Eigenvalues of Robin Laplacians on infinite sectors and application to polygons

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(joint work with Konstantin Pankrashkin)

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Robin eigenvalue problem

Let $\Omega \subset \mathbb{R}^d$ be an open set with a sufficiently regular boundary. We consider the eigenvalue problem :

$$-\Delta\psi = -\left(\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}\right)\psi = E\psi \text{ on } \Omega,$$

$$\frac{\partial\psi}{\partial\nu} = \gamma\psi \text{ on } \partial\Omega,$$

where ν is the **outward** unit normal of $\partial\Omega$, $\gamma > 0$ and E is a discrete eigenvalue.

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More precisely, we study the spectral problem for the self-ajdoint operator T_Ω^γ on $L^2(\Omega)$ associated with the sesquilinear form :

$$t_\Omega^\gamma(\psi,\psi) = \int_\Omega |\nabla\psi|^2 dx - \gamma\int_{\partial\Omega} |\psi|^2 d\sigma, \quad \psi \in H^1(\Omega).$$

Smooth domains

Main goal : Study of $E_n(T_\Omega^\gamma)$ as $\gamma \to +\infty$.

- Change of variables : $E_n(T_\Omega^\gamma) = \gamma^2 E_n(T^1_{\gamma\Omega})$.
- Link with the study of superconductors.

[Lacey-Ockendon-Sabina, 1998 ; Lou-Zhu,2004 ; Levitin-Parnovski 2008, Bruneau-Popoff,2016 ;...]

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[Lacey-Ockendon-Sabina, 1998 ; Lou-Zhu,2004 ; Levitin-Parnovski 2008, Bruneau-Popoff,2016 ;...]

Theorem [Daners-Kennedy, 2010]

If $\partial\Omega$ is C^1, for each fixed $n \in \mathbb{N}$,

$$E_n(T_\Omega^\gamma) = -\gamma^2 + o(\gamma^2), \quad \gamma \to +\infty.$$

Theorem [Exner-Minakov-Parnovski, 2014 ; Pankrashkin-Popoff, 2015]

If $\partial\Omega$ is C^3, for each fixed $n \in \mathbb{N}$,

$$E_n(T_\Omega^\gamma) = -\gamma^2 - (d-1)H_{\max}(\Omega)\gamma + O(\gamma^{\frac{2}{3}}), \quad \gamma \to +\infty,$$

where $H_{\max}(\Omega)$ is the maximum of the mean curvature of $\partial\Omega$.

What happens on non-smooth domains ?

Theorem [Levitin-Parnovski, 2008 ; Bruneau-Popoff, 2016]

If Ω is a 'corner domain' (Lipschitz, piecewise smooth boundary + little more),

$$E_1(T_\Omega^\gamma) = -C\gamma^2 + o(\gamma^2), \quad \gamma \to +\infty,$$

where $C \geq 1$ depends only on the tangent cones of $\partial\Omega$.

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If $\Omega \subset \mathbb{R}^2$ is a **curvilinear polygon**, can we obtain a more detailed eigenvalue asymptotics ?
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Theorem [Pankrashkin,2013]

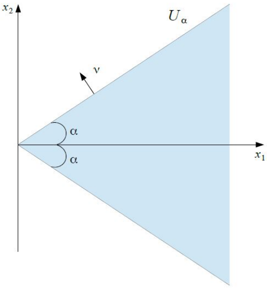
If $\Omega \subset \mathbb{R}^2$ has a piecewise smooth boundary which admits non-convex corners then,

$$E_1(T_\Omega^\gamma) = -\gamma^2 - \kappa_{\max}\gamma + O(\gamma^{\frac{2}{3}}), \quad \gamma \to +\infty.$$

i.e : the non convex corners do not contribute in the asymptotics.

Role of convex corners ?

