Technische Universität Graz
Institut für Angewandte Mathematik
Assoc. Prof. Dr. Günther Of

Technische Numerik 2

Exercise sheet 3, April 29, 2020

Exercise 7: Show that

$$u(x) = \begin{cases} x & \text{for } x \in (-1,0] \\ x^2 & \text{for } x \in (0,1) \end{cases}$$

is in $W_2^1(-1,1)$ but not in $W_2^2(-1,1)$.

Exercise 8: For which $\alpha \in \mathbb{R}$ is

$$u(x) = x^\alpha$$

in $L_2(0,1)$? For which $\alpha \in \mathbb{R}$ is $u' \in L_2(\Omega)$?

Exercise 9:

a) Determine the week derivative of

$$u(x) := |x| - 1$$

und show that $u \in W_2^1(-1,1)$.

b) Show that u satisfies

$$\int_{-1}^{1} u'(x)v'(x)\, dx = -2v(0)$$

for all $v \in \{v \in W_2^1(-1,1) : v(-1) = v(1) = 0\}$ and that

$$u(-1) = u(1) = 0.$$

Exercise 10: Consider $\Omega = (-1,1)$,

$$g(x) := \begin{cases} -1 & \text{for } -1 < x < 0 \\ 5 & \text{for } x = 0 \\ 1 & \text{for } 0 < x < 1 \end{cases}, \qquad u(x) := \int_{-1}^{x}\int_{-1}^{t} g(s)\, ds\, dt \quad \text{for } x \in \Omega$$

and

$$f(x) := u(x) - g(x) \quad \text{for } x \in \Omega.$$

a) Show that u is a weak solution of the boundary value problem

$$\begin{aligned} -u'' + u &= f \quad \text{in } \Omega \\ u(-1) &= 0, \\ u(1) &= -1, \end{aligned}$$

i.e., there holds true

$$\begin{aligned} \int_{-1}^{1} u'(x)v'(x)\, dx + \int_{-1}^{1} u(x)v(x)\, dx &= \int_{-1}^{1} f(x)v(x)\, dx, \\ u(-1) &= 0, \\ u(1) &= -1, \end{aligned}$$

for all $v \in \{v \in W_2^1(-1,1) : v(-1) = v(1) = 0\}$

b) Show that $u \in W_2^1(\Omega)$ but not in $C^2(\Omega)$, i.e., u is not a classical solution.

Exercise 11: Consider the functional $\mathcal{J} \colon \mathbb{R}^M \to \mathbb{R}$ defined by

$$\mathcal{J}(\underline{v}) = \frac{1}{2}\underline{v}^\top A\underline{v} - \underline{v}^\top \underline{f} = \frac{1}{2}\sum_{i=1}^{M}\sum_{k=1}^{M} v_i A[i,k] v_k - \sum_{i=1}^{M} v_i f_i$$

for a symmetric and positive definite matrix $A \in \mathbb{R}^{M\times M}$ and a vector $\underline{f} \in \mathbb{R}^M$. Compute the derivative

$$\frac{\partial}{\partial v_j}\mathcal{J}(\underline{v}),$$

and show that

$$\underline{u} \in \mathbb{R}^M \text{ minimizes } \mathcal{J}(\underline{v}) \quad \Leftrightarrow \quad A\underline{u} = \underline{f}.$$