$1. E \to 2.\ A = \int_{\mathbb{R}} t\, dE(t) \to 3.\ \hat{\Phi}$

Theorem 4.15 (Inversion of the spectral theorem)**.** Let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be a spectral measure with compact support. Then

$$A := \int_{\mathbb{R}} t dE(t)$$

defines a self adjoint operator in $\mathcal{L}(\mathcal{H})$ and the measurable functional calculus $\widehat{\Phi}$ associated to A satisfies

$$\boxed{\widehat{\Phi}(f) = \int_{\mathbb{R}} f dE} \quad \forall f \in B(\mathbb{R}).$$

Without proof.

$E_B = \mathbb{1}_{B \cap \sigma(A)}(A)$

Theorem 4.16. Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to A. Then the following holds for any $\lambda \in \mathbb{R}$:

(i) $\lambda \in \rho(A) \Leftrightarrow \exists$ an open neighborhood $B \subset \mathbb{R}$ of λ with $E_B = 0$.

(ii) $\operatorname{ran} E_{\{\lambda\}} = \ker(A - \lambda)$. In particular, $\lambda \in \sigma_p(A) \Leftrightarrow E_{\{\lambda\}} \neq 0$.

(iii) If λ is an isolated point in $\sigma(A)$, i.e. there exists an open neighborhood $B \subset \mathbb{R}$ of λ with $B \cap \sigma(A) = \{\lambda\}$, then λ is an eigenvalue of A.

Remark: $\lambda \in \sigma_c(A) \Leftrightarrow \forall$ neighborhoods B of $\{\lambda\}$ one has $E_B \neq 0$ and $\underbrace{E_{\{\lambda\}} = 0}_{(*)}$

$(*) \Leftrightarrow E_{(-\infty,\lambda]} - E_{(-\infty,\lambda)} = E_{\{\lambda\}} = 0$

i.e. E is continuous at λ

Proof:

(i) "$\Rightarrow$" Assume $\lambda \in \rho(A)$. Since $\rho(A)$ is open, there exists a neighborhood B of λ s.t. $B \subset \rho(A)$. Then we have $E_B = \mathbb{1}_{\underbrace{B \cap \sigma(A)}_{=\emptyset}}(A) = \mathbb{1}_{\emptyset}(A) = 0$ ✓

"$\Leftarrow$" Assume $\exists$ open neighborhood B of λ s.t.

$E_B = \mathbb{1}_{B \cap \sigma(A)}(A) = 0$. We define

$$g(t) := \begin{cases} \frac{1}{t-\lambda}, & t \in \sigma(A) \setminus B \\ 0, & t \in \sigma(A) \cap B \end{cases}$$

Since $E_B = 0$, we conclude $B \cap \sigma(A) = \emptyset$ and since $\lambda \in B$ and B is open, we have $\mathrm{dist}\{\lambda, \sigma(A)\} > 0$

$\Rightarrow g \in \mathcal{B}(\sigma(A))$. Consider $f(t) = t - \lambda \Rightarrow f \in \mathcal{B}(\sigma(A))$

$$\Rightarrow \underline{f(A) \cdot g(A)} = \hat{\Phi}(f) \cdot \hat{\Phi}(g) = \hat{\Phi}(f \cdot g) = \hat{\Phi}(\mathbb{1}_{\sigma(A) \setminus B})$$

$(A - \lambda)\, g(A)$

$$= \hat{\Phi}(\mathbb{1}_{\sigma(A)} - \mathbb{1}_{B \cap \sigma(A)}) = \underbrace{E_{\sigma(A)}}_{I} - \underbrace{E_B}_{=0} = \underline{I}$$

$$= \dots = \underline{g(A)(A - \lambda)}$$

$\Rightarrow A - \lambda$ is bijective $\Rightarrow \lambda \in \rho(A)$ ✓

(ii) Let $x \in \operatorname{ran} E_{\{\lambda\}} \Rightarrow \exists y \in \mathcal{H}: \underline{x} = E_{\{\lambda\}} y = E_{\{\lambda\}}^2 y = \underline{E_{\{\lambda\}} x}$

$$\Rightarrow \underline{(A - \lambda) x} = (A - \lambda) E_{\{\lambda\}} x = (A - \lambda)\, \mathbb{1}_{\sigma(A) \cap \{\lambda\}}(A) x$$

$$= \hat{\Phi}(\underbrace{(t - \lambda) \cdot \mathbb{1}_{\{\lambda\}}}_{=0}) = \underline{0} \quad \Rightarrow x \in \ker(A - \lambda)$$

$$\Rightarrow \operatorname{ran} E_{\{\lambda\}} \subset \ker(A - \lambda)$$

Conversely, assume $\underline{x \in \ker(A - \lambda)}$, i.e. $Ax = \lambda x$.

