Theorem 4.8. For $A = A^* \in \mathcal{L}(\mathcal{H})$ there exists a unique linear and bounded mapping $\widehat{\Phi} : B(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ with the following properties:

(a) $\widehat{\Phi}(p) = p(A)$ for all $p \in P(\sigma(A))$.

(b) $\widehat{\Phi}$ is multiplicative and an involution. $\widehat{\Phi}(f \cdot g) = \widehat{\Phi}(f) \cdot \widehat{\Phi}(g)$, $\widehat{\Phi}(\overline{f}) = (\widehat{\Phi}(f))^*$

(c) For any sequence $(f_n) \subset B(\sigma(A))$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in \sigma(A)$ imply that

$$\boxed{(\widehat{\Phi}(f_n)x, y) \to (\widehat{\Phi}(f)x, y)} \qquad \forall x, y \in \mathcal{H}.$$

Moreover, for all $f \in C(\sigma(A))$ one has $\widehat{\Phi}(f) = \Phi(f)$.

As for the continuous functional calculus we set for $f \in B(\sigma(A))$

$$\boxed{f(A) := \widehat{\Phi}(f).}$$

The map $\widehat{\Phi}$ is called *measurable functional calculus*.

Goal for today:

$A = A^* \in \mathbb{C}^{n\times n} \Rightarrow A = \sum_{\lambda_n \in \sigma_p(A)} \lambda_n P_{\lambda_n} = \boxed{\int \lambda \, dE_\lambda}$

known from motivation

We want to extend this for $A = A^* \in \mathcal{L}(\mathcal{H})$

Remark 4.9. Condition (c) in Theorem 4.8 can be improved in the following way:

(c') For any sequence $(f_n) \subset B(\sigma(A))$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in \sigma(A)$ imply that

$$\widehat{\Phi}(f_n)x \to \widehat{\Phi}(f)x \qquad \forall x \in \mathcal{H}.$$ convergence in $\mathcal{H}$

Attention: this does not mean, that

$\hat{\Phi}(f_n) \to \hat{\Phi}(f)$ in $\mathcal{L}(\mathcal{H})$

* $B \subset \mathcal{H}$ closed subspace $\Rightarrow$ $\forall x \in \mathcal{H}$ $\exists$ unique $y \in B$, $z \in B^\perp$:

$x = y + z$

$x \mapsto y$ is this orthogonal proj.

Recall: $P \in \mathcal{L}(\mathcal{H})$ is called *orthogonal projection*, if $P = P^* = P^2$. This is equivalent to the fact that ran P is closed and that P is the orthogonal projection in $\mathcal{H}$ onto ran P.*

Lemma 4.10. Let $A = A^* \in \mathcal{L}(\mathcal{H})$. Then the following is true:

$$\mathbb{1}_B(x) = \begin{cases} 1, & x \in B \\ 0, & x \notin B \end{cases}$$

(i) $\mathbb{1}_B(A)(:= \widehat{\Phi}(\mathbb{1}_B))$ is an orthogonal projection for any Borel set $B \subset \sigma(A)$.

(ii) $\mathbb{1}_\emptyset(A) = 0$ and $\mathbb{1}_{\sigma(A)}(A) = I$.

(iii) For any family of pairwise disjoint Borel sets $B_1, B_2, \dots \subset \sigma(A)$ and all $x \in \mathcal{H}$ one has

i.e. sum converges in $\mathcal{H}$

$$\sum_{k=1}^{\infty} \mathbb{1}_{B_k}(A)x = \mathbb{1}_{\cup_{k=1}^{\infty} B_k}(A)x.$$

(iv) For any two Borel sets $B_1, B_2 \subset \sigma(A)$ one has $\mathbb{1}_{B_1}(A)\mathbb{1}_{B_2}(A) = \mathbb{1}_{B_1 \cap B_2}(A)$.

