# Nichtsymmetrische Gleichungssysteme
# Das BiCGStab Verfahren

Lineares Gleichungssystem $A\underline{x} = \underline{f}$, $\quad A$ regulär

Biorthogonale Vektorsysteme

$$\{\underline{p}^\ell\}_{\ell=0}^{n-1}, \{\underline{\widetilde{p}}^\ell\}_{\ell=0}^{n-1}, \qquad (A\underline{p}^k, \underline{\widetilde{p}}^\ell) = (\underline{p}^k, A^\top \underline{\widetilde{p}}^\ell) = 0 \quad \text{für alle } k \neq \ell$$

Konstruktion biorthogonaler Vektoren

Setze

$$\underline{p}^0 = \underline{w}^0, \underline{\widetilde{p}}^0 = \underline{\widetilde{w}}^0,$$

Für $k = 0, \ldots, n-2$ berechne

$$\underline{p}^{k+1} = \underline{w}^{k+1} - \sum_{\ell=0}^{k} \beta_{k,\ell} \underline{p}^\ell, \quad \beta_{k,\ell} = \frac{(A\underline{w}^{k+1}, \underline{\widetilde{p}}^\ell)}{(A\underline{p}^\ell, \underline{\widetilde{p}}^\ell)}$$

$$\underline{\widetilde{p}}^{k+1} = \underline{\widetilde{w}}^{k+1} - \sum_{\ell=0}^{k} \widetilde{\beta}_{k,\ell} \underline{\widetilde{p}}^\ell, \quad \widetilde{\beta}_{k,\ell} = \frac{(A^\top \underline{\widetilde{w}}^{k+1}, \underline{p}^\ell)}{(A^\top \underline{\widetilde{p}}^\ell, \underline{p}^\ell)}$$

Voraussetzung: $(A\underline{p}^\ell, \underline{\widetilde{p}}^\ell) \neq 0$

## Lineares Gleichungssystem

$$A\underline{x} = \underline{f}, \qquad \underline{x} = \underline{x}^0 - \sum_{\ell=0}^{n-1} \alpha_\ell \underline{p}^\ell$$

## Einsetzen

$$A\underline{x}^0 - \sum_{\ell=0}^{n-1} \alpha_\ell A\underline{p}^\ell = \underline{f}$$

## Skalarprodukt mit biorthogonalen Vektoren $\underline{\widetilde{p}}^j$

$$\sum_{\ell=0}^{n-1} \alpha_\ell (A\underline{p}^\ell, \underline{\widetilde{p}}^j) = (A\underline{x}^0 - \underline{f}, \underline{\widetilde{p}}^j)$$

## Zerlegungskoeffizienten

$$\alpha_\ell = \frac{(A\underline{x}^0 - \underline{f}, \underline{\widetilde{p}}^\ell)}{(A\underline{p}^\ell, \underline{\widetilde{p}}^\ell)}$$

## Definition einer Näherungslösung

$$\underline{x}^k = \underline{x}^0 - \sum_{\ell=0}^{k-1} \alpha_\ell \underline{p}^\ell, \qquad \underline{x}^n = \underline{x}, \qquad \alpha_\ell = \frac{(A\underline{x}^0 - \underline{f}, \underline{\widetilde{p}}^\ell)}{(A\underline{p}^\ell, \underline{\widetilde{p}}^\ell)}$$

## Rekursive Definition

$$\begin{aligned} \underline{x}^{k+1} &= \underline{x}^0 - \sum_{\ell=0}^{k} \alpha_\ell \underline{p}^\ell \\ &= \underline{x}^0 - \sum_{\ell=0}^{k-1} \alpha_\ell \underline{p}^\ell - \alpha_k \underline{p}^k = \underline{x}^k - \alpha_k \underline{p}^k, \quad \alpha_k = \frac{(A\underline{x}^0 - \underline{f}, \underline{\widetilde{p}}^k)}{(A\underline{p}^k, \underline{\widetilde{p}}^k)} \end{aligned}$$

## Es ist

$$\begin{aligned} (A\underline{x}^0 - \underline{f}, \underline{\widetilde{p}}^k) &= (A\underline{x}^0 - \underline{f}, \underline{\widetilde{p}}^k) - \sum_{\ell=0}^{k-1} \alpha_\ell (A\underline{p}^\ell, \underline{\widetilde{p}}^k) \\ &= (A\underline{x}^0 - \sum_{\ell=0}^{k-1} \alpha_\ell A\underline{p}^\ell - \underline{f}, \underline{\widetilde{p}}^k) = (A\underline{x}^k - \underline{f}, \underline{\widetilde{p}}^k) = (\underline{r}^k, \underline{\widetilde{p}}^k) \end{aligned}$$

