4.3 The measurable functional calculus

Again, we assume throughout this section that $A = A^* \in \mathcal{L}(\mathcal{H})$. The goal in this section is to extend the continuous functional calculus from the last section for bounded and measurable functions, i.e. to define $f(A)$ for any bounded and measurable function $f : \sigma(A) \to \mathbb{C}$. We set for any compact set $K \subset \mathbb{C}$

$$\boxed{B(K) := \{f : K \to \mathbb{C} : f \text{ is measurable and bounded}\},}$$

which is endowed with the norm $\|\cdot\|_\infty$ a Banach space. The following elementary lemma, which can be found e.g. in [Werner, Lemma VII.1.5], will be very useful in our constructions:

Lemma 4.6. Let $V \subset B(K)$ such that the following holds:

(i) $C(K) \subset V$.

(ii) For any sequence $(f_n) \subset V$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in K$ imply that $f \in V$.

Then $V = B(K)$.

The previous lemma means, roughly speaking, that $B(K)$ is the smallest set of functions, which contains all continuous functions and which is closed with respect to pointwise limits of uniformyl bounded sequences.

In order to formulate the next result, recall that a *complex Borel measure* over $\sigma(A)$ is a map $\mu : \Sigma(\sigma(A)) \to \mathbb{C}$, which is σ-additive (here $\Sigma(\sigma(A))$ is the Borel-σ-algebra over $\sigma(A)$).

Lemma 4.7. Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. Then there exists a complex Borel measure $\mu_{x,y}$ such that

$$\big(f(A)x, y\big) = \int_{\sigma(A)} f\, d\mu_{x,y} \qquad \forall f \in C(\sigma(A)).$$

For any $f \in B(\sigma(A))$ one has

$$\left| \int_{\sigma(A)} f\, d\mu_{x,y} \right| \le \|f\|_\infty \|x\| \cdot \|y\|.$$

In order to prove Lemma 4.7, recall the representation theorem of Markov-Riesz:

Let K be a compact metric space and let $M(K)$ be the space of all complex Borel measures on K endowed with norm

$$\|\mu\| := |\mu|(K) = \sup \sum_{E \in Z} |\mu(E)| \quad , \mu \in M(K),$$

where the supremum is over all finite decompositions Z of K into pairwise disjoint sets. Then the map

$$T: M(K) \to (C(K))' \quad , \boxed{(T_\mu)(f) = \int_K f \, d\mu,}$$

is an isometric isomorphism.

Proof of Lemma 4.7

For fixed $x, y \in \mathcal{H}$ define $\ell_{x,y}: C(K) \to \mathbb{C}$ by

$$\ell_{x,y}(f) := (f(A)x, y)$$

Due to the properties of the continuous functional calculus $\ell_{x,y}$ is linear and

$$|\ell_{x,y}(f)| = |(f(A)x, y)| \leq \|f(A)x\| \cdot \|y\| \leq \underbrace{\|f(A)\|}_{= \|f\|_\infty} \cdot \|x\| \cdot \|y\|$$

$\Rightarrow$ $\ell_{x,y}$ is bounded, $\|\ell_{x,y}\| \leq \|x\| \cdot \|y\|$

By the Markov-Riesz representation theorem there exists $\mu_{x,y} \in M(\sigma(A))$ such that

$$\ell_{x,y}(f) = \int_{\sigma(A)} f \, d\mu_{x,y} \quad [= (f(A)x, y) \;\checkmark] \qquad \underbrace{= \|\ell_{x,y}\|}_{} \leq \|x\| \cdot \|y\|$$

Moreover, for $f \in B(\sigma(A))$: $\left| \int_{\sigma(A)} f \, d\mu_{x,y} \right| \leq \|f\|_\infty \cdot |\mu_{x,y}|(\sigma(A))$ $\square$

Theorem 4.8. For $A = A^* \in \mathcal{L}(\mathcal{H})$ there exists a unique linear and bounded mapping $\widehat{\Phi} : B(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ with the following properties:

(a) $\widehat{\Phi}(p) = p(A)$ for all $p \in P(\sigma(A))$.

(b) $\widehat{\Phi}$ is multiplicative and an involution. i.e. $\hat{\Phi}(f \cdot g) = \hat{\Phi}(f)\hat{\Phi}(g)$, $\hat{\Phi}(\bar{f}) = (\hat{\Phi}(f))^*$

(c) For any sequence $(f_n) \subset B(\sigma(A))$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in \sigma(A)$ imply that

weak convergence of $\hat{\Phi}(f_n)$

$$\boxed{(\widehat{\Phi}(f_n)x, y) \to (\widehat{\Phi}(f)x, y)} \quad \forall x, y \in \mathcal{H}.$$

Moreover, for all $f \in C(\sigma(A))$ one has $\widehat{\Phi}(f) = \Phi(f)$. (i.e. measurable is an extension of continuous functional calc.)

