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Periodic Schrödinger operators with δ'-potentials

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joint work with Pavel Exner

It is known that the spectrum of self-adjoint periodic differential operators has a band structure, i.e. the spectrum is a locally finite union of compact intervals called bands.

The open interval (α,β) is called a gap if $(\alpha,\beta)\cap\sigma(\mathcal{H})=\varnothing$ and $\alpha,\beta\in\sigma(\mathcal{H})$.

In general the presence of gaps in the spectrum is not guaranteed!

Example: $\sigma(-\Delta_{\mathbb{R}^n})=[0,\infty)$.

Problem 1

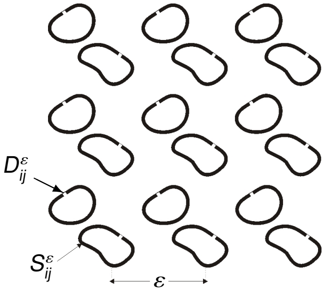
For a given class $\mathcal{L}$ of periodic differential operators to construct the operator $\mathcal{H}\in\mathcal{L}$ with at least one gap in the spectrum

- Scalar elliptic operators of the form $b\nabla^*(a\nabla)$ with periodic a and b
  - A. Figotin, P. Kuchment, *SIAM J. Appl. Math. 56 (1996).*
  - R. Hempel, K. Lienau, *Commun. PDEs 25 (2000).*
  - V. Zhikov, *St. Petersb. Math. J. 16 (2005).*
  - V. Hoang, M. Plum, C. Wieners, *ZAMP 60 (2009).*
- Laplace-Beltrami operators on periodic Riemannian manifolds
  - E. B. Davies, E. M. Harrell, *J. Differ. Equ. 66 (1987).*
  - O. Post, *J. Differ. Equ. 187 (2003).*
  - P. Exner, O. Post, *J. Geom. Phys. 54 (2005).*
- Laplacians posed in noncompact periodic domains
  - S. Nazarov, K. Ruotsalainen, J. Taskinen, *J. Math. Sci. 181(2012).*
- Periodic operators posed in domains with waveguide geometry
  - K. Yoshitomi, *J. Differ. Equ. 142 (1998).*
  - G. Cardone, S. Nazarov, C. Perugia, *Math. Nachr. 283 (2010).*
  - D. Borisov, K. Pankrashkin, *J. Phys. A 46 (2013).*
- Maxwell operators
  - A. Figotin, P. Kuchment, *SIAM J. Appl. Math. 56 (1996)*
  - N. Filonov, *Commun. Math. Phys. 240 (2003).*

Problem 2

To construct the operator $\mathcal{H} \in \mathcal{L}$ having gaps which are close (in some natural sense) to preassigned intervals

- Laplace-Beltrami operators on periodic Riemannian manifolds
  [A. K., J. Differ. Equations 252(3) (2012)]
- Scalar elliptic operators in divergence form
  [A. K., Asympt. Analysis 82(1-2) (2013)]
- Laplacians posed in noncompact periodic domains
  [A. K., E. Khruslov, Math. Meth. Appl. Sci. 38(1) (2015)],
  [A. K., J. Math. Phys. 55(12) (2014)]
- Periodic quantum graphs
  [D. Barseghyan, A. K., J. Phys. A 48(25) (2015)]



- $m \in \mathbb{N}$ is a given number
- the sets S_{ij}^{ε} ($j \in \{1, ..., m\}$ is fixed, $i \in \mathbb{Z}^n$) are distributed ε-periodically in $\mathbb{R}^n$ ($\varepsilon > 0$)
- each set S_{ij}^{ε} has the form $\varepsilon(S_j + i) \setminus D_{ij}^{\varepsilon}$, where S_j are fixed surfaces without a boundary, D_{ij}^{ε} are small "holes"
- the radius of D_{ij}^{ε} is equal to $d_j \varepsilon^{\frac{n}{n-2}}$ ($n \geq 3$) or $e^{-1/d_j \varepsilon^2}$ ($n = 2$)
- $\Omega^{\varepsilon} = \mathbb{R}^n \setminus \left(\bigcup_{i \in \mathbb{Z}^n} \bigcup_{j=1}^{m} S_{ij}^{\varepsilon} \right)$

