Magnetic Schrödinger operators with electric δ-potentials

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Schrödinger operators and boundary value problems,
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Outline

1. Motivation
2. Magnetic Schrödinger operators with δ-potentials
   - The magnetic Schrödinger operator without potential
   - Magnetic Sobolev spaces
   - Definition of the δ-operator
3. Approximation by Hamiltonians with squeezed potentials
4. Exner-Ichinose for homogeneous magnetic fields
5. A quasi boundary triple
6. Outlook

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$$\big(i\partial_t - (-i\nabla_x - A)^2 + V\big)\,\psi(t,x) = 0 \quad + \quad \text{i. c.},$$

where $B = \nabla \times A$, i. e. $A : \mathbb{R}^3 \to \mathbb{R}^3$

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- We consider H in $L^2(\mathbb{R}^d)$ for any $d \geq 2$ (physical meaning for $d = 2, 3$)

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For a zero-set $\Sigma \subset \mathbb{R}^d$ and $\alpha : \Sigma \to \mathbb{R}$ consider

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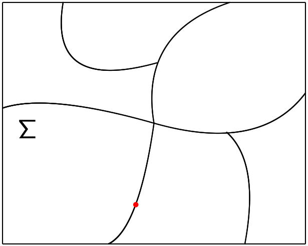
- Description of motion of quantum particle on network of wires in the presence of a magnetic field



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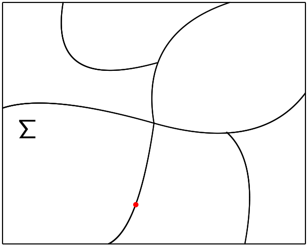

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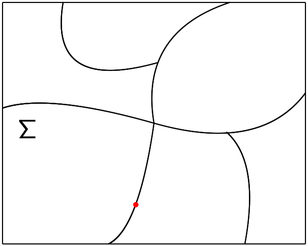

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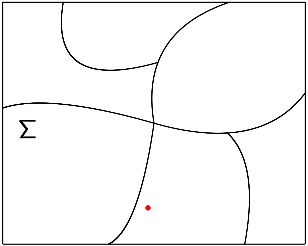

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Conjectures:

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Conjectures:

- For homogeneous magnetic fields ($B = \text{const.}$): same behavior
- For non-homogeneous fields: bound states disappear

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$$\mathfrak{h}_0[f,g] := \big((-i\nabla - A)f, (-i\nabla - A)g\big),$$
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- associated self-adjoint operator

$$H_0 := (-i\nabla - A)^2$$

Definition of magnetic Sobolev spaces

- Problem: for $A \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ we have in general $f \in H^1(\mathbb{R}^d) \not\Rightarrow f \in \mathcal{H}_A^1(\mathbb{R}^d)$

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- $\mathcal{H}^s_A(\Omega)$, equipped with the natural norm, is a Hilbert space

The diamagnetic inequality

Theorem

Let $t > 0$ *and* $f \in L^2(\mathbb{R}^d)$*. Then:*

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Corollary

$\mathcal{H}_A^s(\mathbb{R}^d) \subset H^s(\mathbb{R}^d)$ *for all* $s \geq 0$.

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(diamagnetic inequality)

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- Remark: One can add a form bounded potential Q with relative bound < 1

Outline

1. Motivation
2. Magnetic Schrödinger operators with δ-potentials
   - The magnetic Schrödinger operator without potential
   - Magnetic Sobolev spaces
   - Definition of the δ-operator
3. Approximation by Hamiltonians with squeezed potentials
4. Exner-Ichinose for homogeneous magnetic fields
5. A quasi boundary triple
6. Outlook

Approximation by Hamiltonians with squeezed potentials – the problem

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- Assume $\exists \beta > 0$ such that

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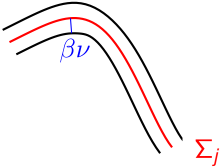

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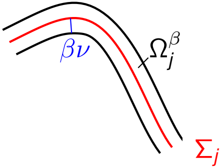

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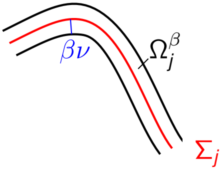

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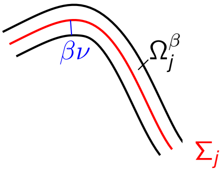

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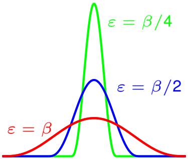

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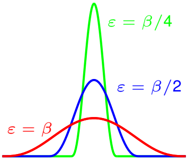

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- $(-i\nabla - A)^2 - \sum_{j=1}^N V_{j,\varepsilon}$ is self-adjoint on $\mathcal{H}_A^2(\mathbb{R}^d)$

The result

Theorem

Define $\alpha \in L^\infty(\Sigma)$ *as*

$$\alpha(x_\Sigma) := \int_{-\beta}^{\beta} V_j(x_\Sigma + s\nu(x_\Sigma))\,\mathrm{d}s, \quad x_\Sigma \in \Sigma_j,$$

and let $\lambda \ll 0$*. Then there exists* $c > 0$ *such that*

$$\left\| \left((-i\nabla - A)^2 - \sum_{j=1}^N V_{j,\varepsilon} - \lambda \right)^{-1} - (H_\alpha - \lambda)^{-1} \right\| \leq c\varepsilon$$

for small $\varepsilon > 0$*. In particular* $(-i\nabla - A)^2 - \sum_{j=1}^N V_{j,\varepsilon}$ *converge to* H_α *in the norm resolvent sense.*

Sketch of the proof

- Let $\mathfrak{h}_\varepsilon[f,g] := \mathfrak{h}_0[f,g] - \sum_{j=1}^N (V_{j,\varepsilon} f, g)$, $\operatorname{dom} \mathfrak{h}_\varepsilon = \mathcal{H}_A^1(\mathbb{R}^d)$

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Markus Holzmann,
Schrödinger operators and boundary value problems, Graz

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where G_λ^A is explicitely given by a combination of
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Proof of Exner-Ichinose

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Outline

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$$\begin{aligned} Tf &:= \left((-i\nabla - A)^2 f_+\right) \oplus \left((-i\nabla - A)^2 f_-\right) \\ \operatorname{dom} T &:= \left\{ f = f_+ \oplus f_- \in \mathcal{H}_A^2(\Omega_+) \oplus \mathcal{H}_A^2(\Omega_-) : f_+|_\Sigma = f_-|_\Sigma \right\} \end{aligned}$$

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The quasi boundary triple and the δ-operator

Define for $\alpha \in \mathbb{R}$ the operator $H_\alpha^Q := T \upharpoonright \ker(\Gamma_0 - \alpha\Gamma_1)$, i.e.

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$$(A_\alpha^Q - \lambda)^{-m} - \left((-i\nabla - A)^2 - \lambda\right)^{-m} \in \mathfrak{S}_{\frac{d-1}{2m+1},\infty}(L^2(\mathbb{R}^d))$$

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- Formulae for scattering theory

Outline

1. Motivation
2. Magnetic Schrödinger operators with δ-potentials
   - The magnetic Schrödinger operator without potential
   - Magnetic Sobolev spaces
   - Definition of the δ-operator
3. Approximation by Hamiltonians with squeezed potentials
4. Exner-Ichinose for homogeneous magnetic fields
5. A quasi boundary triple
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Thank you for your attention!