

Space–Time Methods

Let

$$Y := L^2(0, T; H_0^1(\Omega)), \quad X := \left\{ u \in Y : \partial_t u \in Y', \ u(0) = 0 \text{ in } \Omega \right\},$$

and

$$b(u, v) := \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} \quad \text{for } u \in X, v \in Y.$$

Define

$$\mathcal{B}(u, p; q, v) := \langle \nabla_x p, \nabla_x v \rangle_{L^2(Q)} + b(u, v) + b(q, p) \quad \text{for } u, q \in X, p, v \in Y,$$

and

$$\|(u, p)\|_{X \times Y} := \sqrt{\|u\|_X^2 + \|p\|_Y^2}.$$

11. Prove the boundedness

$$|\mathcal{B}(u, p; q, v)| \leq c_2^B \|(u, p)\|_{X \times Y} \|(q, v)\|_{X \times Y}$$

with a suitable positive constant c_2^B .

12. Prove the inf–sup stability condition

$$c_1^B \|(u, p)\|_{X \times Y} \leq \sup_{(0,0) \neq (q,v) \in X \times Y} \frac{\mathcal{B}(u, p; q, v)}{\|(q, v)\|_{X \times Y}} \quad \text{for all } (u, p) \in X \times Y.$$

Hints: For $u \in X$ define $w \in Y$ such that $\|\partial_t u\|_{Y'} = \|w\|_Y$. For $p \in Y$ and for a suitable chosen $\alpha \in \mathbb{R}_+$ define $\bar{v} := u + w + \alpha p \in Y$ and $\bar{q} := -\alpha u \in X$. Use

$$ab \leq \frac{1}{2\gamma} a^2 + \frac{1}{2} \gamma b^2, \quad \gamma > 0$$

and define γ and α appropriately.

Instead of the heat equation we now consider the diffusion–convection equation

$$-\Delta u(x) + \underline{b}(x) \cdot \nabla u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma := \partial\Omega,$$

where \underline{b} is a given bounded velocity field satisfying $\operatorname{div} \underline{b} = 0$. Define

$$Y := H_0^1(\Omega), \quad X := \left\{ u \in Y : \underline{b} \cdot \nabla u \in Y' \right\}.$$

13. Derive a variational formulation to determine a solution $u \in X$ and prove a related inf–sup stability condition.

14. Derive a related least–squares formulation and prove ellipticity of the associated Schur complement operator which is of the form $S = B'A^{-1}B$.

15. Introduce conforming finite element spaces $X_H \subset X$ and $Y_h \subset Y$ of piecewise linear basis functions, and define an appropriate approximation $\tilde{S}u = B'p_h$ by an approximate solution of the operator equation $Ap = Bu$. Prove discrete ellipticity of \tilde{S} when considering an appropriate choice of the mesh sizes h and H .

Hint: Recall the regularity of piecewise linear finite element functions.