Technische Universität Graz
Institut für Angewandte Mathematik
Univ.–Prof. Dr. Olaf Steinbach
Dipl.–Ing. Richard Löscher

SS 2022
Blatt 2
23.3.2022

Space–Time Methods

For a bounded domain $\Omega \subset \mathbb{R}^m$ we consider a locally quasi–uniform and admissible decomposition into shape regular simplicial elements τ_ℓ of local mesh size h_ℓ. Let $V_h = \text{span}\{\varphi_k\}_{k=1}^M$ be the related space of piecewise linear and continuous basis functions φ_k with support ω_k which covers all finite elements τ_ℓ with x_k as a node, i.e., $\ell \in I(k)$. Hence we can define, for all nodes x_k, the local mesh size

$$\hat{h}_k := \frac{1}{\# I(k)} \sum_{\ell \in I(k)} h_\ell .$$

For the restriction $V_h(\omega_k)$ of the global finite element space onto ω_k we define $Q_h^k : L^2(\omega_k) \to V_h(\omega_k)$ as the local L^2 projection satisfying

$$\langle Q_h^k u, v_h \rangle_{L^2(\omega_k)} = \langle u, v_h \rangle_{L^2(\omega_k)} \quad \text{for all } v_h \in V_h(\omega_k).$$

6. Prove the error estimate, $s = 1, 2$,

$$\| u - Q_h^k u \|_{L^2(\omega_k)} \leq c \, \hat{h}_k^s \, |u|_{H^s(\omega_k)} \quad \text{for all } u \in H^s(\omega_k).$$

7. Prove the stability estimate

$$\| Q_h^k u \|_{H^1(\omega_k)} \leq c \, \|u\|_{H^1(\omega_k)} \, .$$

Next we define the quasi interpolation or Clement operator

$$P_h u(x) = \sum_{k=1}^M (Q_h^k u)(x_k) \, \varphi_k(x) \, .$$

8. Prove that $P_h : V_h \to V_h$ is a projection.

9. Prove the local error estimates, $s = 1, 2$,

$$\| u - P_h u \|_{L^2(\tau_\ell)} \leq c \sum_{x_k \in \tau_\ell} \hat{h}_k^s \, |u|_{H^s(\omega_k)} .$$

10. Prove the stability estimate

$$\| P_h u \|_{H^1(\Omega)} \leq c \, \|u\|_{H^1(\Omega)} .$$