Robin Laplacian on infinite sectors



$$\alpha \in (0, \pi),$$

$$U_\alpha := \left\{ x \in \mathbb{R}^2 : |\arg(x_1 + ix_2)| < \alpha \right\}.$$

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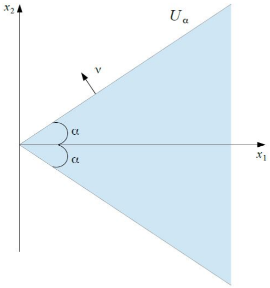

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T_α^γ = Robin Laplacian on $L^2(U_\alpha)$, $\gamma > 0$:

$$T_\alpha^\gamma \psi = -\Delta \psi \text{ on } U_\alpha,$$

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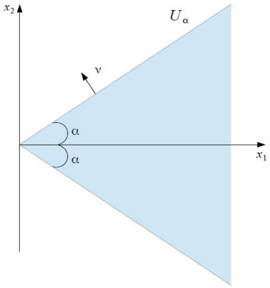

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Behavior of the eigenvalues of T_α^γ with respect to α?

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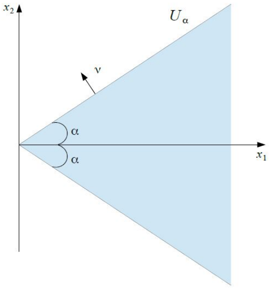

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Behavior of the eigenvalues of T_α^γ with respect to α ?

U_α is invariant by dilations : $E_n(T_\alpha^\gamma) = \gamma^2 E_n(T_\alpha^1)$. In the following : $T_\alpha^1 := T_\alpha$.

Some known results

Proposition [Levitin-Parnovski, 2008]

For all $\alpha \in (0, \pi)$, $\text{spec}_{\text{ess}}(T_\alpha) = [-1, +\infty)$.

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- Is $\mathrm{spec}_{\mathrm{disc}}(T_\alpha^\gamma)$ finite or infinite ?
- What is the behavior (regularity, monotonicity) of the eigenvalues with respect to α ?
- What is their behavior as $\alpha \to 0$?
- What are the properties of the associated eigenfunctions ?

Finiteness of the spectrum and monotonicity

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Proposition

- The eigenvalues of T_α are non-decreasing and continuous with respect to α.
- $(0, \pi/2) \ni \alpha \mapsto N_\alpha$ is decreasing.
- For all $\alpha \geq \pi/6$, $N_\alpha = 1$.

Asymptotic behavior as the angle becomes small

Proposition

There exists $\kappa > 0$ such that $N_\alpha \geq \kappa/\alpha$ as $\alpha \to 0$. In particular,

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Theorem : First order asymptotics

For each $n \in \mathbb{N}$:

$$E_n(T_\alpha) = -\frac{1}{(2n-1)^2\alpha^2} + O(1), \quad \alpha \to 0.$$

The constant can't be improve :

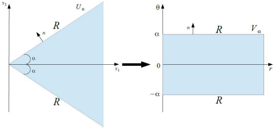
$$E_1(T_\alpha) = -\frac{1}{\alpha^2} - \frac{1}{3} + o(1), \alpha \to 0.$$

Ideas of the proof of the first order asymptotics

To avoid the singularity near the origin we introduce a dense subspace of $H^1(U_\alpha)$:

$$\mathcal{F} := \Big\{ u \in C^\infty(\overline{U_\alpha}) \mid \exists R_1, R_2 > 0 : \ u = 0 \text{ for } |x| < R_1, \text{ and } |x| > R_2 \Big\}.$$

Polar coordinates :



$$\mathcal{U} \colon L^2(U_\alpha, dx) \to L^2(V_\alpha, dr d\theta)$$
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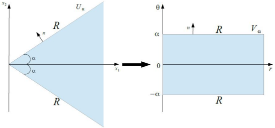

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$$\mathcal{G} := \mathcal{U}(\mathcal{F}) = \{v \in C^\infty(\overline{V_\alpha})| \; \exists R_1, R_2 > 0 : \; u(r,\theta) = 0 \text{ for } r < R_1 \text{ and } r > R_2\}.$$

The unitarily equivalent operator is Q_α associated to

$$q_\alpha(v,v) := t_\alpha\left(\mathcal{U}^*(v),\mathcal{U}^*(v)\right), \quad v \in \mathcal{G},$$

where :

$$q_\alpha(v,v) = \int_{V_\alpha} |v_r|^2 - \frac{1}{4}\frac{|v|^2}{r^2} dr d\theta + \int_{\mathbb{R}^+} \frac{1}{r^2}\left\{\int_{-\alpha}^{\alpha} |v_\theta|^2 d\theta - r|v(r,\alpha)|^2 - r|v(r,-\alpha)|^2\right\} dr.$$