Proof:

$\hat{\Phi}$ involution

(i) $\mathbb{1}_B(A) = \hat{\Phi}(\mathbb{1}_B) = \hat{\Phi}(\overline{\mathbb{1}_B}) \overset{\downarrow}{=} (\hat{\Phi}(\mathbb{1}_B))^* = \mathbb{1}_B(A)^*$

$\mathbb{1}_B(A) = \hat{\Phi}(\mathbb{1}_B) = \hat{\Phi}(\mathbb{1}_B^2) = \hat{\Phi}(\mathbb{1}_B) \cdot \hat{\Phi}(\mathbb{1}_B) = \mathbb{1}_B(A) \cdot \mathbb{1}_B(A)$

$\hat{\Phi}$ is multiplicative

$\Rightarrow \mathbb{1}_B(A)$ is an orthogonal projection ✓

(ii) $\mathbb{1}_\emptyset(A) = \hat{\Phi}(\mathbb{1}_\emptyset) = \hat{\Phi}(0) = 0$ ✓

$(\hat{\Phi}(p) = p(A)$ for any polynomial$)$

$\mathbb{1}_{\sigma(A)}(A) = \hat{\Phi}(\mathbb{1}_{\sigma(A)}) = \hat{\Phi}(\mathbb{1}) = I$ ✓

(iii) Assume that $B_1, B_2, \dots \subset \sigma(A)$ are pairwise disjoint Borel sets. Set $f_n := \sum_{k=1}^{n} \mathbb{1}_{B_k} = \mathbb{1}_{\bigcup_{k=1}^{n} B_k}$

Then $\|f_n\|_\infty = 1 < \infty$ and f_n converges pointwise to $f(t) := \mathbb{1}_{\bigcup_{k=1}^{\infty} B_k}(t)$. By Remark 4.9 this implies for an arbitrary $x \in \mathcal{H}$:

$$\sum_{k=1}^{\infty} \mathbb{1}_{B_k}(A)x = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{1}_{B_k}(A)x = \lim_{n\to\infty} \sum_{k=1}^{n} \hat{\Phi}(\mathbb{1}_{B_k})x$$

$$\overset{(c')}{=} \hat{\Phi}(\mathbb{1}_{\bigcup_{k=1}^{\infty} B_k})x = \mathbb{1}_{\bigcup_{k=1}^{\infty} B_k}(A)x$$

(iv) Note: $\mathbb{1}_{B_1}(t) \cdot \mathbb{1}_{B_2}(t) = \mathbb{1}_{B_1 \cap B_2}(t)$

$$\Rightarrow \underline{\mathbb{1}_{B_1}(A)\, \mathbb{1}_{B_2}(A)} = \hat{\Phi}(\mathbb{1}_{B_1}) \cdot \hat{\Phi}(\mathbb{1}_{B_2}) = \hat{\Phi}(\mathbb{1}_{B_1} \mathbb{1}_{B_2})$$

$\hat{\Phi}$ is multiplicative

$$= \hat{\Phi}(\mathbb{1}_{B_1 \cap B_2}) = \underline{\mathbb{1}_{B_1 \cap B_2}(A)}$$

$\square$

4.4 Spectral measures and integration

Throughout this section Σ is the Borel σ-algebra on $\mathbb{R}$.

Definition 4.11. A map $E : \Sigma \to \mathcal{L}(\mathcal{H})$, $B \mapsto E_B$, is called *spectral measure*, if E_B is an orthogonal projection for all $B \in \Sigma$ and if the following holds:

(i) $E_\emptyset = 0$ and $E_\mathbb{R} = I$.

(ii) For all pairwise disjoint sets $B_1, B_2, \dots \in \Sigma$ and all $x \in \mathcal{H}$ one has the following σ-additivity:

$$\sum_{k=1}^{\infty} E_{B_k} x = E_{\cup_{k=1}^{\infty} B_k} x.$$

(convergence in $\mathcal{H}$)

A spectral measure E has *compact support*, if there exists a compact set $K \subset \mathbb{R}$ such that $E_K = I$.

Properties of spectral measures:

(a) Finite additivity: $E_{B_1} + E_{B_2} = E_{B_1 \cup B_2}$ for all disjoint $B_1, B_2 \in \Sigma$.

(b) $E_{B_1} E_{B_2} = E_{B_1 \cap B_2}$ for all Borel sets B_1, B_2.

