Space–Time Methods

15. Introduce conforming finite element spaces $X_H \subset X$ and $Y_h \subset Y$ of piecewise linear basis functions, and define an appropriate approximation $\widetilde{S}u = B'p_h$ by an approximate solution of the operator equation $Ap = Bu$. Prove discrete ellipticity of $\widetilde{S}$ when considering an appropriate choice of the mesh sizes h and H.

Hint: Recall the regularity of piecewise linear finite element functions.

16. For given $u \in H^{1/2}_{0,}(0,T)$, find $w \in H^{1/2}_{,0}(0,T)$ as solution of the variational problem

$$\langle w, z \rangle_{H^{1/2}_{,0}(0,T)} = \langle \partial_t u, z \rangle_{(0,T)} \quad \text{for all } z \in H^{1/2}_{,0}(0,T).$$

17. Let $u \in H^1_{0,}(0,T)$ be given. Determine the solution $w \in H^1_{,0}(0,T)$ of the variational problem

$$\int_0^T \partial_t w(t) \partial_t v(t)\, dt = - \int_0^T \partial_t u(t) \partial_t v(t)\, dt \quad \text{for all } v \in H^1_{,0}(0,T)$$

by using the ansatz

$$w(t) = \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) .$$

18. Find a closed representation of the solution w as computed in **17.** in terms of u.