## Iterationsvorschrift für Suchrichtungen

$$\underline{p}^0 = \underline{w}^0, \quad \underline{p}^{k+1} = \underline{w}^{k+1} - \sum_{\ell=0}^{k} \beta_{k\ell} \underline{p}^\ell, \quad \beta_{k\ell} = \frac{(A\underline{w}^{k+1}, \widetilde{\underline{p}}^\ell)}{(A\underline{p}^\ell, \widetilde{\underline{p}}^\ell)}$$

## Iterationsvorschrift für Näherungslösungen

$$\underline{x}^{k+1} = \underline{x}^k - \alpha_k \underline{p}^k, \quad \underline{r}^{k+1} = \underline{r}^k - \alpha_k A\underline{p}^k, \quad \alpha_k = \frac{(\underline{r}^k, \widetilde{\underline{p}}^k)}{(A\underline{p}^k, \widetilde{\underline{p}}^k)}$$

## Es gilt

$$\begin{aligned}(\underline{r}^{k+1}, \widetilde{\underline{p}}^k) &= (\underline{r}^k - \alpha_k A\underline{p}^k, \widetilde{\underline{p}}^k) = (\underline{r}^k, \widetilde{\underline{p}}^k) - \alpha_k (A\underline{p}^k, \widetilde{\underline{p}}^k) = 0 \\ (\underline{r}^{k+1}, \widetilde{\underline{p}}^{k-1}) &= (\underline{r}^k, \widetilde{\underline{p}}^{k-1}) - \alpha_k (A\underline{p}^k, \widetilde{\underline{p}}^{k-1}) = 0\end{aligned}$$

## Durch vollständige Induktion folgt

$$(\underline{r}^{k+1}, \widetilde{\underline{p}}^\ell) = 0 \quad \text{für } \ell = 0, \ldots, k$$

## Orthogonalität

$$(\underline{r}^{k+1}, \widetilde{\underline{p}}^{\ell}) \;=\; 0 \quad \text{für } \ell = 0, \ldots, k$$

## Iterationsvorschrift für Suchrichtungen

$$\underline{p}^0 = \underline{w}^0, \quad \underline{p}^{k+1} = \underline{w}^{k+1} - \sum_{\ell=0}^{k} \beta_{k\ell} \underline{p}^{\ell}, \quad \beta_{k\ell} = \frac{(A\underline{w}^{k+1}, \widetilde{\underline{p}}^{\ell})}{(A\underline{p}^{\ell}, \widetilde{\underline{p}}^{\ell})}$$

Dann gilt

$$(\underline{r}^{k+1}, \widetilde{\underline{w}}^{\ell}) = (\underline{r}^{k+1}, \widetilde{\underline{p}}^{\ell} + \sum_{j=0}^{\ell-1} \widetilde{\beta}_{\ell-1,j} \widetilde{\underline{p}}^{j}) = (\underline{r}^{k+1}, \widetilde{\underline{p}}^{\ell}) + \sum_{j=0}^{\ell-1} \beta_{\ell-1,j} (\underline{r}^{k+1}, \widetilde{\underline{p}}^{j}) = 0$$

## Orthogonalität

$$(\underline{r}^{k+1}, \widetilde{\underline{w}}^{\ell}) \;=\; 0 \quad \text{für } \ell = 0, \ldots, k$$

Die Vektoren $\widetilde{\underline{w}}^0, \ldots, \widetilde{\underline{w}}^k, \underline{r}^{k+1}$ sind linear unabhängig: $\widetilde{\underline{w}}^{k+1} = \underline{r}^{k+1}$

## Orthogonalitäten

$$(\underline{r}^{k+1}, \widetilde{\underline{p}}^{\ell}) = 0, \quad (\underline{r}^{k+1}, \widetilde{\underline{w}}^{\ell}) = (\underline{r}^{k+1}, \underline{r}^{\ell}) = 0 \quad \text{für } \ell = 0, \ldots, k$$

Dann gilt

$$(\underline{r}^k, \widetilde{\underline{p}}^k) = (\underline{r}^k, \widetilde{\underline{w}}^k - \sum_{\ell=0}^{k-1} \widetilde{\beta}_{k-1,\ell} \widetilde{\underline{p}}^{\ell}) = (\underline{r}^k, \underline{r}^k) - \sum_{\ell=0}^{k-1} \widetilde{\beta}_{k-1,\ell} (\underline{r}^k, \widetilde{\underline{p}}^{\ell}) = (\underline{r}^k, \underline{r}^k)$$

und somit

$$\alpha_k = \frac{(\underline{r}^k, \underline{r}^k)}{(A\underline{p}^k, \widetilde{\underline{p}}^k)} \neq 0, \quad \alpha_k = 0: \quad \underline{r}^k = \underline{0}, \quad \underline{x}^k = \underline{x}$$