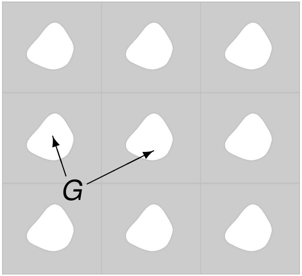
We denote by $\mathcal{H}^{\varepsilon} = -\Delta_{\Omega^{\varepsilon}}$ the Neumann Laplacian in Ω^{ε}.

One has [A.K., 2014]:

- the operator $\mathcal{H}^{\varepsilon}$ has at least m gaps as ε is small enough,
- the first m gaps converge as $\varepsilon \to 0$ to certain intervals (a_j, b_j), whose closures are pairwise disjoint; the next gaps (if any) go to infinity,
- one can completely control the location of the intervals (a_j, b_j) via a suitable choice of the numbers d_j and the surfaces S_j.

The goal

To study this problem for periodic Schrödinger operators with singular potentials



- $B \subset (0,1)^n$ – an open domain
- $G := \bigcup\limits_{i \in \mathbb{Z}^n} (B + i)$
- $\Omega := \mathbb{R}^n \setminus \overline{G}$
- $\mathcal{H}^\varepsilon := -\Delta_{\mathbb{R}^n} + \varepsilon^{-1} 1_\Omega,\ \varepsilon > 0.$
- $\mathcal{H}$ is the Dirichlet Laplacian in G

One can prove[1] that $\mathcal{H}^\varepsilon$ norm resolvent converges to $\mathcal{H}$.

Since $\sigma(\mathcal{H}) = \bigcup\limits_{k=1}^{\infty} \{\lambda_k\}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \to \infty$, the spectrum of $\mathcal{H}^\varepsilon$ has at least m gaps provided ε is small enough.

[1] R. Hempel, I. Herbst, Commun. Math. Phys. 169 (1995), 237–259

Notations:

- $m \in \mathbb{N}$
- $\varepsilon > 0$ - a small parameter
- $Y := (0,1)^n$
- B_j, $j = 1, \dots, m$ be Lipschitz domains satisfying

$$\overline{B_{j_1}} \cap \overline{B_{j_2}} = \varnothing, \quad \bigcup_{j=1}^m B_j \subset Y$$

- $B_0 := Y \setminus \bigcup_{j=1}^m \overline{B_j}$
- $S_j := \partial B_j$, $j = 1, \dots, m$
- $B_{ij}^\varepsilon := \varepsilon(B_j + i)$, $i \in \mathbb{Z}^n$, $j = 1, \dots, m$
- $S_{ij}^\varepsilon := \partial B_{ij}^\varepsilon$ – the surfaces supporting our potential
- $\Omega^\varepsilon := \mathbb{R}^n \setminus \overline{\bigcup_{ij} B_{ij}^\varepsilon}$

Let us define accurately the Schrödinger operator $\mathcal{H}^\varepsilon = -\Delta + V^\varepsilon$ with a singular potential defined by the following formal expression:

$$V^\varepsilon = \sum_{i\in\mathbb{Z}^n}\sum_{j=1}^{m} q_j\varepsilon^{-1}\langle\delta'_{S_{ij}^\varepsilon},\cdot\rangle\delta'_{S_{ij}^\varepsilon},\ q_j \text{ are positive constants.}$$

In what follows by $(f)_{ij}^+$ (respectively, $(f)_{ij}^+$) we denote the trace of the function f on S_{ij}^ε, when we approach this surface from outside (respectively, inside).