As for the continuous functional calculus we set for $f \in B(\sigma(A))$

$$\boxed{f(A) := \widehat{\Phi}(f).}$$

The map $\widehat{\Phi}$ is called *measurable functional calculus.*

Proof of Theorem 4.8:

Step 1: Fix $f \in B(\sigma(A))$. We show:

$$\mathcal{H} \ni (x,y) \mapsto \int_{\sigma(A)} f \, d\mu_{x,y} \qquad (*)$$

is a bounded sesquilinear form on $\mathcal{H}$

[recall: $a : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is a bounded sesquilinear form, if
- $a[\alpha x + \beta y, z] = \alpha\, a[x,z] + \beta\, a[y,z]$
- $a[x, \alpha y + \beta z] = \bar{\alpha}\, a[x,y] + \bar{\beta}\, a[x,z]$
- $|a[x,y]| \le c \cdot \|x\| \cdot \|y\|$]

Here $\mu_{x,y}$ is the measure from Lemma 4.7

Additivity w.r.t. 1st entry

Let $x, z \in \mathcal{H}$ and

$$V := \left\{ h \in B(\sigma(A)) : \int_{\sigma(A)} h \, d\mu_{x+z,y} = \int_{\sigma(A)} h \, d\mu_{x,y} + \int_{\sigma(A)} h \, d\mu_{z,y} \quad \forall y \in \mathcal{H} \right\}$$

Note that $C(k) \subset V$, as $f \in C(k)$:

$$\int_{\sigma(A)} f \, d\mu_{x+z,y} \overset{\text{Lemma 4.7}}{=} (f(A)(x+z), y) = (f(A)x, y) + (f(A)z, y) = \int_{\sigma(A)} f \, d\mu_{x,y} + \int_{\sigma(A)} f \, d\mu_{z,y}$$

Let $\underline{(h_n) \subset V}$ such that $\sup_{n \in \mathbb{N}} \|h_n\|_\infty < \infty$ and such that $h(t) := \lim_{n \to \infty} h_n(t)$ exists for all $t \in \sigma(A)$.

We have to show $h \in V$.

dominated convergence

$\mu_{x+z,y}$ is a bounded measure, (h_n) is uniformly bounded

$\Rightarrow |h_n(t)| \leq C \in L^1(\sigma(A), d\mu_{x+z,y})$

$$\underline{\int_{\sigma(A)} h \, d\mu_{x+z,y}} = \lim_{n \to \infty} \int_{\sigma(A)} h_n \, d\mu_{x+z,y} \overset{h_n \in V}{=} \lim_{n \to \infty} \left(\int_{\sigma(A)} h_n \, d\mu_{x,y} + \int_{\sigma(A)} h_n \, d\mu_{z,y} \right)$$

$$= \underline{\int_{\sigma(A)} h \, d\mu_{x,y} + \int_{\sigma(A)} h \, d\mu_{z,y}} \quad \Rightarrow h \in V$$

By Lemma 4.6 we get $V = B(\sigma(A))$.

With similar arguments, the map in $(*)$ is sesquilinear.

Moreover: $\left| \int_{\sigma(A)} f \, d\mu_{x,y} \right| \overset{\text{Lemma 4.7}}{\leq} \|f\|_\infty \cdot \|x\| \cdot \|y\|$

$\Rightarrow$ the sesquilinear form in $(*)$ is bounded ✓

Step 2: Construction of $\hat{\Phi}$:

We use the following theorem of Lax-Milgram:
Let a be a bounded sesquilinear form. Then there exists a unique $A \in \mathcal{L}(\mathcal{H})$ s.t. $a[x,y] = (Ax,y)$

$\rightsquigarrow$ exercise

By the above Lax-Milgram theorem and the result of step 1 there exists for any $f \in B(\sigma(A))$ a unique operator $\hat{\Phi}(f) \in \mathcal{L}(\mathcal{H})$ s.t. $(\hat{\Phi}(f)x,y) = \int_{\sigma(A)} f \, d\mu_{x,y} \quad \forall x,y \in \mathcal{H}$

with the help of Lemma 4.7 we have $\|\hat{\Phi}(f)\| \leq \|f\|_\infty$

$\Rightarrow \hat{\Phi}: B(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ is bounded and by construction $\hat{\Phi}(f) = \Phi(f) = f(A)$ for $f \in C(\sigma(A))$

Step 3: We show (b) and (c).

