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Numerics and Simulation
Elective subject mathematics (DDM)

Exercise sheet 6, June 1, 2022

Exercise 23: Show that the discrete version of the alternating Schwarz method from the first lecture can be written as a multiplicative Schwarz method and the related error propagation operator, respectively:

$$u^{n+1} - u = (I - P_2)(I - P_1)(u^n - u).$$

Exercise 24: Prove that for the multiplicative Schwarz operator

$$P_{mu} = I - (I - P_N) \cdots (I - P_0) = C_{mu}^{-1} A$$

the preconditioning $\underline{y} = C_{mu}^{-1} \underline{x}$ can be realized by

$\underline{y} = R_0^\top \tilde{A}_0^{-1} R_0 \underline{x}$
for $i = 1, \ldots, N$ **do**
  $\underline{y} = \underline{y} + R_i^\top \tilde{A}_i^{-1} R_i (\underline{x} - A\underline{y})$
end for

Exercise 25: Find a realization of the preconditioning A_{h1}^{-1} of the hybrid Schwarz method

$$P_{h1} = P_0 + (I - P_0) \left(\sum_{i=1}^{N} P_i \right) (I - P_0) = A_{h1}^{-1} A.$$

Exercise 26: Prove with the assumptions 2 and 3 from the lecture

$$\begin{aligned} a(P_i u, v) &\leq a(P_i u, u)^{1/2} a(P_i v, v)^{1/2}, \\ a(P_i u, P_j v) &\leq \omega \varepsilon_{ij} a(P_i u, u)^{1/2} a(P_j v, v)^{1/2} \end{aligned}$$

for $i, j = 0, 1, \ldots, N$ and for all $u, v \in V$.

Exercise 27: Prove that there holds for the error propagation operator E_N of the multiplicative Schwarz method:

$$\|E_{N+1}\|_a \leq \|E_N\|_a,$$

where E_{N+1} is the modified operator with an additional subspace V_{N+1} satisfying Assumption 3 with $\omega \in (0, 2)$.

Exercise 28: The eigenvectors $\underline{v}_\ell$ of the one-dimensional FE stiffness matrix

$$A_h = \frac{1}{h} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(M-1)\times(M-1)}$$

are given by

$$\underline{v}_\ell[i] = \sqrt{2} \sin\left(\frac{i\ell\pi}{M}\right)$$

with eigenvalues $\lambda_\ell = 4h^{-1} \sin^2\left(\frac{\ell\pi}{2M}\right)$.

a) Show that, there holds a representation $\underline{z}^k = S_\omega^k \underline{z}^0$ for the error $\underline{z}^k = \underline{u} - \underline{u}^k$ in the damped Jacobi method.

b) Find the damping parameter ω such that S_ω damps the high frequency error parts $(M/2 \leq \ell \leq M-1)$ of the error $\underline{z}^0 = \sum_{\ell=1}^{M-1} \alpha_\ell \underline{v}_\ell$ down maximally.