In the space $L^2(\mathbb{R}^n)$ we define the sesquilinear form $\mathfrak{h}^\varepsilon$ by

$$\mathfrak{h}^\varepsilon[u,v] = \int_{\mathbb{R}^n}\nabla u\cdot\nabla\bar{v}\mathrm{d}x + \sum_{i\in\mathbb{Z}^n}\sum_{j=1}^{m} q_j^{-1}\varepsilon\int_{S_{ij}^\varepsilon}\left((u)_{ij}^+ - (u)_{ij}^-\right)\overline{\left((v)_{ij}^+ - (u)_{ij}^-\right)}\mathrm{d}s$$

with $\mathrm{dom}(\mathfrak{h}^\varepsilon) = \widetilde{H}^1(\mathbb{R}^n) := H^1(\Omega^\varepsilon)\bigoplus_{i,j} H^1(B_{ij}^\varepsilon)$.

The form $\mathfrak{h}^\varepsilon$ is symmetric, densely defined, closed and positive.

By $\mathcal{H}^\varepsilon$ we denote the operator associated with the form $\mathfrak{h}^\varepsilon$, i.e.

$$(\mathcal{H}^\varepsilon u, v)_{L^2(\mathbb{R}^n)} = \mathfrak{h}^\varepsilon[u, v], \quad \forall u \in \mathrm{dom}(\mathcal{H}^\varepsilon),\ \forall v \in \mathrm{dom}(\mathfrak{h}^\varepsilon).$$

The functions u from $\mathrm{dom}(\mathcal{H}^\varepsilon)$ satisfy (see, e.g., [2]):

- $u \in \widetilde{H}^1(\mathbb{R}^n), \quad \Delta u \in L^2(\mathbb{R}^n), \quad \left(\frac{\partial u}{\partial n}\right)^{\pm}_{ij} \in L^2(S^\varepsilon_{ij}),$
- $\mathcal{H}^\varepsilon u = -\Delta u,$
- $\left(\frac{\partial u}{\partial n}\right)^{+}_{ij} = \left(\frac{\partial u}{\partial n}\right)^{-}_{ij} =: \left(\frac{\partial u}{\partial n}\right)_{ij}, \quad q_j \varepsilon^{-1} \left(\frac{\partial u}{\partial n}\right)_{ij} + \left((u)^{-}_{ij} - (u)^{+}_{ij}\right) = 0.$

[2] J. Behrndt, P. Exner, V. Lotoreichik, Rev. Math. Phys. 26 (2014), 1450015.

For $j = 1, \ldots, m$ we set:

$$a_j := q_j^{-1}|S_j||B_j|^{-1}.$$

It is assumed that the numbers a_j are pairwise non-equivalent. We renumber them in the ascending order: $a_j < a_{j+1}$, $j = 1, \ldots, m+1$.

We consider the following equation (with unknown $\lambda \in \mathbb{C}$):

$$\mathcal{F}(\lambda) = 0, \text{ where } \mathcal{F}(\lambda) := 1 + \frac{1}{|B_0|}\sum_{i=1}^{m}\frac{q_j^{-1}|S_j|}{\lambda - q_j^{-1}|S_j||B_j|^{-1}}.$$

It has exactly m roots b_j satisfying (after appropriate renumbering)

$$a_j < b_j < a_{j+1},\ j = 1, \ldots, m-1, \quad a_m < b_m < \infty.$$

Theorem 1

Let $L > 0$ be an arbitrary number. Then the spectrum of the operator $\mathcal{H}^\varepsilon$ in $[0, L]$ has the following structure for ε small enough:

$$\sigma(\mathcal{H}^\varepsilon) \cap [0, L] = [0, L] \setminus \bigcup_{j=1}^{m} \big(a_j(\varepsilon), b_j(\varepsilon)\big),$$

where the endpoints of the intervals $\big(a_j(\varepsilon), b_j(\varepsilon)\big)$ satisfy the relations

$$\lim_{\varepsilon\to 0} a_j(\varepsilon) = a_j, \quad \lim_{\varepsilon\to 0} b_j(\varepsilon) = b_j, \ j = 1, \dots, m.$$

Theorem 2

Let $L > 0$ be an arbitrarily large number and let (α_j,β_j), $j = 1,\dots,m$ be any arbitrary intervals satisfying

$$0 < \alpha_1, \quad \alpha_j < \beta_j < \alpha_{j+1},\ j = \overline{1, m-1}, \quad \alpha_m < \beta_m < L.$$

