Exercise 22: Let X be a Hilbert space and $(x_n)_n \in X$ an orthonormal system. Prove that

a) $(x_n)_n$ does not converge (and neither does any of its subsequences).

b) $(x_n)_n$ weakly converges to zero.

Exercise 23: Let X be a reflexive Banach space and X' its dual space. Prove that the weak convergence $f = \operatorname*{w-lim}_{n\to\infty} f_n$ in X' is equivalent to the pointwise limit

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in X.$$

Exercise 24: Let X be a reflexive Banach space. Prove that every closed linear subspace $U \subseteq X$ is reflexive as well.

Hint: Use the following two consequence of the Hahn-Banach theorem:

- For every $x_0 \in X \setminus U$ there exists some $f \in V'$ with $f|_U = 0$ and $f(x_0) \neq 0$.
- For every $g \in U'$ there exists some $f \in V'$ with $f|_U = g$.

Exercise 25: Let X, Y be reflexive Banach spaces. Prove that the product space $X \times Y$ equipped with the norm $\|(x,y)\| := \|x\| + \|y\|$ is reflexive as well.

Exercise 26: Show that for any $p \in (1,\infty)$ the sequence space

$$l^p = \Big\{ (x_n)_n \in \mathbb{C} \,\Big|\, \sum_{n\in\mathbb{N}} |x_n|^p < \infty \Big\} \quad \text{with norm} \quad \|(x_n)_n\|_{l^p} = \Big(\sum_{n\in\mathbb{N}} |x_n|^p \Big)^{\frac{1}{p}},$$

is reflexive.

Hint: Show that for $q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, the mapping $\varphi : l^p \to (l^q)'$ defined by

$$(\varphi(x_n)_n)(y_n)_n := \sum_{n\in\mathbb{N}} x_n y_n, \qquad (x_n)_n \in l^p,\ (y_n)_n \in l^q,$$

is bounded and surjective. For the Surjectivity choose $x_n := f((\delta_{nk})_k)$ for every $f \in (l^q)'$.

Exercise 27: Let X be a Banach space and $f : X \to \mathbb{R}$ a mapping. Prove that the following assertions are equivalent.

(i) f is lower semicontiuous.

(ii) For every $x \in X$ and every $\varepsilon > 0$ there exists some $\delta > 0$, such that

$$f(y) > f(x) - \varepsilon, \qquad y \in B_\delta(x).$$

(iii) For every convergent sequence $x = \lim_{n\to\infty} x_n$ one has

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Exercise 28: Determine all values $\alpha, \beta, \gamma \in \mathbb{R}$ for which the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \begin{cases} \alpha, & \text{if } x < 0, \\ \beta, & \text{if } x = 0, \\ \gamma, & \text{if } x > 0, \end{cases}$$

is lower semi continuous.

Exercise 29: Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open. Prove that for every $f \in L^2(\Omega)$ the minimization problem

$$F(u) = \int_\Omega (|\nabla u|^2 + fu) dx = \min!, \qquad u \in \mathring{W}_1^2(\Omega) \text{ with } u \geq 0 \text{ almost everywhere,}$$

has a unique solution.

Exercise 30: Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open and bounded, $f, g \in W_1^2(\Omega)$ and $\alpha \in \mathbb{R}$. Prove that the minimization problem

$$F(u) = \int_\Omega |\nabla(u - f)|^2 dx = \min!, \qquad u - g \in \mathring{W}_1^2(\Omega) \text{ and } \int_\Omega u \, dx = \alpha,$$

has a unique solution.

Exercise 31: Let X be a reflexive Banach space and $A, B \subseteq X$ closed, convex, with $A \cap B = \emptyset$ and either A or B bounded. Show that in this case A and B have positive distance

$$\text{dist}(A, B) := \inf_{x \in A, y \in B} \|x - y\| > 0.$$