## Rekursionsvorschrift

$$\widetilde{\underline{r}}^0, \quad \widetilde{\underline{r}}^{k+1} = \widetilde{\underline{r}}^k - \widetilde{\alpha}_k A^\top \widetilde{\underline{p}}^k, \quad \widetilde{\alpha}_k = \frac{(\widetilde{\underline{r}}^k, \underline{p}^k)}{(A^\top \widetilde{\underline{p}}^k, \underline{p}^k)}$$

## Orthogonalitäten

$$(\widetilde{\underline{r}}^{k+1}, \underline{p}^{\ell}) = (\widetilde{\underline{r}}^{k+1}, \underline{w}^{\ell}) = 0 \quad \text{für } \ell = 0, \ldots, k$$

Die Vektoren $\underline{w}^0, \ldots, \underline{w}^k, \widetilde{\underline{r}}^{k+1}$ sind linear unabhängig: $\underline{w}^{k+1} = \widetilde{\underline{r}}^{k+1}$

## Orthogonalitäten

$$(\underline{\widetilde{r}}^{k+1}, \underline{p}^{\ell}) = 0, \quad (\underline{\widetilde{r}}^{k+1}, \underline{w}^{\ell}) = (\underline{r}^{k+1}, \underline{\widetilde{r}}^{\ell}) = 0 \quad \text{für } \ell = 0, \ldots, k$$

Dann gilt

$$(\underline{\widetilde{r}}^{k}, \underline{p}^{k}) = (\underline{\widetilde{r}}^{k}, \underline{w}^{k} - \sum_{\ell=0}^{k-1} \beta_{k-1,\ell}\underline{p}^{\ell}) = (\underline{\widetilde{r}}^{k}, \underline{\widetilde{r}}^{k}) - \sum_{\ell=0}^{k-1} \beta_{k-1,\ell}(\underline{\widetilde{r}}^{k}, \underline{p}^{\ell}) = (\underline{\widetilde{r}}^{k}, \underline{\widetilde{r}}^{k})$$

und somit

$$\widetilde{\alpha}_k = \frac{(\underline{\widetilde{r}}^{k}, \underline{\widetilde{r}}^{k})}{(A\underline{p}^{k}, \underline{\widetilde{p}}^{k})} \neq 0, \quad \widetilde{\alpha}_k = 0 : \quad \underline{\widetilde{r}}^{k} = \underline{0}$$

Berechnung von

$$\beta_{k\ell} = \frac{(A\underline{\widetilde{r}}^{k+1}, \underline{\widetilde{p}}^{\ell})}{(A\underline{p}^{\ell}, \underline{\widetilde{p}}^{\ell})}$$

Für den Zähler von $\beta_{k\ell}$ ist

$$(\underline{\widetilde{r}}^{k+1}, A^\top \underline{\widetilde{p}}^{\ell}) = \frac{1}{\widetilde{\alpha}_\ell}(\underline{\widetilde{r}}^{k+1}, \underline{\widetilde{r}}^{\ell} - \underline{\widetilde{r}}^{\ell+1}) = \begin{cases} 0 & \text{für } \ell < k \\ -\frac{1}{\widetilde{\alpha}_k}(\underline{\widetilde{r}}^{k+1}, \underline{\widetilde{r}}^{k+1}) & \text{für } \ell = k \end{cases}$$

Weiterhin ist

$$\widetilde{\alpha}_k(\underline{p}^k, A^\top \underline{\widetilde{p}}^k) = (\underline{p}^k, \underline{\widetilde{r}}^k - \underline{\widetilde{r}}^{k+1}) = (\underline{p}^k, \underline{\widetilde{r}}^k) = (\underline{\widetilde{r}}^k, \underline{\widetilde{r}}^k)$$

Suchrichtung

$$\underline{p}^{k+1} = \underline{\widetilde{r}}^{k+1} + \beta_k \underline{p}^k, \quad \beta_k = \frac{(\underline{\widetilde{r}}^{k+1}, \underline{\widetilde{r}}^{k+1})}{(\underline{\widetilde{r}}^k, \underline{\widetilde{r}}^k)}$$

# Gradientenverfahren biorthogonaler Richtungen (1)

Für eine beliebig gegebene Startnäherung $\underline{x}^0 \in \mathbb{R}^n$ sei $\underline{r}^0 = A\underline{x}^0 - \underline{f}$.
Wähle $\underline{\widetilde{r}}^0$. Setze $\underline{p}^0 = \underline{\widetilde{r}}^0, \underline{\widetilde{p}}^0 = \underline{r}^0$ und berechne $\varrho_0 = (\underline{r}^0, \underline{r}^0), \widetilde{\varrho}_0 = (\underline{\widetilde{r}}^0, \underline{\widetilde{r}}^0)$.
Stoppe, falls $\varrho_0 < \varepsilon^2$ mit einer vorgegebenen Fehlergenauigkeit $\varepsilon$ erreicht ist.

