A Lieb-Thirring inequality for Schrödinger operators with δ-potentials supported on a hyperplane

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LT-inequality for classical Potentials

For a regular potential $V : \mathbb{R}^d \to \mathbb{R}$, consider

Schrödinger operator

$$H_V = -\Delta + V$$

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Let $(E_i)_i$ be the negative eigenvalues of H_V, then

Lieb-Thirring Inequaltiy, 1976

$$\sum_i |E_i|^\gamma \leq c_\gamma \int_{\mathbb{R}^d} V_-^{\frac{d}{2}+\gamma} dx$$

for $\gamma > \frac{1}{2}$ if $d = 1$
$\gamma > 0$ if $d \geq 2$

Also $\gamma = \frac{1}{2}$ if $d = 1$ (Weidl 1996)
$\gamma = 0$ if $d \geq 3$ (Cwikel 1977, Lieb 1980, Rosenblum 1976)

Schrödinger operator with δ-potential

Let $\Sigma \subseteq \mathbb{R}^d$ a hypersurface and $\alpha : \Sigma \to \mathbb{R}$. Then consider

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defined by the bilinear form

$$a_\alpha(f, g) = \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^d)} - \int_\Sigma \alpha\, f|_\Sigma\, \bar{g}|_\Sigma\, d\sigma$$

Goal: For $\gamma > 0$ (maybe $\gamma \geq 0$) an inequaltiy of the form

$$\sum_i |E_i|^\gamma \leq c_\gamma \int_\Sigma \alpha_-^{\frac{d}{2}+\gamma} d\sigma$$

with $(E_i)_i$ the negative eigenvalues of $-\Delta_\alpha$.

Weyl-function and number of eigenvalues

For $\lambda < 0$, the Weyl-function looks like

$$M_\lambda f(x) = \int_\Sigma G_\lambda(x-y)f(y)d\sigma(y) : L^2(\Sigma) \to L^2(\Sigma)$$

with $G_\lambda : \mathbb{R}^d \to \mathbb{C}$, the integral kernel of the resolvent of the laplacian.

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Consider now the number of eigenvalues

- N_ε = number of eigenvalues of $-\Delta_\alpha$ smaller than $(-\varepsilon)$.
- B_ε = number of eigenvalues of $\alpha M_{-\varepsilon}$ larger than 1.

Birman-Schwinger principle

$$N_\varepsilon = B_\varepsilon$$

Birman-Schwinger bound

Brasche, Exner, Kuperin, Seba, 1994

$$N_\varepsilon \leq \left(\sup_{x \in \mathbb{R}^d} \int_\Sigma G_{-\varepsilon}(x-y)^q |\alpha(y)| d\sigma(y) \right)^{\frac{1}{q-1}} \int_\Sigma \alpha_- d\sigma$$

$$N_0 \leq \left(\sup_{x \in \mathbb{R}^d} \int_\Sigma G_0(x-y)^q |\alpha(y)| d\sigma(y) \right)^{\frac{1}{q-1}} \int_\Sigma \alpha_- d\sigma$$

for any $q \in (1, 2]$.

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$$\sum_i |E_i|^\gamma \leq c_\gamma \int_\Sigma \alpha_-^{\frac{d}{2}+\gamma} d\sigma$$

with $(E_i)_i$ the negative eigenvalues of $-\Delta_\alpha$.

Way to achieve this

The way to the inequality is mainly splitted into three steps

$$\sum_i |E_i|^\gamma \overset{(1)}{\longleftrightarrow} N_\varepsilon \overset{(2)}{\longleftrightarrow} B_\varepsilon \overset{(3)}{\longleftrightarrow} \int_\Sigma |\alpha|^{\frac{d}{2}+\gamma} d\sigma$$

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(3) $B_\varepsilon \leq \mathrm{tr}(\alpha^\eta M^\eta_{-\varepsilon})$ difficult

for some specific $\eta \geq 1$.

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For some integral kernel $G^{(\eta)}_{-\varepsilon}$ depending on ε and η.

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For some integral kernel $G_{-\varepsilon}^{(\eta)}$ depending on ε and η.

Needed property

$$G_{-\varepsilon}^{(\eta)}(0) \stackrel{?}{=} \mathcal{O}(\varepsilon^{\frac{d}{2}-\eta})$$

Special case: Hyperplane

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$$-\Delta_\alpha = -\Delta + \underbrace{(\alpha_0 + \alpha_1)\delta_\Sigma}_{\text{not compact}}$$

New Weyl-function

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New Boundary triplet

$$\Gamma_0' = \Gamma_0 + \alpha_0 \Gamma_1 \qquad \text{and} \qquad \Gamma_1' = \Gamma_1$$

is a boundary triplet of $-\Delta + \alpha_0 \delta_\Sigma$

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Weyl-function

$$M_\lambda' = M_\lambda (1 + \alpha_0 M_\lambda)^{-1} = \frac{1}{2}\left((-\Delta_{d-1} - \lambda)^{\frac{1}{2}} + \frac{\alpha_0}{2} \right)^{-1}$$

Powers of the Weyl-function

Using Fourier transformation to calculate powers of $M'_{-\varepsilon}$.

$$\Rightarrow M'^{\eta}_{-\varepsilon} = \frac{1}{2^\eta}\left((-\Delta_{d-1}+\varepsilon)^{\frac{1}{2}} + \frac{\alpha_0}{2}\right)^{-\eta}$$

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$$\Rightarrow FM'^{\eta}_{-\varepsilon}F^{-1} = \frac{1}{2^\eta}\left((|k|^2+\varepsilon)^{\frac{1}{2}}+\frac{\alpha_0}{2}\right)^{-\eta}$$

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with

Integral kernel of Weyl-function

$$G'^{(\eta)}_{-\varepsilon}(x) = \frac{1}{2^\eta(2\pi)^{d-1}}\int\limits_{\mathbb{R}^{d-1}} \frac{1}{\left((|k|^2+\varepsilon)^{\frac{1}{2}}+\frac{\alpha_0}{2}\right)^{\eta}} e^{ikx}dk$$

For $\varepsilon > \frac{\alpha_0^2}{4}$ we get as before

$$B_\varepsilon \leq \mathrm{tr}(\alpha_1^\eta M'^\eta_{-\varepsilon}) = G'^{(\eta)}_{-\varepsilon}(0) \int_{\mathbb{R}^{d-1}} |\alpha_1|^\eta d\sigma$$

with

$$G'^{(\eta)}_{-\varepsilon}(0) = \frac{1}{2^\eta (2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{1}{\left((|k|^2+\varepsilon)^{\frac{1}{2}} + \frac{\alpha_0}{2}\right)^\eta} dk$$

Open question

What is $G_{-\varepsilon}^{(\eta)}$ for general Σ ?