Calculus of variations

Exercise 10: Let $\mathcal{C}([0,1])$ be the space of continuous functions with norm $\|u\|_\infty = \sup_{x\in[0,1]} |u(x)|$. Furthermore, let $F : \mathcal{C}([0,1]) \to \mathbb{C}$ be the mapping

$$F(u) = \int_0^1 (u(x)-1)^2 dx, \qquad u \in \mathcal{C}([0,1]).$$

a) Calculate for every $u_0 \in \mathcal{C}([0,1])$ the Gâteaux derivative $F'(u_0)$.

b) Is F Fréchet differentiable?

Exercise 11: Let X be a normed space, $U \subseteq X$ open and $F : U \to \mathbb{R}$ Gâteaux differentiable in $u_0 \in U$. Prove that

$$F \text{ is Fréchet differentiable in } u_0$$
$$\Updownarrow$$
$$(F'(u_0))(h) = \lim_{t\to 0} \tfrac{F(u_0+th)-F(u_0)}{t} \text{ is uniform for all } \|h\| \leq 1$$

Exercise 12: Let X be a normed space. Prove, that every linear functional $F : X \to \mathbb{C}$ is Fréchet differentiable.

Exercise 13: Let $a < b \in \mathbb{R}$ and $L : [a,b]\times\mathbb{R}^{n+1} \to \mathbb{R}$ sufficiently often continuously differentiable. Consider the functional $F : \mathcal{C}^n([a,b]) \to \mathbb{R}$ defined as

$$F(u) = \int_a^b L\big(x, u(x), \dots, u^{(n)}(x)\big) dx, \qquad u \in \mathcal{C}^n([a,b]). \tag{1}$$

Calculate $\delta F(u;h)$ and $\delta^2 F(u;h)$ for every $u, h \in \mathcal{C}^n([a,b])$.

Exercise 14: Let F be the functional in (1) and $u \in \mathcal{C}^n([a,b])$ fixed. Prove that if $\delta F(u;h) = 0$ for every $h \in \mathcal{C}^n([a,b])$, then

$$\sum_{k=0}^n (-1)^k \frac{d^k}{dx^k} L_{u_k}\big(x, u(x), \dots, u^{(n)}(x)\big) = 0, \qquad x \in [a,b], \tag{2}$$

where we used the notation L_{u_k} for the partial derivative $L_{u_k}(x, u_0, \dots, u_n) = \frac{\partial}{\partial u_k} L(x, u_0, \dots, u_n)$.

Exercise 15: Consider the minimization problem

$$F(u) = \int_0^1 (u'(x)^2 + u(x)) dx = \min!, \qquad u \in \mathcal{C}^2([0,1]),\ u(0) = u(1) = 0.$$

a) Derive a differential equation for the solution u using the Euler Lagrange equation (2).

b) Find the solution u.

c) Use the second variation $\delta^2 F(u;h)$ to show that this solution is indeed a local minimum.
Hint: Use the Poincaré inequality.

Exercise 16: For $x_0, y_0 > 0$ consider the problem of the Brachiostochrone

$$F(u) = \int_0^{x_0} \sqrt{\frac{1+u'(x)^2}{-u(x)}} dx = \min!, \qquad u \in \mathcal{C}^2([0,x_0]),\ u(0)=0,\ u(x_0) = -y_0.$$

a) Derive a differential equation for the solution u using the Euler Lagrange equation (2).

b) Show that the solution (in parameter form) is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c \begin{pmatrix} t - \sin(t) \\ \cos(t) - 1 \end{pmatrix}$, $t \in [0, t_0]$.

c) How do the free parameters c and t_0 depend on the coefficients x_0 and y_0.

Exercise 17: Let $G \subseteq \mathbb{R}^n$ be open and bounded and $L : \overline{G} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ sufficiently often continuously differentiable. Consider the functional $F : \mathcal{C}^1(\overline{G}) \to \mathbb{R}$ defined as

$$F(u) = \int_G L\big(x, u(x), \nabla u(x)\big) dx, \qquad u \in \mathcal{C}^1(\overline{G}). \tag{3}$$

Calculate $\delta F(u;h)$ and $\delta^2 F(u;h)$ for every $u, h \in \mathcal{C}^1(\overline{G})$.

Exercise 18: Let F be the functional in (3) and $u \in \mathcal{C}^1(\overline{G})$ fixed. Prove that if $\delta F(u;h) = 0$ for every $h \in \mathcal{C}^1(\overline{G})$, then

$$\sum_{k=1}^{n} \frac{d}{dx_k} L_{y_k}\big(x, u(x), \nabla u(x)\big) = L_u\big(x, u(x), \nabla u(x)\big), \qquad x \in \overline{G}, \tag{4}$$

where we used the notation $L_u(x,u,y) = \frac{\partial}{\partial u} L(x,u,y)$ and $L_{y_k} = \frac{\partial}{\partial y_k} L(x,u,y_1,\ldots,y_n)$.

Exercise 19: Let $G \subseteq \mathbb{R}^2$ be open and bounded, $f \in \mathcal{C}^1(\overline{G})$, $F \in \mathcal{C}^2(\mathbb{R})$. Derive the Euler-Lagrange equation (4) for the functional

$$F(u) = \int_G \Big(F\big(|\nabla u(x)|^2\big) - 2f(x)u(x)\Big) dx.$$

Exercise 20: Let X be a real Banach space, $\alpha \in \mathbb{R} \setminus \{0\}$, $F : X \to \mathbb{R}$ be Gâteaux differentiable and $G : X \to \mathbb{R}$ be a linear functional. If u_0 is a solution of the minimization problem

$$F(u) = \min!, \qquad u \in X \text{ with } G(u) = \alpha,$$

then there exists some $\lambda \in \mathbb{R}$, such that

$$F'(u_0) + \lambda G'(u_0) = 0.$$

Exercise 21: For some fixed $c \in (0, \frac{\pi}{2})$ consider the minimization problem

$$\int_0^2 \sqrt{1+u'(x)^2} dx = \min!, \qquad u \in \mathcal{C}_0^1([0,2]) \text{ with } \int_0^2 u(x) dx = c.$$

a) Give a geometric interpretation.

b) Find the solution.

c) What is the interpretation of the upper bound $\frac{\pi}{2}$ of the constant c.