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Robin Laplacian $B_{\alpha,r}$ acting on $L^2(-\alpha,\alpha)$, $r \in \mathbb{R}_+$:

$$B_{\alpha,r} u = -u'' \text{ sur } (-\alpha,\alpha)$$
$$\pm u'(\pm\alpha) = ru(\pm\alpha).$$

First eigenvalue : $E_1(\alpha, r)$ associated to the eigenfunction ϕ_α.

Reduction of the dimension : we apply q_α on functions of the form $v(r,\theta) = f(r)\phi_\alpha(r,\theta)$:

$$q_\alpha(v,v) = \left\{ \int_{\mathbb{R}_+} |f'(r)|^2 - \frac{1}{4r^2}|f(r)|^2 - \frac{1}{\alpha r}|f(r)|^2 dr \right\} + \int_{\mathbb{R}_+} K_\alpha(r)|f(r)|^2 dr.$$

We define the operator H_a acting on $L^2(\mathbb{R}_+)$ by

$$(H_a)(v) = \left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - \frac{1}{ar} \right) v(r), \quad v \in C_c^\infty(\mathbb{R}_+),$$

and H_a^∞ its Friedrichs extension. Then, $\mathrm{spec}_{\mathrm{ess}}(H_a^\infty) = [0,+\infty)$ and its discrete eigenvalues are :

$$\mathcal{E}_n(a) = -\frac{1}{(2n-1)^2 a^2}, \quad n \in \mathbb{N}.$$

Orthogonal projections :

$\Pi v(r,\theta) := f(r)\Phi_\alpha(r,\theta), \quad f(r) := \int_{-\alpha}^{\alpha} v(r,\theta)\Phi(r,\theta)d\theta$ and

$Pv(r,\theta) := v(r,\theta) - \Pi v(r,\theta)$.

For all $\alpha \in (0,1)$:

$$(1-\alpha^2)\mathcal{I}^* \begin{pmatrix} H^\infty_{\alpha(1-\alpha^2)} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{I} \begin{pmatrix} \Pi v \\ Pv \end{pmatrix} - M \le Q_\alpha \le \mathcal{I}^* \begin{pmatrix} H^\infty_\alpha & 0 \\ 0 & 0 \end{pmatrix} \mathcal{I} \begin{pmatrix} \Pi v \\ Pv \end{pmatrix} + M,$$

$M \in \mathbb{R}_+$, $\mathcal{I}$ is the unitary operator satisfying $\mathcal{I}(\Pi v, Pv) = (f, Pv)$.

We conclude with the min-max principle.

Theorem : Complete asymptotic expansion

For each $n \in \mathbb{N}$, there exists $\lambda_{j,n} \in \mathbb{R}$, $j \in \mathbb{N} \cup \{0\}$, such that for all $N \in \mathbb{N} \cup \{0\}$:

$$E_n(T_\alpha) = \frac{1}{\alpha^2} \sum_{j=0}^{N} \lambda_{j,n} \alpha^{2j} + O(\alpha^{2N}), \quad \alpha \to 0,$$

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Proof : standard perturbation theory, each eigenvalue is simple as $\alpha \to 0$.

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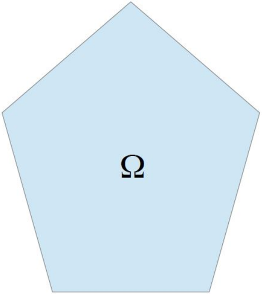
Theorem : An Agmon-type estimate for the eigenfunctions

Let E be a discrete eigenvalue of T_α and $\mathcal{V}$ be an associated eigenfunction. Then, for all $\epsilon \in (0,1)$,

$$\int_{U_\alpha} \left(|\nabla \mathcal{V}|^2 + |\mathcal{V}|^2 \right) e^{2(1-\epsilon)\sqrt{-1-E}|x|} dx < +\infty.$$