$\Rightarrow f(A)x = f(\lambda)x$ for any $f \in C(\sigma(A))$ [exercise]

Consider
$$V = \{ f \in \mathcal{B}(\sigma(A)) : f(A)x = f(\lambda)x \}$$

$\Rightarrow C(\sigma(A)) \subset V$. Let $(f_n) \subset V$ s.t. $\sup_n \|f_n\|_\infty < \infty$

and $f(t) := \lim_{n \to \infty} f_n(t)$ exists $\forall t \in \sigma(A)$.

$$\Rightarrow \underline{f(A)x} \overset{\text{Remark 4.4}}{=} \lim_{n \to \infty} f_n(A)x \overset{f_n \in V}{=} \lim_{n \to \infty} f_n(\lambda)x = \underline{f(\lambda)x}$$

$\Rightarrow f \in V$. By Lemma 4.6 we conclude $V = \mathcal{B}(\sigma(A))$.

In particular, for $f = \mathbb{1}_{\{\lambda\}} \in \mathcal{B}(\sigma(A))$ we get

$\underline{E_{\{\lambda\}} x} = \mathbb{1}_{\{\lambda\}}(A)x = \mathbb{1}_{\{\lambda\}}(\lambda)x = \underline{x} \Rightarrow \underline{x \in \operatorname{ran} E_{\{\lambda\}}}$

$\Rightarrow \ker(A - \lambda) \subset \operatorname{ran} E_{\{\lambda\}} \Rightarrow \ker(A - \lambda) = \operatorname{ran} E_{\{\lambda\}}$

(iii) Let B be open with $B \cap \sigma(A) = \{\lambda\} \Rightarrow \underbrace{\mathbb{1}_{B \setminus \{\lambda\}}}_{\subset \rho(A)}(A) = 0$

If $E_{\{\lambda\}} = 0 \Rightarrow E_{\{\lambda\}} + E_{B\setminus\{\lambda\}} = E_B = 0$, i.e. by (i) $\lambda \in \rho(A)$ ↯

$\Rightarrow E_{\{\lambda\}} \neq 0$, i.e. by (ii) $\lambda \in \sigma_p(A)$ $\square$

Theorem 4.17. Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to A. Moreover, let $B \in \Sigma$ and set $\mathcal{H}_B := \operatorname{ran} E_B$. Then the following holds:

$E_B = \mathbb{1}_{B \cap \sigma(A)}(A)$

(i) $A\mathcal{H}_B \subset \mathcal{H}_B$, $\mathcal{H}_B^\perp = \operatorname{ran} E_{\mathbb{R}\setminus B}$ and $A\mathcal{H}_B^\perp \subset \mathcal{H}_B^\perp$.

(ii) $A_B := A \restriction \mathcal{H}_B$ is bounded and self-adjoint in $\mathcal{H}_B$.

(iii) $(\sigma(A) \cap B^\circ) \subset \sigma(A_B) \subset (\sigma(A) \cap \overline{B})$.

In particular, $A = A_B \oplus A_{\mathbb{R}\setminus B}$.

i.e. $\mathcal{H}_B, \mathcal{H}_B^\perp$ are invariant subspaces for A

i.e. $A_B : \mathcal{H}_B \to \mathcal{H}_B$ is well-defined, $A_B = A_B^*$ and $A_B \in \mathcal{L}(\mathcal{H}_B)$

B° ... interior of B

Proof: exercise

In the following we use the statement, that for a fixed $x \in \mathcal{H}$ and any spectral measure the map

$\Sigma \ni B \mapsto (E_B x, x) \in \mathbb{R}$

is a Borel measure

4.6 Spectral theorem for unbounded self-adjoint operators

First, we discuss, how $f(A)$ can be constructed, if $A = A^* \in \mathcal{L}(\mathcal{H})$ and $f : \mathbb{R} \to \mathbb{C}$ is measurable, but unbounded.

Proposition 4.18. Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $f : \mathbb{R} \to \mathbb{C}$ be measurable. Define

$$f_n(\lambda) := f(\lambda)\mathbb{1}_{|f|\le n}(\lambda) = \begin{cases} f(\lambda), & \text{if } |f(\lambda)| \le n, \\ 0, & \text{if } |f(\lambda)| > n. \end{cases}$$

Then f_n is measurable and bounded for all $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} f_n(A)x$ exists, if and only if $x \in D_f := \{x \in \mathcal{H} : \int_{\mathbb{R}} |f(\lambda)|^2 d(E(\lambda)x, x) < \infty\}$. In particular,

$$f(A)x := \lim_{n\to\infty} f_n(A)x, \quad x \in \operatorname{dom} f(A) := D_f,$$

is a well-defined linear operator in $\mathcal{H}$. If f is real-valued, then $f(A)$ is self adjoint.

Notation: We set

$$\boxed{\int_{\mathbb{R}} f dEx} = \lim_{n\to\infty} f_n(A)x = \lim_{n\to\infty} \int_{\mathbb{R}} f_n dEx,$$

where the limit is w.r.t. the norm in $\mathcal{H}$, so that $\int_{\mathbb{R}} f dEx = f(A)x$ for all $x \in D_f$.