Suppose that the sets B_j, $j = 1,\dots,m$, satisfy

$$|B_j| = \left(1 - \sum_{j=1}^{m}|B_j|\right)\frac{\beta_j - \alpha_j}{\alpha_j}\prod_{i=\overline{1,m}|i\neq j}\left(\frac{\beta_i - \alpha_j}{\alpha_i - \alpha_j}\right).$$

Then one has

$$a_j = \alpha_j,\ b_j = \beta_j, \quad j = 1,\dots,m$$

provided

$$q_j = \frac{|B_j|}{\alpha_j|S_j|},\ j = 1,\dots,m.$$

We rescale the problem to Y-periodic. Namely, we consider the operator

$$\widetilde{\mathcal{H}^\varepsilon} = -\varepsilon^{-2}\Delta + \sum_{i\in\mathbb{Z}^n}\sum_{j=1}^{m} q_j\langle\delta'_{S_{ij}},\cdot\rangle\delta'_{S_{ij}},$$

where $S_{ij} := S_j + i$. It is clear that $\sigma(\widetilde{\mathcal{H}^\varepsilon}) = \sigma(\mathcal{H}^\varepsilon)$.

We introduce the following forms in $L^2(Y)$:

$$\mathfrak{h}^{\varepsilon,N}:\ \mathrm{dom}(\mathfrak{h}^{\varepsilon,N}) = \left\{u\in L^2(Y) : u\in H^1(B_j),\ j=\overline{1,m},\ u\in H^1(Y\setminus\bigcup_{j=1}^m \overline{B_j})\right\},$$

$$\mathfrak{h}^\varepsilon_N[u,v] = \frac{1}{\varepsilon^2}\int_Y \nabla u\cdot\nabla\overline{v}\,\mathrm{d}x + \sum_{j=1}^m \frac{1}{q_j}\int_{S_j}\left((u)^+_j-(u)^-_j\right)\overline{\left((v)^+_j-(v)^-_j\right)}\mathrm{d}s$$

$$\mathfrak{h}^{\varepsilon,D}:\ \mathrm{dom}(\mathfrak{h}^{\varepsilon,D}) = \left\{u\in\mathfrak{h}^{\varepsilon,N}:\ u|_{\partial Y}=0\right\},\quad \mathfrak{h}^\varepsilon_D[u,v]=\mathfrak{h}^\varepsilon_N[u,v]$$

$$\mathfrak{h}^{\varepsilon,+}:\ \mathrm{dom}(\mathfrak{h}^{\varepsilon,+}) = \left\{u\in\mathfrak{h}^{\varepsilon,N}:\ u \text{ is periodic}\right\},\quad \mathfrak{h}^\varepsilon_+[u,v]=\mathfrak{h}^\varepsilon_N[u,v]$$

$$\mathfrak{h}^{\varepsilon,-}:\ \mathrm{dom}(\mathfrak{h}^{\varepsilon,-}) = \left\{u\in\mathfrak{h}^{\varepsilon,N}:\ u \text{ is antiperiodic}\right\},\quad \mathfrak{h}^\varepsilon_-[u,v]=\mathfrak{h}^\varepsilon_N[u,v]$$

We denote by $\mathcal{H}^{\varepsilon,N}$, $\mathcal{H}^{\varepsilon,D}$, $\mathcal{H}^{\varepsilon,+}$, $\mathcal{H}^{\varepsilon,-}$ the operators associated with these forms. The spectra of these operators are purely discrete.

We denote by $\left\{\lambda_k^{\varepsilon,N}\right\}_{k\in\mathbb{N}}$, $\left\{\lambda_k^{\varepsilon,D}\right\}_{k\in\mathbb{N}}$, $\left\{\lambda_k^{\varepsilon,+}\right\}_{k\in\mathbb{N}}$, $\left\{\lambda_k^{\varepsilon,-}\right\}_{k\in\mathbb{N}}$ the corresponding sequences of eigenvalues, renumbered in the ascending order and with account of multiplicity.