(simple exercise)

The following corollary is an immediate consequence of Lemma 4.10 and Def. 4.11

Corollary 4.12. Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $\widehat{\Phi}$ be the associated measurable calculus. Then the map

$$E : \Sigma \to \mathcal{L}(\mathcal{H}), \quad B \mapsto E_B := \mathbb{1}_{B\cap\sigma(A)}(A) = \widehat{\Phi}(\mathbb{1}_{B\cap\sigma(A)})$$

is a spectral measure with compact support. The above map E is called *spectral measure associated to A.* $\hookrightarrow$ (support $\subseteq \sigma(A)$)

Integration with respect to spectral measures

In the following let E be a fixed spectral measure.

Step 1: integration of simple functions: Let $f = \sum_{k=1}^{n} \alpha_k \mathbb{1}_{B_K}$ for $\alpha_k \in \mathbb{C}$ and pairwise disjoint sets $B_k \in \Sigma$, $k \in \{1, \dots, n\}$. Then we define

$$\int_{\mathbb{R}} f dE := \sum_{k=1}^{n} \alpha_k E_{B_k}.$$

One verifes that the above definition of the integral is independent of the representation of f.

Step 2: bounded and measurable functions

Lemma 4.13. Let E be a spectral measure. Then one has

$$\left\| \int_{\mathbb{R}} f dE \right\| \leq \|f\|_\infty$$

for all simple functions f. Inparticular, the map $f \mapsto \int_{\mathbb{R}} f dE$, defined on the set of all simple functions, is a bounded, densely defined, and linear operator from $B(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$.

Consequence: There exists a unique continuation of the integral with respect to E to the space $B(\mathbb{R})$, which is again a bounded linear map. For an arbitrary $f \in B(\mathbb{R})$ we denote this extension applied to f by $\int_{\mathbb{R}} f dE \in \mathcal{L}(\mathcal{H})$ (or sometimes $\int_{\mathbb{R}} f(t) dE(t)$) and this operator is defined by

$$\int_{\mathbb{R}} f dE := \lim_{n\to\infty} \int_{\mathbb{R}} f_n dE,$$

where (f_n) is any sequence of simple functions with $\|f_n - f\|_\infty \to 0$, as $n \to \infty$.

If f is defined on $\sigma(A)$, then we identify this function with its zero continuation.

Proof of Lemma 4.13:

Linearity of the integral of simple functions is clear by definition. Moreover, the simple functions are dense in $\mathcal{B}(\mathbb{R})$ w.r.t. $\|\cdot\|_\infty$ (see measure theory lecture).

Boundedness: $f = \sum_{k=1}^n \alpha_k \mathbb{1}_{B_k}$, B_k ... pairwise disjoint

$$\left\|\left(\int_{\mathbb{R}} f\,dE\right)x\right\|^2 = \left\|\sum_{k=1}^n \alpha_k E_{B_k} x\right\|^2$$

$$= \left(\sum_{k=1}^n \alpha_k E_{B_k} x, \sum_{\ell=1}^n \alpha_\ell E_{B_\ell} x\right)$$

$$= \sum_{k=1}^n \sum_{\ell=1}^n \alpha_k \overline{\alpha_\ell} \left(E_{B_k} x, E_{B_\ell} x\right)$$

$$\overset{E_{B_\ell}^* = E_{B_\ell}}{=} \sum_{k=1}^n \sum_{\ell=1}^n \alpha_k \overline{\alpha_\ell} \Big(\underbrace{E_{B_\ell} E_{B_k}}_{E_{B_k \cap B_\ell} = \begin{cases} E_{B_k}, & k=\ell \\ 0, & k \neq \ell \end{cases}} x, x\Big)$$

$$= \sum_{k=1}^n |\alpha_k|^2 \Big(\underbrace{E_{B_k}}_{= E_{B_k}^* E_{B_k}} x, x\Big) = \sum_{k=1}^n \underbrace{|\alpha_k|^2}_{\leq \max_{k \in \{1,\dots,n\}} |\alpha_k|^2 = \|f\|_\infty^2} \|E_{B_k} x\|^2$$