Berechne für $k = 0, 1, \ldots, n-2$ :

$$\underline{s}^k = A\underline{p}^k, \underline{\widetilde{s}}^k = A^\top \underline{\widetilde{p}}^k,\ \sigma_k = (\underline{s}^k, \underline{\widetilde{p}}^k),\ \alpha_k = \frac{\varrho_k}{\sigma_k}, \widetilde{\alpha}_k = \frac{\widetilde{\varrho}_k}{\sigma_k}$$

$$\underline{x}^{k+1} = \underline{x}^k - \alpha_k \underline{p}^k$$

$$\underline{r}^{k+1} = \underline{r}^k - \alpha_k \underline{s}^k$$

$$\underline{\widetilde{r}}^{k+1} = \underline{\widetilde{r}}^k - \widetilde{\alpha}_k \underline{\widetilde{s}}^k$$

$$\varrho_{k+1} = (\underline{r}^{k+1}, \underline{r}^{k+1}), \widetilde{\varrho}_{k+1} = (\underline{\widetilde{r}}^{k+1}, \underline{\widetilde{r}}^{k+1}),$$

Stoppe, falls $\varrho_{k+1} < \varepsilon^2 \varrho_0$ mit einer vorgegebenen Fehlergenauigkeit $\varepsilon$ erreicht ist. Berechne andernfalls die neuen Suchrichtungen

$$\underline{p}^{k+1} = \underline{\widetilde{r}}^{k+1} + \beta_k \underline{p}^k,\ \beta_k = \frac{\widetilde{\varrho}_{k+1}}{\widetilde{\varrho}_k}; \quad \underline{\widetilde{p}}^{k+1} = \underline{r}^{k+1} + \widetilde{\beta}_k \underline{\widetilde{p}}^k,\ \widetilde{\beta}_k = \frac{\varrho_{k+1}}{\varrho_k}$$

$\underline{\widetilde{r}}^0 = \underline{r}^0$, $A = A^\top$: CG Verfahren

## Parameter

$$\varrho_k = (\underline{r}^k, \underline{r}^k), \quad \widetilde{\varrho}_k = (\widetilde{\underline{r}}^k, \widetilde{\underline{r}}^k), \quad \alpha_k, \widetilde{\alpha}_k, \quad \beta_k, \widetilde{\beta}_k$$

## Wählen jetzt

$$\underline{w}^{k+1} = \underline{r}^{k+1}, \quad \widetilde{\underline{w}}^{k+1} = \widetilde{\underline{r}}^{k+1}$$

Lineare Unabhängigkeit kann nicht mehr gewährleistet werden!

## Folgerung

$$(\underline{r}^k, \widetilde{\underline{r}}^\ell) = 0 \quad \text{für } k \neq \ell$$

## Folgerung

$$\alpha_k = \widetilde{\alpha}_k = \frac{(\underline{r}^k, \widetilde{\underline{r}}^k)}{(A\underline{p}^k, \widetilde{\underline{p}}^k)}, \quad \beta_k = \widetilde{\beta}_k = \frac{(\underline{r}^{k+1}, \widetilde{\underline{r}}^{k+1})}{(\underline{r}^k, \widetilde{\underline{r}}^k)}$$

## Gradientenverfahren biorthogonaler Richtungen (2)

Für eine beliebig gegebene Startnäherung $\underline{x}^0 \in \mathbb{R}^n$ sei $\underline{r}^0 = A\underline{x}^0 - \underline{f}$.
Wähle $\underline{\widetilde{r}}^0$. Setze $\underline{p}^0 = \underline{r}^0, \underline{\widetilde{p}}^0 = \underline{\widetilde{r}}^0$ und berechne $\varrho_0 = (\underline{r}^0, \underline{\widetilde{r}}^0)$.
Stoppe, falls $\varrho_0 < \varepsilon^2$ mit einer vorgegebenen Fehlergenauigkeit $\varepsilon$ erreicht ist.

Berechne für $k = 0, 1, \ldots, n-2$ :

$$\underline{s}^k = A\underline{p}^k, \underline{\widetilde{s}}^k = A^\top \underline{\widetilde{p}}^k, \; \sigma_k = (\underline{s}^k, \underline{\widetilde{p}}^k), \; \alpha_k = \frac{\varrho_k}{\sigma_k}$$

$$\underline{x}^{k+1} = \underline{x}^k - \alpha_k \underline{p}^k$$

$$\underline{r}^{k+1} = \underline{r}^k - \alpha_k \underline{s}^k$$

$$\underline{\widetilde{r}}^{k+1} = \underline{\widetilde{r}}^k - \alpha_k \underline{\widetilde{s}}^k$$

$$\varrho_{k+1} = (\underline{r}^{k+1}, \underline{\widetilde{r}}^{k+1}),$$

Stoppe, falls $\|\underline{r}^{k+1}\|_2 < \varepsilon \|\underline{r}^0\|_2$ mit einer vorgegebenen Fehlergenauigkeit $\varepsilon$ erreicht ist. Berechne andernfalls die neuen Suchrichtungen

$$\underline{p}^{k+1} = \underline{r}^{k+1} + \beta_k \underline{p}^k, \quad \underline{\widetilde{p}}^{k+1} = \underline{\widetilde{r}}^{k+1} + \beta_k \underline{\widetilde{p}}^k, \quad \beta_k = \frac{\varrho_{k+1}}{\varrho_k}$$