Note: for unbounded f $\int_{\mathbb{R}} f dE$ does not belong to $\mathcal{L}(\mathcal{H})$ in general

Proof:
First, note that for any $g \in \mathcal{B}(\mathbb{R})$ and all $x \in \mathcal{H}$:

$$\|g(A)x\|^2 = \int_{\mathbb{R}} |g(\lambda)|^2 d(E(\lambda)x, x) \qquad (*)$$

(exercise)

Assume that $x \in D_f$, i.e. $\int_{\mathbb{R}} |f(\lambda)|^2 d(E(\lambda)x,x) < \infty$.

$(*)$ for $g = f_n - f_m$ reads

$$(**) \quad \|f_n(A)x - f_m(A)x\|^2 = \int_{\mathbb{R}} |f_n(\lambda) - f_m(\lambda)|^2 d(E(\lambda)x,x)$$

Since $f \in D_f$, we have that $|f|^2 \in L^1(\mathbb{R}, (Ex,x))$.

Moreover, by definition of f_n we have

- $|f_n(\lambda) - f_m(\lambda)|^2 \to 0$ f.a.e. $\lambda \in \mathbb{R}$
- $|f_n(\lambda) - f_m(\lambda)|^2 \leq |f(\lambda)|^2$

By the dominated convergence theorem we have that the r.h.s. in $(**)$ converges to 0 and thus $(f_n(A) - f_m(A))x \to 0$ in $\mathcal{H}$, i.e. $(f_n(A)x)$ is a Cauchy-sequence, i.e. $\lim\limits_{n\to\infty} f_n(A)x$ exists ✓

Conversely, assume that $\lim\limits_{n\to\infty} f_n(A)x$ exists.
By $(**)$ (f_n) is a Cauchy sequence in $L^2(\mathbb{R}, (E x, x))$
The limit element must coincide with the pointwise limit $f \Rightarrow f \in L^2(\mathbb{R}, (Ex, x))$

$$\Rightarrow \int_{\mathbb{R}} |f(\lambda)|^2 d(Ex, x) < \infty \Rightarrow x \in D_f \quad ✓$$

In the following assume that f is real-valued

- $f(A)$ is symmetric. Let $x, y \in \operatorname{dom} f(A) = D_f$

$$\Rightarrow \underline{(f(A)x, y)} \overset{x \in D_f}{=} \lim_{n\to\infty} (f_n(A)x, y) \overset{\substack{f_n \in B(\mathbb{R}) \\ f_n \text{ real-valued}}}{=} \lim_{n\to\infty} (x, f_n(A)y)$$

$$\overset{y \in D_f}{=} \underline{(x, f(A)y)} \quad ✓$$

$\Rightarrow f(A) \subset f(A)^*$.
In order to show $f(A)^* \subset f(A)$, we verify $\operatorname{dom} f(A)^* \subset \operatorname{dom} f(A)$
Let $\underline{y \in \operatorname{dom} f(A)^*}$. Note: for any $\underline{x \in \mathcal{H}}$ the element $\mathbb{1}_{\{|f| \leq m\}}(A)x \in \operatorname{dom} f(A)$. Indeed, for sufficiently large $n \geq m$ one has $f_n(A)\mathbb{1}_{\{|f| \leq m\}}(A)x = \Phi(f \underbrace{\mathbb{1}_{\{|f| \leq n\}} \cdot \mathbb{1}_{\{|f| \leq m\}}}_{\mathbb{1}_{\{|f| \leq m\}}})x$

$= f_m(A)x$

Since $f_m(A)x$ is independent of n, we have
$\lim\limits_{n\to\infty} f_n(A)\mathbb{1}_{\{|f| \leq m\}}(A)x = \lim\limits_{n\to\infty} f_m(A)x = f_m(A)x$

From this, we get

$$\underline{\left(\mathbb{1}_{\{|f|\le m\}}(A)\, f(A)^* y,\ x \right)} = \left(f(A)^* y,\ \underbrace{\mathbb{1}_{\{|f|\le m\}}(A) x}_{\in \operatorname{dom} f(A)} \right)$$

$$= \Big(y,\ \underbrace{f(A)\, \mathbb{1}_{\{|f|\le m\}}(A)}_{f_m(A)}\, x \Big) = \left(y, f_m(A) x \right)$$

$f_m(A) = f_m(A)^* \in \mathcal{L}(\mathcal{H})$

$$\underset{=}{} \quad \underline{\left(f_m(A) y, x \right)}$$

$(\square)$

$$\Rightarrow \underline{\mathbb{1}_{\{|f|\le m\}}(A)\, f(A)^* y} = f_m(A) y$$

Note $h_m(t) := \mathbb{1}_{\{|f|\le m\}}(t) \Rightarrow h_m(t) \to 1 \quad \forall t \in \mathbb{R}$

$\sup_m \|h_m\|_\infty = 1 < \infty$

Hence, by Remark 4.9 [condition (c')] we get

$$\lim_{m\to\infty} h_m(A)\, f(A)^* y \overset{(\square)}{=} \lim_{m\to\infty} f_m(A) y \quad \text{exists}$$

$\Rightarrow y \in D_f = \operatorname{dom} f(A) \quad \Rightarrow \operatorname{dom} f(A)^* \subset \operatorname{dom} f(A)$

$\Rightarrow f(A)$ is selfadjoint [since we already know that $f(A)$ is symmetric!] $\square$