Application to Robin Laplacians on polygons



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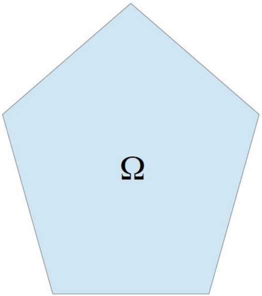

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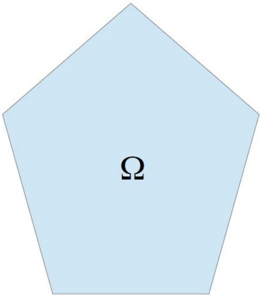

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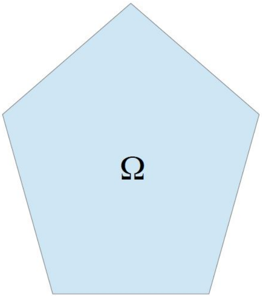

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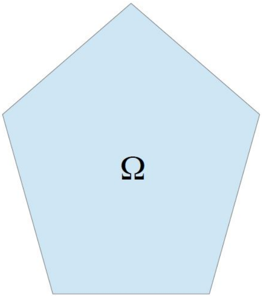

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Behavior of $E_n(Q^\gamma)$ as $\gamma \to +\infty$?

Proposition [Levitin-Parnovski,2008 ; Bruneau-Popoff,2016]

$$E_1(Q^\gamma) = -\frac{\gamma^2}{\sin^2\left(\min_{v\in\mathcal{V}} \alpha_v\right)} + o(\gamma^2), \quad \gamma \to +\infty.$$

Model operator

We define $T^{\oplus}$ the Laplacian acting on $\bigoplus_{v\in\mathcal{V}} L^2(U_{\alpha_v})$ and defined by :

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Then,

- $\mathrm{spec}(T^{\oplus}) = \bigcup_{v\in\mathcal{V}} \mathrm{spec}(T_{\alpha_v})$,
- $\mathrm{spec}_{\mathrm{ess}}(T^{\oplus}) = [-1, +\infty)$,
- $N^{\oplus} := \#\{n \in \mathbb{N}, E_n(T^{\oplus}) < -1\} = \sum_{v\in\mathcal{V}} N_{\alpha_v} < +\infty$,
- $E_1(T^{\oplus}) = -\dfrac{1}{\sin^2(\min_{v\in\mathcal{V}} \alpha_v)}$.

Asymptotics of the first eigenvalues of Q^γ

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Ideas of the proof [Bonnaillie-Noël-Dauge, 2006] :

- Construction of quasi-modes :
for $v \in \mathcal{V}$, let $\psi_n^{\gamma,v}$ be a normalized eigenfunction of $T_{\alpha_v}^\gamma$ and χ_v a smooth radial cut-off function such that $\operatorname{supp} \chi_v \subset B(v, r)$. We define

$$\phi_n^{\gamma,v} := \psi_n^{\gamma,v} \chi_v.$$

Asymptotics of the first eigenvalues of Q^γ

Theorem

For all $n \le N^\oplus$,

$$E_n(Q^\gamma) = \gamma^2 E_n(T^\oplus) + O(e^{-c\gamma}), \quad \gamma \to +\infty.$$

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- $\phi_n^{\gamma,v} \in D(Q^\gamma)$ and

$$\frac{\|Q^\gamma \phi_n^{\gamma,v} - \gamma^2 E_n(T_{\alpha_v})\|^2}{\|\phi_n^{\gamma,v}\|^2} = O(e^{-c\gamma}), \quad \gamma \to +\infty.$$

Spectral theorem implies

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For γ large enough,

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Proof : Localization property of $\psi_n^{\gamma,v}$.