## Iterationsvorschrift für Residuum

$$\underline{r}^{k+1} = \varphi_{k+1}(A)\underline{r}^0 = [\varphi_k(A) - \alpha_k A \psi_k(A)]\,\underline{r}^0, \quad \widetilde{\underline{r}}^{k+1} = \varphi_{k+1}(A^\top)\widetilde{\underline{r}}^0$$

## Iterationsvorschrift für Suchrichtung

$$\underline{p}^{k+1} = \psi_{k+1}(A)\underline{r}^0 = [\varphi_{k+1}(A) + \beta_k \psi_k(A)]\,\underline{r}^0, \quad \widetilde{\underline{p}}^{k+1} = \psi_{k+1}(A^\top)\widetilde{\underline{r}}^0$$

## Parameter

$$\varrho_k = (\underline{r}^k, \widetilde{\underline{r}}^k) = (\varphi_k(A)\underline{r}^0, \varphi_k(A^\top)\widetilde{\underline{r}}^0) = (\varphi_k^2(A)\underline{r}^0, \widetilde{\underline{r}}^0) = (\hat{\underline{r}}^k, \widetilde{\underline{r}}^0)$$

$$\sigma_k = (A\underline{p}^k, \widetilde{\underline{p}}^k) = (A\psi_k(A)\underline{r}^0, \psi_k(A^\top)\widetilde{\underline{r}}^0) = (A\psi_k^2(A)\underline{r}^0, \widetilde{\underline{r}}^0) = (A\hat{\underline{p}}^k, \widetilde{\underline{r}}^0)$$

mit

$$\hat{\underline{r}}^k = \varphi_k^2(A)\underline{r}^0, \qquad \hat{\underline{p}}^k = \psi_k^2(A)\underline{r}^0, \quad \hat{\underline{r}}^0 = \hat{\underline{p}}^0 = \underline{r}^0$$

## modifiziertes Residuum

$$
\begin{aligned}
\underline{\hat{r}}^{k+1} &= \varphi_{k+1}^2(A)\underline{r}^0 = [\varphi_k(A) - \alpha_k A \psi_k(A)]^2 \, \underline{r}^0 \\
&= \left[\varphi_k^2(A) - 2\alpha_k A \varphi_k(A)\psi_k(A) + \alpha_k^2 A^2 \psi_k^2(A)\right] \underline{r}^0 \\
&= \left[\varphi_k^2(A) - 2\alpha_k A \varphi_k(A)[\varphi_k(A) + \beta_{k-1}\psi_{k-1}(A)] + \alpha_k^2 A^2 \psi_k^2(A)\right] \underline{r}^0 \\
&= (I - 2\alpha_k A)\varphi_k^2(A)\underline{r}^0 - 2\alpha_k \beta_{k-1} A \varphi_k(A)\psi_{k-1}(A)\underline{r}^0 + \alpha_k^2 A^2 \psi_k^2(A)\underline{r}^0 \\
&= (I - 2\alpha_k A)\underline{\hat{r}}^k - 2\alpha_k \beta_{k-1} A \underline{\hat{q}}^k + \alpha_k^2 A^2 \underline{\hat{p}}^k
\end{aligned}
$$

mit

$$
\underline{\hat{q}}^k = \varphi_k(A)\psi_{k-1}(A)\underline{r}^0, \quad \underline{q}^0 = \underline{0}
$$

## Rekursionsvorschrift

$$\begin{aligned}
\underline{\hat{q}}^{k+1} &= \varphi_{k+1}(A)\psi_k(A)\underline{r}^0 \\
&= [\varphi_k(A) - \alpha_k A\psi_k(A)]\,\psi_k(A)\underline{r}^0 \\
&= \varphi_k(A)\psi_k(A)\underline{r}^0 - \alpha_k A\psi_k^2(A)\underline{r}^0 \\
&= \varphi_k(A)\,[\varphi_k(A) + \beta_{k-1}\psi_{k-1}(A)]\,\underline{r}^0 - \alpha_k A\psi_k^2(A)\underline{r}^0 \\
&= \varphi_k^2(A)\underline{r}^0 + \beta_{k-1}\varphi_k(A)\psi_{k-1}(A)\underline{r}^0 - \alpha_k A\psi_k^2(A)\underline{r}^0 \\
&= \underline{\hat{r}}^k + \beta_{k-1}\underline{\hat{q}}^k - \alpha_k A\underline{\hat{p}}^k
\end{aligned}$$

