3 Symmetric and self-adjoint operators

In this chapter $\mathcal{H}$ is always a Hilbert space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with scalar product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$.

Definition 3.1. Let S be a densely defined operator in $\mathcal{H}$, i.e. $\overline{\operatorname{dom} S} = \mathcal{H}$. Then the *adjoint operator* S^* of S is defined by

$$\operatorname{dom} S^* = \{g \in \mathcal{H} : \exists g' \in \mathcal{H} : (Sf, g) = (f, g')\,\forall f \in \operatorname{dom} S\},$$
$$S^* g = g'.$$

$\hookrightarrow (Sf, g) = (f, S^* g) \ \forall f \in \operatorname{dom} S$

In the following $\mathcal{H} \times \mathcal{H}$ is endowed with the inner product

$$\big((f, f'), (g, g')\big) := (f, g) + (f', g'), \quad (f, f'), (g, g') \in \mathcal{H} \times \mathcal{H},$$

and we denote by $()^\perp$ the orthogonal complement in $\mathcal{H} \times \mathcal{H}$ w.r.t. the above inner product.

Lemma 3.2. Define the operator $\mathcal{U} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$,

$$\mathcal{U}(h, h') := (h', -h), \qquad (h, h') \in \mathcal{H} \times \mathcal{H}.$$

Then for any densely defined operator S in $\mathcal{H}$ one has $\mathcal{G}(S^*) = (\mathcal{U}\mathcal{G}(S))^\perp = \mathcal{U}(\mathcal{G}(S))^\perp$.

Proposition 3.3. Let S be a densely defined operator in $\mathcal{H}$. Then the following holds:

(i) $S^* \in \mathcal{C}(\mathcal{H})$.

(ii) S is closable $\Leftrightarrow$ $\operatorname{dom} S^*$ is dense in $\mathcal{H}$. In this case one has

$$(\overline{S})^* = S^* \quad \text{and} \quad \overline{S} = S^{**}.$$

(iii) $S \subset T \Rightarrow T^* \subset S^*$.

Lemma 3.4. Let S be a densely defined operator in $\mathcal{H}$. Then one has for any $\lambda \in \mathbb{K}$

(i) $\big(\operatorname{ran}(S - \lambda)\big)^\perp = \ker(S^* - \overline{\lambda})$ and

(ii) $\overline{\operatorname{ran}(S - \lambda)} = \big(\ker(S^* - \overline{\lambda})\big)^\perp$.

Definition 3.5. A densely defined operator S is called

(i) *symmetric*, if $S \subset S^*$; i.e. $\operatorname{dom} S \subset \operatorname{dom} S^*$, $Sf = S^*f \ \forall f \in \operatorname{dom} S$

(ii) *self adjoint*, if $S = S^*$; i.e. $\operatorname{dom} S = \operatorname{dom} S^*$, $Sf = S^*f \ \forall f \in \operatorname{dom} S$

(iii) *essentially self adjoint*, if $\overline{S}$ is self adjoint, i.e. if $\overline{S} = S^*$.

Lemma 3.6. Let S be a densely defined operator in $\mathcal{H}$. Then the following are equivalent:

(i) S is symmetric.

(ii) $(Sf, g) = (f, Sg)$ for all $f, g \in \operatorname{dom} S$.

If $\mathbb{K} = \mathbb{C}$, then (i) and (ii) are equivalent to

(iii) $(Sf, f) \in \mathbb{R}$ for all $f \in \operatorname{dom} S$.

Lemma 3.7. (i) Each symmetric operator S is closable and $\overline{S}$ is also symmetric.

(ii) Each self adjoint operator is closed.

Proof: (i) Since S is symmetric, S is densely defined (by def.)
S symmetric $\Rightarrow S \subset S^* \Rightarrow \operatorname{dom} S \subset \operatorname{dom} S^* \Rightarrow S^*$ is densely defined. By Prop. 3.3 (ii) we have that S is closable and $(\overline{S})^* = S^* \Rightarrow \overline{S} \subset S^*$ [since $\overline{S}$ is the smallest closed extension of S and S^* is another] ✓

(ii) By Prop. 3.3 (i) $S^* \in C(\mathcal{H}) \Rightarrow$ For any self adjoint op. S one has $S = S^* \in C(\mathcal{H})$ □

Proposition 3.8. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let S be symmetric and closed. Then the following holds:

(i) $\mathbb{C} \setminus \mathbb{R} \subset r(S)$ and $\operatorname{ran}(S - \lambda)$ is closed for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

points of regular type, i.e. $(S-\lambda)^{-1}$ bdd.

