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Numerics and Simulation
Elective subject mathematics (DDM)

Exercise sheet 3, April 6, 2022

Exercise 11: Set up the Steklov Poincare interface equation for the Neumann boundary value problem

$$-\Delta u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega$$

for two subdomains. Consider this equation in the space

$$\widehat{H}^{1/2}(\Sigma) := \{\mu \in H^{1/2}(\Sigma) : \int_\Gamma \mu ds_x = 0\}$$

where $\Sigma = \partial\Omega \cup \Gamma$. Show unique solvability of the variational formulation. Show that the solution satisfies the variational formulation even for test functions in $H^{1/2}(\Sigma)$ (Take the solvability condition of the Neumann boundary value problem into account).

Exercise 12: Formulate the discrete version of the Robin Robin method for $\gamma_1 = \gamma_2 = \gamma > 0$ using the flux approximation (1.12). Rewrite the system in terms of the Schur complement matrices $S^{(i)}$ and interpret the resulting system als forward block Gauss Seidel method.

Exercise 13: Show that the Dirichlet Neumann method with automatic selection of the relaxation parameter is equivalent to the original method, if the same relaxation parameters are chosen. Hint: Show the following statements:

- u_2^n is solution of the original "Neumann" bvp.
- $\tilde{u}_2^n$ is the harmonic extension of $\gamma_0^{\text{int}} u_2^n$.
- $\tilde{u}_1^n$ is the harmonic extension of $\gamma_0^{\text{int}} u_1^n$.
- u_1^n is solution of the original Dirichlet bvp.

Exercise 14: Consider the classical alternating Schwarz method for the model problem of Section 1.10. Show that the convergence rate is

$$\varrho(k, \delta) = \exp(-\lambda(k)\delta)$$

with $\lambda(k) = \sqrt{\eta + k^2}$. What happens in the case of no overlap?

Exercise 15: Consider the optimized Schwarz method for the Dirichlet boundary value problem and a non-overlapping domain decomposition

$$\begin{pmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} + S^{(2)} \end{pmatrix} \begin{pmatrix} \underline{u}_{I,1}^{n+1} \\ \underline{u}_{\Gamma,1}^{n+1} \end{pmatrix} = \begin{pmatrix} \underline{f}_I^{(1)} \\ \underline{f}_\Gamma + S^{(2)} \underline{u}_{\Gamma,2}^n - A_{\Gamma I}^{(2)} \underline{u}_{I,2}^n - A_{\Gamma\Gamma}^{(2)} \underline{u}_{\Gamma,2}^n \end{pmatrix}$$
$$\begin{pmatrix} A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} + S^{(1)} \end{pmatrix} \begin{pmatrix} \underline{u}_{I,2}^{n+1} \\ \underline{u}_{\Gamma,2}^{n+1} \end{pmatrix} = \begin{pmatrix} \underline{f}_I^{(2)} \\ \underline{f}_\Gamma + S^{(1)} \underline{u}_{\Gamma,1}^n - A_{\Gamma I}^{(1)} \underline{u}_{I,1}^n - A_{\Gamma\Gamma}^{(1)} \underline{u}_{\Gamma,1}^n \end{pmatrix}.$$

Check if the two local problems are uniquely solvable. Show that this algorithm converges in two steps. Show that the second rows correspond to the transmission conditions

$$\begin{aligned}
\frac{\partial u_1^{n+1}}{\partial n_1} + S_2 u_1^{n+1} &= -\frac{\partial u_2^n}{\partial n_2} + S_2 u_2^n \\
\frac{\partial u_2^{n+1}}{\partial n_2} + S_1 u_2^{n+1} &= -\frac{\partial u_1^n}{\partial n_1} + S_1 u_1^n
\end{aligned}$$

of the optimal Schwarz method.