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Space–Time Methods

Let $X \subset H \subset X^*$ and $Y \subset H \subset Y^*$ be Gelfand triples of Hilbert spaces, where X^*, Y^* are the duals of X, Y with respect to H. Let $A : Y \to Y^*$ and $B : X \to Y^*$ be bounded and invertible linear operators. In particular we assume that A is elliptic in Y. We consider the minimization problem to find the minimizer $(u, z) \in X \times Y^*$ of the functional

$$\mathcal{J}(u,z) = \frac{1}{2}\,\|u-\overline{u}\|_H^2 + \frac{1}{2}\,\varrho\,\|z\|_{Y^*}^2 \quad \text{subject to } Bu = z,$$

when $\overline{u} \in H$ is given, and $\varrho \in \mathbb{R}_+$ is some regularization parameter, and

$$\|z\|_{Y^*} = \sqrt{\langle A^{-1}z, z\rangle_H} \quad \text{for } z \in Y^*.$$

24. The solution of the state equation $Bu = z$ defines the operator $u = \mathcal{S}z$ with $\mathcal{S} : H \to X \subset H$. Write the reduced functional $\mathcal{J}(\mathcal{S}z, z) =: \widetilde{\mathcal{J}}(z)$ and determine the minimizer z of the reduced functional. Write the resulting optimality system, namely the primal problem, the gradient equation, and the adjoint problem involving the adjoint operator $\mathcal{S}^*$.

25. Consider $H = L^2(\Omega)$, $X = Y = H_0^1(\Omega)$, and $A = B = -\Delta$, i.e., consider the minimization problem subject to the Dirichlet boundary value problem

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \partial\Omega.$$

Derive the boundary value problem for the adjoint operator $\varphi = \mathcal{S}^*\psi$, write the resulting optimality system, and eliminate the control z and the adjoint variable.

26. Consider $H = L^2(\Omega)$, $X = H_0^1(\Omega)$, $Y = L^2(\Omega)$, $A = I$, $B = -\Delta$, i.e., consider the minimization problem subject to the Dirichlet boundary value problem

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \partial\Omega.$$

Derive the boundary value problem for the adjoint operator $\varphi = \mathcal{S}^*\psi$, write the resulting optimality system, and eliminate the control z and the adjoint variable.

27. Let $Q = \Omega \times (0,T)$. Consider $H = L^2(Q)$, $Y = L^2(0,T; H_0^1(\Omega))$, $X = \{u \in Y : \partial_t \in Y^*, u(0) = 0\}$, and $A = -\Delta_x$, $B = \partial_t - \Delta_x$, i.e., consider the minimization problem subject to the heat equation

$$\partial_t u(x,t) - \Delta_x u(x,t) = z(x,t) \quad \text{for } (x,t) \in Q, \quad u(x,t) = 0 \quad \text{for } (x,t) \in \Sigma, \quad u(x,0) = 0 \quad \text{for } x \in \Omega.$$

Derive the boundary value problem for the adjoint operator $\varphi = \mathcal{S}^*\psi$, write the resulting optimality system, and eliminate the control z and the adjoint variable.

28. Let $Q = \Omega \times (0,T)$. Consider $H = L^2(Q)$, $Y = L^2(Q)$, $X = H^{1,1}_{0;0,}(Q)$, and $A = I$, $B = \partial_{tt} - \Delta_x$, i.e., consider the minimization problem subject to the wave equation

$$\partial_{tt} u - \Delta_x u = z \quad \text{in } Q, \quad u = 0 \quad \text{on } \Sigma, \quad u(0) = \partial_t u(t)_{|t=0} = 0 \quad \text{in } \Omega.$$

Derive the boundary value problem for the adjoint operator $\varphi = \mathcal{S}^*\psi$, write the resulting optimality system, and eliminate the control z and the adjoint variable.