(c) Assume $(f_n) \subset B(K)$ s.t. $\sup_n \|f_n\|_\infty < \infty$ and $f_n(t) \to f(t)$ for all $t \in \sigma(A)$. Then one can use dominated convergence as in step 1 and gets

$$(\hat{\Phi}(f)x,y) \underset{\text{Def. } \hat{\Phi}}{=} \int_{\sigma(A)} f \, d\mu_{x,y} = \lim_{n\to\infty} \int_{\sigma(A)} f_n \, d\mu_{x,y} = \lim_{n\to\infty} (\hat{\Phi}(f_n)x,y)$$

Linearity of $\hat{\Phi}$:

$$(\hat{\Phi}(f+g)x,y) = \int_{\sigma(A)} (f+g) \, d\mu_{x,y} = \int_{\sigma(A)} f \, d\mu_{x,y} + \int_{\sigma(A)} g \, d\mu_{x,y}$$
$$= (\hat{\Phi}(f)x,y) + (\hat{\Phi}(g)x,y)$$

Similar: $\hat{\Phi}(\alpha f) = \alpha \hat{\Phi}(f)$

$\hat{\Phi}$ is multiplicative, i.e. $\hat{\Phi}(f \cdot g) = \hat{\Phi}(f)\,\hat{\Phi}(g)$

Define for a fixed $g \in C(\sigma(A))$

$$V := \{ h \in B(\sigma(A)) : \hat{\Phi}(hg) = \hat{\Phi}(h)\,\hat{\Phi}(g) \}$$

Note $C(\sigma(A)) \subset V$ (Φ is multiplicative by Theorem 4.3)

Let $(f_n) \subset V$ s.t. $\sup_n \|f_n\|_\infty < \infty$ and $f_n(t) \to f(t)$ $\forall t \in \sigma(A)$.

Then for arbitrary x, y:

$$(\hat{\Phi}(f_n)\,\hat{\Phi}(g)x, y) \xrightarrow[n\to\infty]{(c)} \underline{(\hat{\Phi}(f)\,\hat{\Phi}(g)x, y)}$$

$$\overset{f_n \in V}{\|}$$

$$(\hat{\Phi}(f_n \cdot g)x, y) \xrightarrow{(c)} \underline{(\hat{\Phi}(f \cdot g)x, y)}$$

$\Rightarrow f \in V \Rightarrow$ By Lemma 4.6 $V = B(\sigma(A))$

Let $f \in B(\sigma(A))$. Set

$$V := \{ h \in B(\sigma(A)) : \hat{\Phi}(f \cdot h) = \hat{\Phi}(f)\,\hat{\Phi}(h) \}$$

By our previous considerations: $C(\sigma(A)) \subset V$

As before one shows that for $(h_n) \subset V$ s.t. $\sup_n \|h_n\|_\infty < \infty$ and $h_n(t) \to h(t)$, then $h \in V$

By Lemma 4.6 $V = B(\sigma(A))$. ✓

$\hat{\Phi}$ is an involution: similar (exercise)

Step 4: Uniqueness

$\hat{\Phi}(f)$ is uniquely determined by the continuous functional calculus ~~for $f \in C(\sigma(A))$~~. With (c) and Lemma 4.6 you prove that the set of all f, where $\hat{\Phi}(f)$ is uniquely determined, is closed w.r.t. pointwise limit of uniformly bounded sequences and hence $\hat{\Phi}(f)$ is uniquely determined $\forall f \in B(\sigma(A))$ □

Remark 4.9. Condition (c) in Theorem 4.8 can be improved in the following way:

(c') For any sequence $(f_n) \subset B(\sigma(A))$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in \sigma(A)$ imply that

$$\widehat{\Phi}(f_n)x \to \widehat{\Phi}(f)x \qquad \forall x \in \mathcal{H}.$$

strong convergence

Proof:

$$\|\widehat{\Phi}(f_n)x\|^2 = (\widehat{\Phi}(f_n)x, \widehat{\Phi}(f_n)x) = (\widehat{\Phi}(\overline{f_n})\cdot\widehat{\Phi}(f_n)x, x)$$
$$= (\widehat{\Phi}(|f_n|^2)x, x) \xrightarrow{(c)} (\widehat{\Phi}(|f|^2)x, x) = \ldots = \|\widehat{\Phi}(f)x\|^2$$

$$\Rightarrow \|\widehat{\Phi}(f_n)x - \widehat{\Phi}(f)x\|^2 = \|\widehat{\Phi}(f_n)x\|^2 - (\widehat{\Phi}(f_n)x, \widehat{\Phi}(f)x)$$
$$- (\widehat{\Phi}(f)x, \widehat{\Phi}(f_n)x) + \|\widehat{\Phi}(f)x\|^2$$

$$\xrightarrow[n\to\infty]{(c)} \cancel{\|\widehat{\Phi}(f)x\|^2} - \cancel{(\widehat{\Phi}(f)x, \widehat{\Phi}(f)x)} - \cancel{(\widehat{\Phi}(f)x, \widehat{\Phi}(f)x)} + \cancel{\|\widehat{\Phi}(f)x\|^2} = 0$$