Using Floquet-Bloch theory and minimax principle we get:

$$\sigma(\mathcal{H}^\varepsilon) = \bigcup_{k\in\mathbb{N}} L_k^\varepsilon,\ L_k^\varepsilon \text{ are compact intervals satisfying} \\ [\lambda_k^{\varepsilon,+}, \lambda_k^{\varepsilon,-}] \subset L_k^\varepsilon \subset [\lambda_k^{\varepsilon,N}, \lambda_k^{\varepsilon,D}] \quad (1)$$

Our goal it to prove that

$$\lim_{\varepsilon\to 0} \lambda_k^{\varepsilon,N} = \lim_{\varepsilon\to 0} \lambda_k^{\varepsilon,+} = \begin{cases} 0, & k = 1 \\ b_{k-1}, & 2 \leq k \leq m+1 \\ \infty, & k \geq m+2 \end{cases} \quad (2)$$

$$\lim_{\varepsilon\to 0} \lambda_k^{\varepsilon,D} = \lim_{\varepsilon\to 0} \lambda_k^{\varepsilon,-} = \begin{cases} a_k, & 1 \leq k \leq m \\ \infty, & k \geq m+1 \end{cases}$$

$$(1) + (2) \implies \text{Theorem 1}$$

The forms $\mathfrak{h}^{\varepsilon,\bullet}$ increases monotonically as ε decreases. We introduce the limit forms $\mathfrak{h}^{\bullet}$ by

$$\mathrm{dom}(\mathfrak{h}^{\bullet}) = \left\{u \in \mathrm{dom}(\mathfrak{h}^{\varepsilon,\bullet}) :\ \sup_{\varepsilon} \mathfrak{h}^{\varepsilon,\bullet}[u,u] < \infty\right\},$$
$$\mathfrak{h}^{\bullet}[u,v] := \lim_{\varepsilon\to 0} \mathfrak{h}^{\varepsilon,\bullet}[u,v]$$

The forms $\mathfrak{h}^{\bullet}$ are positive and closed (see [3]).

We denote by $\mathcal{H}^{\bullet}$ the operators acting in $\overline{\mathrm{dom}(\mathfrak{h}^{\bullet})}^{L^2(Y)}$ being associated with these forms.

[3] B. Simon, J. Funct. Anal. 28 (1978), no. 3, 377–385.

Finally, we define the “resolvents” of these operators:

$$R^\bullet := \begin{cases} (\mathcal{H}^\bullet + I)^{-1} & \text{on } \overline{\mathrm{dom}(\mathcal{H}^\bullet)}^{L^2(Y)} \\ 0 & \text{on } L^2(Y) \ominus \overline{\mathrm{dom}(\mathcal{H}^\bullet)}^{L^2(Y)} \end{cases}$$

Then (again see [2])

$$\forall f \in L^2(Y): \ \left\|(\mathcal{H}^{\varepsilon,\bullet} + I)^{-1} f - R^\bullet f\right\|_{L^2(Y)} \to 0 \text{ as } \varepsilon \to 0. \quad (3)$$

Moreover, since $(\mathcal{H}^{\varepsilon_1,\bullet} + I)^{-1} \geq (\mathcal{H}^{\varepsilon_2,\bullet} + I)^{-1}$ as $\varepsilon_1 \geq \varepsilon_2$, and $(\mathcal{H}^{\varepsilon,\bullet} + I)^{-1}$ and $R^\bullet$ are compact operators, one can upgrade (3) to norm convergence (see Theorem VIII-3.5 from [4]):

$$\left\|(\mathcal{H}^{\varepsilon,\bullet} + I)^{-1} - R^\bullet\right\|_{\mathcal{L}(L^2(Y))} \to 0 \text{ as } \varepsilon \to 0. \quad (4)$$

[4] T. Kato, Perturbation theory for linear operators, Springer, New-York, 1966.