(ii) $\sigma_p(S) \cup \sigma_c(S) \subset \mathbb{R}$.

(iii) For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ one has $\|(S - \lambda)^{-1}\| \leq \frac{1}{|\operatorname{Im} \lambda|}$. (also true for S symmetric, but not closed)

Proof: Let $f \in \operatorname{dom} S$ and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then

$$\|(S-\lambda)f\|^2 = ((S-\lambda)f, (S-\lambda)f)$$

$$= (Sf, Sf) - \lambda \underbrace{\overbrace{(f, Sf)}^{=(Sf,f)}}_{\text{Lemma 3.6 } \in \mathbb{R}} - \bar{\lambda}(Sf, f) + |\lambda|^2 (f, f)$$

$$= \|Sf\|^2 - \underbrace{(\lambda + \bar{\lambda})}_{2 \operatorname{Re} \lambda}(Sf, f) + \underbrace{|\lambda|^2}_{(\operatorname{Re}\lambda)^2 + (\operatorname{Im}\lambda)^2} \|f\|^2$$

$$= \|Sf\|^2 - \operatorname{Re}\lambda\,(Sf, f) - \operatorname{Re}\lambda\,(f, Sf) + (\operatorname{Re}\lambda)^2 \|f\|^2 + (\operatorname{Im}\lambda)^2 \|f\|^2$$

$$= \underbrace{\|(S - \operatorname{Re}\lambda) f\|^2}_{\geq 0} + (\operatorname{Im}\lambda)^2 \|f\|^2 \geq (\operatorname{Im}\lambda)^2 \|f\|^2$$

$\Rightarrow \quad \|(S-\lambda)f\| \geq |\operatorname{Im}\lambda| \cdot \|f\| \qquad (\text{for } \lambda \in \mathbb{C}\setminus\mathbb{R}) \quad (*)$

$\Rightarrow \quad \ker(S-\lambda) = \{0\}$ and $\|(S-\lambda)^{-1}g\| \leq \frac{1}{|\operatorname{Im}\lambda|}\|g\|$

(put $g = (S-\lambda)f$ in $(*)$)

Consequences:

- $\lambda \notin \sigma_p(S)$ and $\lambda \notin \sigma_c(S)$ [as $(S-\lambda)^{-1}$ is bounded]
- $\lambda \in r(S)$, as $(S-\lambda)^{-1}$ is bdd.
- Since $(S-\lambda)^{-1}$ is closed (exercise!) and bounded, $\operatorname{dom}(S-\lambda)^{-1} = \operatorname{ran}(S-\lambda)$ is closed! (Thm 1.6)

$\square$

Lemma 3.9. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let $S \subset S^*$. If $\operatorname{ran}(S - \lambda) = \mathcal{H}$ for a $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $S \in \mathcal{C}(\mathcal{H})$.

Proof: As in the proof of Prop. 3.8 one shows that $S-\lambda$ is injective and $(S-\lambda)^{-1}$ is bounded for $\lambda \in \mathbb{C}\setminus\mathbb{R}$.
If $\operatorname{ran}(S-\lambda) = \mathcal{H}$, then $\operatorname{ran}(S-\lambda) = \operatorname{dom}(S-\lambda)^{-1} = \mathcal{H}$.
$\Rightarrow$ By Theorem 1.6 we get $(S-\lambda)^{-1}$ must be closed
$\Rightarrow S \in C(\mathcal{H})$ $\square$

Theorem 3.10. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$, let S be a symmetric operator in $\mathcal{H}$, and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the following are equivalent:

(i) S is self adjoint.

(ii) $S \in \mathcal{C}(\mathcal{H})$ and $\boxed{\ker(S^* - \lambda) = \{0\}} = \ker(S^* - \overline{\lambda})$.

