) *applied to the reflection* (13)*, i.e., for $\varphi \in L^2(0,T)$ we have $\mathcal{H}_T\varphi = -\mathcal{H}\overline{\varphi} + B\varphi$ in $L^2(0,T)$.*

5. Conclusions

The relation between the classical and the modified Hilbert transformations $\mathcal{H}$ and $\mathcal{H}_T$ provides a link between the coercivity results for the wave single layer boundary integral operator as considered in [3] in the case of an unbounded time interval $(0,\infty)$, and in [11] for a bounded time interval $(0,T)$. More general, this result may be helpful to prove coercivity estimates for time-dependent parabolic and hyperbolic partial differential equations in bounded and unbounded time intervals, both for related domain bilinear forms and boundary integral operators. But also from a computational point of view this result will be important. Following [9], we may replace the modified Hilbert transform $\mathcal{H}_T$ by the classical Hilbert transform $\mathcal{H}$ taking into account the extension (13), since the latter is probably simpler to implement which is important when considering completely unstructured simplicial meshes in space-time finite and boundary element methods. Finally, and following [2], one may generalize the relations and representations of the classical Hilbert transform and the fractional derivative to the modified Hilbert transform, in particular when considering problems in finite time intervals.

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