One has:

$$\mathrm{dom}(\mathcal{H}^N) = \mathrm{dom}(\mathcal{H}^+) = \left\{ u(x) = \sum_{j=0}^{m} \mathbf{u}_j 1_{B_j}(x),\ \mathbf{u}_j \text{ are constants} \right\}$$

$$\mathcal{H}^N u = \mathcal{H}^+ u = \left(\sum_{k=1}^{m} \frac{|S_j|}{q_j |B_0|} (\mathbf{u}_0 - \mathbf{u}_k) \right) 1_{B_0}(x) + \sum_{j=1}^{m} \frac{|S_j|}{q_j |B_j|} (\mathbf{u}_j - \mathbf{u}_0) 1_{B_j}(x)$$

$$\mathcal{H}^D u = \mathcal{H}^- u = \mathrm{dom}(\mathcal{H}^-) = \left\{ u(x) = \sum_{j=1}^{m} \mathbf{u}_j 1_{B_j}(x),\ \mathbf{u}_j \text{ are constants} \right\}$$

$$\mathcal{H}^D u = \mathcal{H}^- u = \sum_{j=1}^{m} \frac{|S_j|}{q_j |B_j|} \mathbf{u}_j 1_{B_j}(x)$$

We denote the eigenvalues of these operators by

$$\lambda_k^N,\ \lambda_k^+,\ k = \overline{1, m+1}, \quad \lambda_k^D,\ \lambda_k^-,\ k = \overline{1, m}.$$

It follows from (3) that

$$\lim_{\varepsilon\to 0}(\lambda_k^{\varepsilon,N/+}+1)^{-1}=\begin{cases}(\lambda_k^{N/+}+1)^{-1}, & 1\le k\le m+1\\ 0, & k\ge m+2\end{cases}$$

$$\lim_{\varepsilon\to 0}(\lambda_k^{\varepsilon,D/-}+1)^{-1}=\begin{cases}(\lambda_k^{D/-}+1)^{-1}, & 1\le k\le m\\ 0, & k\ge m+1\end{cases}$$

or, equivalently,

$$\lim_{\varepsilon\to 0}\lambda_k^{\varepsilon,N/+}=\begin{cases}\lambda_k^{N/+}, & 1\le k\le m+1\\ \infty, & k\ge m+2\end{cases}$$

$$\lim_{\varepsilon\to 0}\lambda_k^{\varepsilon,D/-}=\begin{cases}\lambda_k^{D/-}, & 1\le k\le m\\ \infty, & k\ge m+1\end{cases}$$

It is easy to see that $\lambda_k^{\varepsilon,D} = \lambda_k^{\varepsilon,-} = q_k^{-1}|S_k||B_k|^{-1} = a_k$.

The eigenvalues $\lambda_k^{\varepsilon,N} = \lambda_k^{\varepsilon,+}$ are the roots of the equation

$$\det(H^N - \lambda I) = 0,$$

where the matrix H is as follows:

$$H := \begin{pmatrix} \sum\limits_{j=1}^m q_j^{-1}|S_j||B_0|^{-1} & -q_1^{-1}|S_1||B_0|^{-1} & -q_2^{-1}|S_2||B_0|^{-1} & \dots & -q_m^{-1}|S_m||B_0|^{-1} \\ -q_1^{-1}|S_1||B_1|^{-1} & q_1^{-1}|S_1||B_1|^{-1} & 0 & \dots & 0 \\ -q_2^{-1}|S_2||B_2|^{-1} & 0 & q_2^{-1}|S_2||B_2|^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_m^{-1}|S_m||B_m|^{-1} & 0 & 0 & \dots & q_m^{-1}|S_m||B_m|^{-1} \end{pmatrix}$$

After some algebra we obtain:

$$\det(H^N - \lambda I) = -\lambda\left(\prod_{j=1}^m \left(q_j^{-1}|S_j||B_j|^{-1} - \lambda\right)\right)\left(1 + \frac{1}{|B_0|}\sum_{i=1}^m \frac{q_j^{-1}|S_j|}{\lambda - q_j^{-1}|S_j||B_j|^{-1}}\right),$$

whence $\lambda_1^{N/+} = 0$, $\lambda_k^{N/+} = b_{k-1}$ as $k = 2, \dots, m+1$.

Thank you for your attention!