(iii) $\operatorname{ran}(S - \lambda) = \mathcal{H} = \operatorname{ran}(S - \overline{\lambda})$.

(iv) $S \in \mathcal{C}(\mathcal{H})$ and $\lambda, \overline{\lambda} \in \rho(S)$.

Remark: If one of the assertions (ii), (iii) or (iv) from Theorem 3.10 hold for one $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then due to their equivalence to (i) these assertions hold for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, see also the proof.

Proof: (ii) $\Leftrightarrow$ (iii)
By Lemma 3.4(i) one has $\ker(S^*-\overline{\lambda}) = (\operatorname{ran}(S-\lambda))^\perp$
If (iii) holds, then by Lemma 3.9 $S \in C(\mathcal{H})$ and

$$\ker(S^*-\overline{\lambda}) = (\operatorname{ran}(S-\lambda))^\perp \overset{(iii)}{=} \mathcal{H}^\perp = \{0\}$$
$$\ker(S^*-\lambda) = (\operatorname{ran}(S-\overline{\lambda}))^\perp = \mathcal{H}^\perp = \{0\} \Rightarrow \text{(ii)} \checkmark$$

If (ii) holds, then $S \in C(\mathcal{H})$ and by Prop. 3.8 $\operatorname{ran}(S-\lambda), \operatorname{ran}(S-\overline{\lambda})$ closed

$$\Rightarrow \operatorname{ran}(S-\lambda) = \overline{\operatorname{ran}(S-\lambda)} = (\ker(S^*-\overline{\lambda}))^\perp = \mathcal{H}$$
$$\operatorname{ran}(S-\overline{\lambda}) = \overline{\operatorname{ran}(S-\overline{\lambda})} = (\ker(S^*-\lambda))^\perp = \mathcal{H} \Rightarrow \text{(iii)} \checkmark$$

(i) $\Rightarrow$ (ii) If $S=S^*$, then $S \in \mathcal{C}(\mathcal{H})$ [Lemma 3.7]

$\ker(S^*-\bar{\lambda}) \overset{S=S^*}{=} \ker(S-\bar{\lambda}) \overset{\text{Prop. 3.8}}{=} \{0\} \Rightarrow$ (ii) ✓

(iii) $\Rightarrow$ (i)

Since S is symmetric by assumption, we have $S \subset S^*$. Hence it suffices to show that $\operatorname{dom} S^* \subset \operatorname{dom} S$.

Take an arbitrary $f \in \operatorname{dom} S^* \subset \mathcal{H}$.

$\overset{\text{(iii)}}{\Rightarrow} \exists g \in \operatorname{dom} S: \underbrace{(S-\lambda)g}_{\in \operatorname{ran}(S-\lambda)} = \underbrace{(S^*-\lambda)f}_{\in \mathcal{H}}$

Since $S \subset S^*$, we have $Sg = S^*g$ and $g \in \operatorname{dom} S^*$

$\Rightarrow (S^*-\lambda)(f-g) = 0$

$\Rightarrow f-g \in \ker(S^*-\lambda) \overset{\text{Lemma 3.4}}{=} (\operatorname{ran}(S-\bar{\lambda}))^{\perp} \overset{\text{(iii)}}{=} \mathcal{H}^{\perp} = \{0\}$

$\Rightarrow f = g \in \operatorname{dom} S \Rightarrow \operatorname{dom} S^* \subset \operatorname{dom} S \Rightarrow S^* \subset S$

$\Rightarrow S = S^*$ ✓

(iii) $\Rightarrow$ (iv) Assume $\operatorname{ran}(S-\lambda) = \operatorname{ran}(S-\bar{\lambda}) = \mathcal{H}$

By Lemma 3.9 this shows $S \in \mathcal{C}(\mathcal{H})$. As in the proof of Prop. 3.8 we have $\ker(S-\lambda) = \ker(S-\bar{\lambda}) = \{0\}$.