Lemma

For all $1 \leq l \leq K$ and for γ large enough,

$$E_{\kappa_1+\ldots+\kappa_l}(Q^\gamma) \leq \gamma^2 \lambda_l + C\gamma^2 e^{-c\gamma},$$
$$E_{\kappa_1+\ldots+\kappa_l+1}(Q^\gamma) \geq \gamma^2 \lambda_{l+1} - C.$$

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Cluster of eigenvalues

For $1 \leq n \leq \kappa_1$,

$$-C\gamma^{\frac{4}{3}} \leq E_n(Q^\gamma) - \gamma^2 E_n(T^\oplus) \leq C\gamma^2 e^{-c\gamma}.$$

For $\kappa_1 < n \leq N^\oplus$,

$$-C \leq E_n(Q^\gamma) - \gamma^2 E_n(T^\oplus) \leq C\gamma^2 e^{-c\gamma}.$$

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Spectral approximation

Let A be a self-adjoint operator acting on a Hilbert space H and $\lambda \in \mathbb{R}$. If there exists $\psi_1, ..., \psi_n \in D(A)$ linearly independent and $\eta > 0$ such that

$$\|(A - \lambda)\psi_j\| \leq \eta\|\psi_j\|, \quad j = 1, ..., n,$$

then,

$$\dim \operatorname{Ran} P_A(\lambda - C\eta, \lambda + C\eta) \geq n,$$

where $P_A(a, b)$= spectral projection of A on (a, b), $C > 0$ depends on the Gramian matrix of $(\psi_j)_j$.
In particular, if $\operatorname{spec}_{\mathrm{ess}}(A) \cap (\lambda - C\eta, \lambda + C\eta) = \emptyset$, there exist at least n eigenvalues in $(\lambda - c\eta, \lambda + c\eta)$.

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In our case :

- $\mathrm{spec}_{\mathrm{ess}}(Q^\gamma) = \emptyset$,
- $\eta = O(e^{-c\gamma})$.

Work in progress

The asymptotics remains true for curvilinear polygons but the remainders are polynomials.

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Important remark

Proof $\implies$ $E_{N^\oplus+1} \geq -\gamma^2 - \kappa_{\max}\gamma + O(\gamma^{\frac{2}{3}}), \quad \gamma \to +\infty.$

What happens for $E_{N^\oplus+j}$?

Weyl asymptotics

We want to study $N(Q^\gamma, c\gamma^2) := \#\{n \in \mathbb{N}, E_n(Q^\gamma) < c\gamma^2\}$ as $\gamma \to +\infty$.
What are the interesting constants $c \in \mathbb{R}$ **?**

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Theorem [Helffer-Kachmar-Raymond, 2017]

Let $D \subset \mathbb{R}^2$ be an open, bounded connected set such that ∂D is C^4 smooth, and T^γ be the Robin Laplacian acting on $L^2(D)$. Then, for all $\lambda \in \mathbb{R}$,

$$N(T_D^\gamma, -\gamma^2 + \lambda\gamma) \underset{\gamma \to +\infty}{\sim} \frac{\sqrt{\gamma}}{\pi} \int_{\partial D} \sqrt{(\kappa(s) + \lambda)_+} d\sigma,$$

and for all $E \in (-1, 0)$, $N(T_D^\gamma, E\gamma^2) \underset{\gamma \to +\infty}{\sim} \dfrac{|\partial D|}{\pi} \gamma \sqrt{E+1}$, where $\partial D \ni s \mapsto \kappa(s)$ is the curvature of ∂D.

Remark. For $E < -1$, $\lim_{\gamma \to +\infty} N(T_D^\gamma, E\gamma^2) = 0$.

Work in progress

The Weyl formulae remain true for curvilinear polygons.

- There is no contribution of the vertices in the asymptotics.
- If Ω is a polygon with straight edges,

$$\lim_{\gamma \to +\infty} \frac{N(Q^\gamma, -\gamma^2)}{\sqrt{\gamma}} = 0.$$

- Ideas of the proof :

  Upper bound : partition of unity adapted to truncated sectors : the truncated sectors do not contribute, the 'regular' part gives the asymptotics.
  Lower bound : Dirichlet bracketing.

What comes next ?

- Asymptotics of eigenvalues on circular cones as the angle goes to 0 ?
- What happens for the next eigenvalues, i.e : for $j \in \mathbb{N}$,

$$E_{N^{\oplus}+j}(Q^{\gamma}) \xrightarrow[\gamma \to +\infty]{} ?$$

- Can we adapt the proof in higher dimension ? Study of polyhedra ?

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Thank you for your attention