## modifzierte Suchrichtung

$$\begin{aligned}
\underline{\hat{p}}^{k+1} &= \psi_{k+1}^2(A)\underline{r}^0 = [\varphi_{k+1}(A) + \beta_k\psi_k(A)]^2\,\underline{r}^0 \\
&= \varphi_{k+1}^2(A)\underline{r}^0 + 2\beta_k\varphi_{k+1}(A)\psi_k(A)\underline{r}^0 + \beta_k^2\psi_k^2(A)\underline{r}^0 \\
&= \underline{\hat{r}}^{k+1} + 2\beta_k\underline{\hat{q}}^{k+1} + \beta_k^2\underline{\hat{p}}^k
\end{aligned}$$

## Residuum

$$\underline{\hat{r}}^{k+1} = \underline{\hat{r}}^{k} - \alpha_k A \left[ 2\underline{\hat{r}}^{k} + 2\beta_{k-1}\underline{\hat{q}}^{k} - \alpha_k A \underline{\hat{p}}^{k} \right] = \underline{\hat{r}}^{k} - \alpha_k A \underline{\bar{w}}^{k}$$

## Näherungslösung

$$\underline{\hat{x}}^{k+1} = \underline{\hat{x}}^{k} - \alpha_k \left[ 2\underline{\hat{r}}^{k} + 2\beta_{k-1}\underline{\hat{q}}^{k} - \alpha_k A \underline{\hat{p}}^{k} \right] = \underline{\hat{x}}^{k} - \alpha_k \underline{\bar{w}}^{k}$$

# CGS Verfahren (Conjugate Gradient Squared)

Für eine beliebig gegebene Startnäherung $\underline{x}^0 \in \mathbb{R}^n$ sei $\underline{r}^0 = A\underline{x}^0 - \underline{f}$.
Wähle $\underline{\widetilde{r}}^0$. Setze $\underline{p}^0 = \underline{r}^0, \underline{q}^0 = \underline{0},\ \beta_{-1} = 0$ und berechne $\varrho_0 = (\underline{r}^0, \underline{\widetilde{r}}^0)$.
Stoppe, falls $\|\underline{r}\|_2 < \varepsilon$ mit einer vorgegebenen Genauigkeit $\varepsilon$ erreicht ist.

Berechne für $k = 0, 1, \ldots, n-2$ :

$$\underline{s}^k = A\underline{p}^k,\ \sigma_k = (\underline{s}^k, \underline{\widetilde{r}}^0),\ \alpha_k = \frac{\varrho_k}{\sigma_k}$$

$$\underline{w}^k = \underline{r}^k + \beta_{k-1}\underline{q}^k,\ \underline{q}^{k+1} = \underline{w}^k - \alpha_k\underline{s}^k,\ \underline{\bar{w}}^k = \underline{q}^{k+1} + \underline{w}^k,\ \underline{\bar{s}}^k = A\underline{\bar{w}}^k$$

$$\underline{x}^{k+1} = \underline{x}^k - \alpha_k\underline{\bar{w}}^k$$

$$\underline{r}^{k+1} = \underline{r}^k - \alpha_k\underline{\bar{s}}^k$$

$$\varrho_{k+1} = (\underline{r}^{k+1}, \underline{\widetilde{r}}^0),$$

Stoppe, falls $\|\underline{r}^{k+1}\|_2 < \varepsilon\|\underline{r}^0\|_2$ mit einer vorgegebenen Genauigkeit $\varepsilon$ erreicht ist. Berechne andernfalls die neuen Suchrichtungen

$$\beta_k = \frac{\varrho_{k+1}}{\varrho_k},\ \underline{p}^{k+1} = \underline{r}^{k+1} + \beta_k(2\underline{q}^{k+1} + \beta_k\underline{p}^k).$$

## CGS Verfahren

$$\underline{\hat{r}}^{k+1} = \varphi_{k+1}^2(A)\underline{r}^0 = \varphi_{k+1}(A)\underline{r}^{k+1}$$

## Polynom

$$\theta_{k+1}(A) = \prod_{\ell=0}^{k}(I - \omega_\ell A) = (I - \omega_k A)\theta_k(A)$$

## Glättung des Konvergenzverhaltens des Residuums

$$\underline{\hat{r}}^{k+1} = \theta_{k+1}(A)\underline{r}^{k+1} = (I - \omega_k A)\theta_k(A)\underline{r}^k = (I - \omega_k A)\underline{\hat{r}}^k$$