$\Rightarrow (S-\lambda)^{-1}, (S-\bar{\lambda})^{-1}$ exist, are closed, are defined on $\mathcal{H}$

$\Rightarrow$ By the closed graph thm. we get $(S-\lambda)^{-1}, (S-\bar{\lambda})^{-1} \in \mathcal{L}(\mathcal{H})$

$\Rightarrow \lambda, \bar{\lambda} \in \rho(S)$

(iv) $\Rightarrow$ (iii) Assume that $\lambda, \bar{\lambda} \in \rho(S)$

$\Rightarrow (S-\lambda)^{-1}, (S-\bar{\lambda})^{-1} \in \mathcal{L}(\mathcal{H})$

$\Rightarrow \operatorname{dom}(S-\lambda)^{-1} = \operatorname{ran}(S-\lambda) = \mathcal{H}$

$\operatorname{dom}(S-\bar{\lambda})^{-1} = \operatorname{ran}(S-\bar{\lambda}) = \mathcal{H}$ $\Rightarrow$ (iii) ✓ $\square$

Proposition 3.11. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let S be a self adjoint operator in $\mathcal{H}$. Then the following holds:

(i) $\sigma(S) \subset \mathbb{R}$ and $\sigma_r(S) = \emptyset$.

(ii) $\lambda \in \sigma(S) \Leftrightarrow$ there exists a sequence $(x_n)_n \subset \operatorname{dom} S$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ such that $\|(S - \lambda)x_n\| \to 0$ for $n \to \infty$.

"approximating eigensequence"

Proof: (i) Let $\lambda \in \sigma_r(S)$, i.e. $S-\lambda$ is injective, $\overline{\operatorname{ran}(S-\lambda)} \neq \mathcal{H}$

- Case 1: $\lambda \in \mathbb{C}\setminus\mathbb{R}$. By Prop. 3.8 we know $\ker(S-\bar\lambda) = \{0\}$

$\Rightarrow \mathcal{H} = (\{0\})^\perp = (\ker(\overset{S^*}{\underset{\shortparallel}{S}} - \bar\lambda))^\perp \overset{\text{Lemma 3.4}}{=} \overline{(\operatorname{ran}(S-\lambda))} \neq \mathcal{H}$ ↯

- Case 2: $\lambda \in \mathbb{R} \cap \sigma_r(S)$

$\Rightarrow \mathcal{H} \neq \overline{\operatorname{ran}(S-\lambda)} \overset{S=S^*,\ \lambda\in\mathbb{R}}{=} \overline{(\operatorname{ran}(S-\bar\lambda))} \overset{\text{Lemma 3.4}}{=} (\ker(S^*-\lambda))^\perp \overset{S=S^*}{=} (\ker(S-\lambda))^\perp$

$\Rightarrow \ker(S-\lambda) \neq \{0\}$ ↯

$\Rightarrow \sigma_r(S) = \emptyset$

By Prop. 3.8 $\mathbb{C}\setminus\mathbb{R} \subset r(S) \overset{\sigma_r(S)=\emptyset}{=} \rho(S) \Rightarrow \sigma(S) \subset \mathbb{R}$

(ii) "$\Rightarrow$" If $\lambda \in \sigma_p(S) \Rightarrow \exists x \in \operatorname{dom} S : \|x\| = 1,\ (S-\lambda)x = 0$. Set $x_n := x$

$\Rightarrow x_n$ fulfills $\|x_n\| = 1,\ \|(S-\lambda)x_n\| = 0 \xrightarrow{n\to\infty} 0$ ✓

If $\lambda \in \sigma_c(S) \Rightarrow (S-\lambda)^{-1}$ is unbounded

$\Rightarrow \forall n \in \mathbb{N} : \exists x_n \in \operatorname{dom}(S-\lambda)^{-1} = \operatorname{ran}(S-\lambda) : \|(S-\lambda)^{-1}x_n\| > n\|x_n\|$

Set $z_n := \dfrac{(S-\lambda)^{-1}x_n}{\|(S-\lambda)^{-1}x_n\|} \Rightarrow \|z_n\| = 1,\ \|(S-\lambda)z_n\| = \left\|\dfrac{x_n}{\|(S-\lambda)^{-1}x_n\|}\right\| < \dfrac{1}{n} \to 0$