$$\leq \|f\|_\infty^2 \sum_{k=1}^n \|E_{B_k} x\|^2 = \dots = \|f\|_\infty^2 \left\|\sum_{k=1}^n E_{B_k} x\right\|^2$$

$$= \|f\|_\infty^2 \Big\|\underbrace{E_{\bigcup_{k=1}^n B_k}}_{\text{orth. proj.}} x\Big\|^2 \leq \|f\|_\infty^2 \|x\|^2 \quad \forall x \in \mathcal{H}$$

$$\Rightarrow \left\|\int_{\mathbb{R}} f\,dE\right\| \leq \|f\|_\infty \quad \checkmark$$

$\square$

4.5 Spectral theorem for bounded self adjoint operators

Theorem 4.14 (Spectral theorem for bounded self-adjoint operators)**.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$, let $\widehat{\Phi}$ be the measurable functional calculus, and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to A. Then

$$\boxed{\widehat{\Phi}(f) = \int_{\mathbb{R}} f dE}$$

holds for all $f \in B(\sigma(A))$. In particular, one has

$f(t) = t$

$$\boxed{A = \int_{\mathbb{R}} t dE(t).}$$

Proof:

Let $f \in B(\sigma(A))$ and denote its extension by 0 onto $\mathbb{R}$ also by f.
Take a sequence (f_n) of simple functions such that $\|f_n - f\|_\infty \to 0$, as $n \to \infty$. Then, by the continuity of $\widehat{\Phi} : B(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ and by the construction of the integral w.r.t. E we have

$$\int_{\mathbb{R}} f dE = \lim_{n\to\infty} \int_{\mathbb{R}} f_n dE = \lim_{n\to\infty} \sum_{k=1}^{m(n)} \alpha_k^{(n)} E_{B_k^{(n)}}$$

$$f_n = \sum_{k=1}^{m(n)} \alpha_k^{(n)} 1\!\!1_{B_k^{(n)}}$$

$$E_B = 1\!\!1_B(A)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{m(n)} \alpha_k^{(n)} 1\!\!1_{B_k^{(n)}}(A)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{m(n)} \alpha_k^{(n)} \widehat{\Phi}(1\!\!1_{B_k^{(n)}})$$

$$= \lim_{n\to\infty} \widehat{\Phi}\Big(\underbrace{\sum_{k=1}^{m(n)} \alpha_k^{(n)} 1\!\!1_{B_k^{(n)}}}_{f_n}\Big)$$

$$= \lim_{n\to\infty} \widehat{\Phi}(f_n) = \widehat{\Phi}(f)$$

2. $A = A^* \in \mathcal{L}(\mathcal{H})$ 1.

spectral theorem (Thm 4.14)

Inversion (Thm 4.15)

3. $\widehat{\Phi}$ 2. ... measurable functional calculus

1. E, $\int f dE$

3. $\mathbb{R}$

Theorem 4.15 (Inversion of the spectral theorem)**.** Let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be a spectral measure with compact support. Then

$$\boxed{A := \int_{\mathbb{R}} t dE(t)}$$

defines a self adjoint operator in $\mathcal{L}(\mathcal{H})$ and the measurable functional calculus $\widehat{\Phi}$ associated to A satisfies

$$\widehat{\Phi}(f) = \int_{\mathbb{R}} f dE \qquad \forall f \in B(\mathbb{R}).$$

Without proof.

Theorem 4.16. Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to A. Then the following holds for any $\lambda \in \mathbb{R}$:

(i) $\lambda \in \rho(A) \Leftrightarrow \exists$ an open neighborhood $B \subset \mathbb{R}$ of λ with $E_B = 0$.

(ii) $\operatorname{ran} E_{\{\lambda\}} = \ker(A - \lambda)$. In particular, $\lambda \in \sigma_p(A) \Leftrightarrow E_{\{\lambda\}} \neq 0$.

(iii) If λ is an isolated point in $\sigma(A)$, i.e. there exists an open neighborhood $B \subset \mathbb{R}$ of λ with $B \cap \sigma(A) = \{\lambda\}$, then λ is an eigenvalue of A.