## Berechnung des Koeffizienten $\omega_k$

$$\omega_k = \arg\min \|(I - \omega A)\underline{\hat{r}}^k\|_2$$

$$\omega_k = \frac{(A\underline{\hat{r}}^k, \underline{\hat{r}}^k)}{(A\underline{\hat{r}}^k, A\underline{\hat{r}}^k)}$$

## modifiziertes Residuum

$$\begin{aligned}\underline{\hat{r}}^{k+1} &= \theta_{k+1}(A)\underline{r}^{k+1} = (I-\omega_k A)\theta_k(A)\left[\underline{r}^k - \alpha_k A\underline{p}^k\right] \\ &= (I-\omega_k A)\theta_k(A)\underline{r}^k - \alpha_k(I-\omega_k A)A\theta_k(A)\underline{p}^k \\ &= (I-\omega_k A)\underline{\hat{r}}^k - \alpha_k(I-\omega_k A)A\underline{\hat{p}}^k\end{aligned}$$

## modifizierte Suchrichtung

$$\begin{aligned}\underline{\hat{p}}^{k+1} &= \theta_{k+1}(A)\underline{p}^{k+1} = \theta_{k+1}(A)\left[\underline{r}^{k+1} + \beta_k\underline{p}^k\right] \\ &= \theta_{k+1}(A)\underline{r}^{k+1} + \beta_k(I-\omega_k A)\theta_k(A)\underline{p}^k = \underline{\hat{r}}^{k+1} + \beta_k(I-\omega_k A)\underline{\hat{p}}^k\end{aligned}$$

Für die Berechnung von

$$\varrho_{k+1} = (\underline{r}^{k+1}, \underline{\widetilde{r}}^{k+1}) = (\underline{r}^{k+1}, \varphi_{k+1}(A^\top)\underline{\widetilde{r}}^0)$$

folgt

$$\begin{aligned}
\varphi_{k+1}(A^\top)\underline{\widetilde{r}}^0 &= \underline{\widetilde{r}}^{k+1} = \underline{\widetilde{r}}^k - \alpha_k A^\top \underline{\widetilde{p}}^k = \underline{\widetilde{r}}^k - \alpha_k A^\top [\underline{\widetilde{r}}^k + \beta_{k-1}\underline{\widetilde{p}}^{k-1}] \\
&= -\alpha_k A^\top \underline{\widetilde{r}}^k + \underline{\widetilde{r}}^k - \alpha_k \beta_{k-1} A^\top \underline{\widetilde{p}}^{k-1} \\
&= -\alpha_k A^\top \varphi_k(A^\top)\underline{\widetilde{r}}^0 + \varphi_k(A^\top)\underline{\widetilde{r}}^0 - \alpha_k\beta_{k-1}A^\top \psi_{k-1}(A^\top)\underline{\widetilde{r}}^0 \\
&= -\alpha_k A^\top \varphi_k(A^\top)\underline{\widetilde{r}}^0 + \phi_k(A^\top)\underline{\widetilde{r}}^0
\end{aligned}$$

rekursive Anwendung

$$\varphi_{k+1}(A^\top)\underline{\widetilde{r}}^0 = \prod_{\ell=0}^{k}(-\alpha_\ell)(A^\top)^{k+1}\underline{\widetilde{r}}^0 + \widetilde{\phi}_k(A^\top)\underline{\widetilde{r}}^0$$

Mit Orthogonalität ist

$$\varrho_{k+1} = (\underline{r}^{k+1}, \varphi_{k+1}(A^\top)\underline{\widetilde{r}}^0) = \prod_{\ell=0}^{k}(-\alpha_\ell)(\underline{r}^{k+1}, (A^\top)^{k+1}\underline{\widetilde{r}}^0).$$

Andererseits ist

$$\hat{\varrho}_{k+1} = (\hat{\underline{r}}^{k+1}, \tilde{\underline{r}}^0) = (\theta_{k+1}(A)\underline{r}^{k+1}, \tilde{\underline{r}}^0) = (\underline{r}^{k+1}, \theta_{k+1}(A^\top)\tilde{\underline{r}}^0)$$

und

$$\theta_{k+1}(A^\top)\tilde{\underline{r}}^0 = \prod_{\ell=0}^{k}(I - \omega_\ell A^\top)\tilde{\underline{r}}^0 = \prod_{\ell=0}^{k}(-\omega_\ell)(A^\top)^{k+1}\tilde{\underline{r}}^0 + \hat{\phi}_k(A^\top)\tilde{\underline{r}}^0$$

Mit Orthogonalität ist

$$\bar{\varrho}_{k+1} = (\underline{r}^{k+1}, \theta_{k+1}(A^\top)\tilde{\underline{r}}^0) = \prod_{\ell=0}^{k}(-\omega_\ell)(\underline{r}^{k+1}, (A^\top)^{k+1}\tilde{\underline{r}}^0).$$