$\Rightarrow (z_n)$ is the approximating eigensequence ✓

"$\Leftarrow$" $\exists (x_n)$: $\|x_n\| = 1$ and $(S-\lambda)x_n \to 0$

If $S-\lambda$ not injective $\Rightarrow$ $\lambda \in \sigma_p(S) \subset \sigma(S)$

If $S-\lambda$ is injective, then $(S-\lambda)^{-1}$ must be unbounded, as

$y_n = (S-\lambda)x_n \Rightarrow \frac{\|(S-\lambda)^{-1} y_n\|}{\|y_n\|} = \frac{\|x_n\|}{\|(S-\lambda)x_n\|} \xrightarrow[n\to\infty]{} \infty$ $\square$

$\Rightarrow \lambda \in \sigma(S)$

Example 3.12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and define the operator $T : L^2(\mathbb{R}) \supset \operatorname{dom} T \to L^2(\mathbb{R})$ by

$$(Tg)(x) = f(x)g(x) \quad \text{for } x \in \mathbb{R}, \quad g \in \operatorname{dom} T := \{g \in L^2(\mathbb{R}) : fg \in L^2(\mathbb{R})\}.$$

Then one has:

(i) $T = T^*$.

(ii) $\sigma(T) = \overline{\{f(x) : x \in \mathbb{R}\}}$ and T is bounded, if and only if f is bounded.

(iii) $\sigma_p(T) = \{\mu \in \mathbb{R} : |f^{-1}(\{\mu\})| > 0\}$.

(i) • T is symmetric:

$(Tg, g) = \int_{\mathbb{R}} (Tg)(x) \cdot \overline{g(x)}\, dx = \int_{\mathbb{R}} \underbrace{f(x)}_{\in \mathbb{R}} |g(x)|^2 dx \in \mathbb{R}$

$\Rightarrow T \subset T^*$ by Lemma 3.6

• For $\lambda \notin \overline{\{f(x) : x \in \mathbb{R}\}}$: $T - \lambda$ is bijective

$T-\lambda$ injective: $(T-\lambda)g = 0$

$\Leftrightarrow (f-\lambda)g = 0$ in $L^2(\mathbb{R})$

$\Leftrightarrow (f(x)-\lambda)g(x) = 0$ f.a.e. $x \in \mathbb{R}$

$\Rightarrow g(x) = 0$ f.a.e. $x \in \mathbb{R}$

$\Rightarrow \ker(T-\lambda) = \{0\}$ ✓

• $(T-\lambda)$ surjective:

For $\lambda \notin \overline{\{f(x) : x \in \mathbb{R}\}}$ there exists $\varepsilon > 0$: $|f(x) - \lambda| > \varepsilon$ $\forall x \in \mathbb{R}$

Define for $\underline{g \in L^2(\mathbb{R})}$ the function $k := \frac{1}{f-\lambda} \cdot g$

$k \in \operatorname{dom} T$:

$k \in L^2(\mathbb{R})$: $\int_{\mathbb{R}} |k(x)|^2 dx = \int_{\mathbb{R}} \underbrace{\frac{1}{|f(x)-\lambda|^2}}_{<\varepsilon^2} |g(x)|^2 dx < \frac{1}{\varepsilon^2} \|g\|_{L^2}^2 < \infty$ ✓

Moreover: $\int_{\mathbb{R}} |f \cdot k|^2 dx = \int_{\mathbb{R}} \Big| \underbrace{\frac{f(x)}{(f(x)-\lambda)}}_{|\cdot| < M} \cdot g(x) \Big|^2 dx < M \cdot \|g\|_{L^2}^2$

$\Rightarrow k \in \operatorname{dom} T$ ✓

By def. of T: $(T-\lambda) k = \frac{f-\lambda}{f-\lambda} g = g$

$\Rightarrow \underline{g \in \operatorname{ran}(T-\lambda)} \Rightarrow \operatorname{ran}(T-\lambda) = L^2(\mathbb{R})$ for all $\underbrace{\lambda \notin \{f(x) : x \in \mathbb{R}\}}$

Thm 3.10
$\Rightarrow$ $T = T^*$