Daraus folgt

$$\varrho_{k+1} = \left[\prod_{\ell=0}^{k}\frac{\alpha_\ell}{\omega_\ell}\right]\hat{\varrho}_{k+1}$$

und somit

$$\beta_k = \frac{\varrho_{k+1}}{\varrho_k} = \frac{\hat{\varrho}_{k+1}}{\hat{\varrho}_k}\frac{\alpha_k}{\omega_k}$$

## Berechnung von $\sigma_k$

$$\begin{aligned}\sigma_k &= (A\underline{p}^k, \underline{\widetilde{p}}^k) = (A\underline{p}^k, \psi_k(A^\top)\underline{\widetilde{r}}^0) = (A\underline{p}^k, [\varphi_k(A^\top) + \beta_{k-1}\psi_{k-1}(A^\top)]\underline{\widetilde{r}}^0)\\ &= (A\underline{p}^k, \varphi_k(A^\top)\underline{\widetilde{r}}^0) = \prod_{\ell=0}^{k-1}(-\alpha_\ell)(A\underline{p}^k, (A^\top)^k\underline{\widetilde{r}}^0)\end{aligned}$$

## Andererseits ist

$$\begin{aligned}\hat{\sigma}_k &= (A\underline{\hat{p}}^k, \underline{\widetilde{r}}^0) = (A\theta_k(A)\underline{p}^k, \underline{\widetilde{r}}^0) = (A\underline{p}^k, \theta_k(A^\top)\underline{\widetilde{r}}^0)\\ &= (A\underline{p}^k, \prod_{\ell=0}^{k-1}(I - \omega_\ell A^\top)\underline{\widetilde{r}}^0) = \prod_{\ell=0}^{k-1}(-\omega_\ell)(A\underline{p}^k, (A^\top)^k\underline{\widetilde{r}}^0)\end{aligned}$$

## Somit gilt

$$\sigma_k = \hat{\sigma}_k \prod_{\ell=0}^{k-1} \frac{\alpha_\ell}{\omega_\ell}.$$

Berechnung von $\alpha_k$

$$\alpha_k = \frac{\varrho_k}{\sigma_k} = \frac{\hat{\varrho}_k}{\hat{\sigma}_k}$$

modifiziertes Residuum

$$\underline{\hat{r}}^{k+1} = (I - \omega_k A)\underline{\hat{r}}^k = \underline{\hat{r}}^k - \omega_k A \underline{\hat{r}}^k$$

modifizierte Näherungslösung

$$\underline{\hat{x}}^{k+1} = \underline{\hat{x}}^k - \omega_k \underline{\hat{r}}^k$$

# Stabilisiertes Gradientenverfahren biorthogonaler Richtungen (BiCGStab)

Für eine beliebig gegebene Startnäherung $\underline{x}^0 \in \mathbb{R}^n$ sei $\underline{r}^0 = A\underline{x}^0 - \underline{f}$.
Wähle $\underline{\widetilde{r}}^0 = \underline{r}^0$. Setze $\underline{p}^0 = \underline{r}^0$, und berechne $\varrho_0 = (\underline{r}^0, \underline{\widetilde{r}}^0)$.
Stoppe, falls $\|\underline{r}\|_2 < \varepsilon$ mit einer vorgegebenen Genauigkeit $\varepsilon$ erreicht ist.

Berechne für $k = 0, 1, \ldots, n-2$ :

$\underline{s}^k = A\underline{p}^k,\ \sigma_k = (\underline{s}^k, \underline{\widetilde{r}}^0)$. Stoppe, falls $\sigma_k = 0$.

$$\alpha_k = \frac{\varrho_k}{\sigma_k}, \quad \underline{w}^k = \underline{r}^k - \alpha_k \underline{s}^k, \quad \underline{v}^k = A\underline{w}^k, \quad \omega_k = \frac{(\underline{v}^k, \underline{w}^k)}{(\underline{v}^k, \underline{v}^k)}$$

$$\underline{x}^{k+1} = \underline{x}^k - \alpha_k \underline{p}^k - \omega_k \underline{w}^k$$

$$\underline{r}^{k+1} = \underline{r}^k - \alpha_k \underline{s}^k - \omega_k \underline{v}^k$$

$$\varrho_{k+1} = (\underline{r}^{k+1}, \underline{\widetilde{r}}^0),$$

Stoppe, falls $\|\underline{r}^{k+1}\|_2 < \varepsilon \|\underline{r}^0\|_2$ mit einer vorgegebenen Genauigkeit $\varepsilon$ erreicht ist. Berechne andernfalls die neuen Suchrichtungen

$$\beta_k = \frac{\varrho_{k+1}}{\varrho_k} \frac{\alpha_k}{\omega_k}, \quad \underline{p}^{k+1} = \underline{r}^{k+1} + \beta_k(\underline{p}^k - \omega_k \underline{s}^k).$$