Theorem 5.5 (Wüst)**.** Let $\overline{A} = A^*$ in $\mathcal{H}$ and let V be a symmetric operator in $\mathcal{H}$ such that $\operatorname{dom} A \subset \operatorname{dom} V$. If there exists an $a \geq 0$ such that $\|Vx\| \leq a\|x\| + \|Ax\|$ holds for all $x \in \operatorname{dom} A$, then $A + V$ is essentially self adjoint.

Caution: The condition in Wüst's theorem is not equivalent to

$\|Vx\| \leq a\|x\| + b\|Ax\|$

"V is A-bounded with A-bound 1."

A-bound ~ inf b

Under the last condition, the statement of the theorem is not true in general!

Definition 5.6. A self adjoint operator A in $\mathcal{H}$ is called *semibounded from below*, if there exists a $\gamma \in \mathbb{R}$ such that

$$(Ax, x) \geq \gamma\|x\|^2$$

holds for all $x \in \operatorname{dom} A$. Each such γ is called *lower bound of* A and we write $A \geq \gamma$ in this case.

Lemma 5.7. Let A be a self adjoint operator in $\mathcal{H}$. Then $A \geq \gamma$ if and only if $\sigma(A) \subset [\gamma, \infty)$.

Proof:

"$\Rightarrow$" easy

"$\Leftarrow$" For $x \in \operatorname{dom} A$

$$((A-\gamma)x, x) = \int_{\mathbb{R}} (t-\gamma)\, d(Ex, x)$$

$\sigma(A) \subset [\gamma, \infty)$

$\Rightarrow E_{(-\infty, \gamma)} = 0$

(Theorem 4.16)

$$= \int_{\gamma}^{\infty} \underbrace{(t-\gamma)}_{\geq 0}\, d(E(t)x, x) \geq 0$$

$\square$

Theorem 5.8. Let $A = A^*$ in $\mathcal{H}$ be bounded from below, $A \geq \gamma_A$. Assume that V is a symmetric operator in $\mathcal{H}$ that is A-bounded with A-bound less than one, i.e. there exist $a \geq 0$ and $b \in (0,1)$ such that

$$\|Vx\| \leq a\|x\| + b\|Ax\|$$

holds for all $x \in \operatorname{dom} A$. Then $A + V$ is bounded from below and

s.a. by Kato Rellich

$$\gamma := \gamma_A - \max\left\{\frac{a}{1-b}, a + b|\gamma_A|\right\}$$

is a lower bound for $A + V$.

Proof:

By Kato-Rellich we know that $A+V$ is self adjoint.
Let γ be defined as in the theorem. By Lemma 5.7 it suffices to show that $(-\infty, \gamma) \subset \rho(A+V)$.
For that we use Lemma 5.3 and prove for $\lambda < \gamma \leq \gamma_A$ that

$$\|V(A-\lambda)^{-1}\| < 1 \qquad (*)$$

To show $(*)$ we note first

$$\|V(A-\lambda)^{-1}x\| \leq a\|(A-\lambda)^{-1}x\| + b\|A(A-\lambda)^{-1}x\| \quad \forall x \in \mathcal{H}$$

as $(A-\lambda)^{-1}x \in \operatorname{dom} A$ for all $x \in \mathcal{H}$.

$$\Rightarrow \underline{\|V(A-\lambda)^{-1}\|} \le a\,\|(A-\lambda)^{-1}\| + b\,\|A(A-\lambda)^{-1}\|$$

spectral theorem

$$= a\,\Big\|\underbrace{\int_{\gamma_A}^{\infty} \frac{1}{t-\lambda}\, dE(t)}_{=(A-\lambda)^{-1}}\Big\| + b\,\Big\|\underbrace{\int_{\gamma_A}^{\infty} \frac{t}{t-\lambda}\, dE(t)}_{A(A-\lambda)^{-1}}\Big\|$$

Lemma 4.13

$$\le a \sup_{\gamma_A \le t < \infty} \left|\frac{1}{t-\lambda}\right| + b \sup_{\gamma_A \le t<\infty} \left|\underbrace{\frac{t-\lambda+\lambda}{t-\lambda}}_{1+\frac{\lambda}{t-\lambda}}\right|$$

$$= \underline{a\,\frac{1}{\gamma_A-\lambda} + b\max\left\{1, \frac{|\gamma_A|}{\gamma_A-\lambda}\right\}} \qquad (\square)$$

Since $\lambda < \gamma$ and due to the def. of γ we claim the rhs < 1: In fact:

$$\lambda < \gamma = \gamma_A - \max\left\{\frac{a}{1-b},\ a+b|\gamma_A|\right\}$$

$$\Rightarrow \lambda < \gamma_A - \frac{a}{1-b} \Rightarrow \frac{a}{1-b} < \gamma_A - \lambda \Rightarrow \frac{1-b}{a} > \frac{1}{\gamma_A-\lambda}$$

$$\Rightarrow 1-b > a\cdot\frac{1}{\gamma_A-\lambda} \Rightarrow \underline{1 > b + \frac{a}{\gamma_A-\lambda}} \qquad (1)$$

$$\Rightarrow \lambda < \gamma_A - (a+b|\gamma_A|) \Rightarrow a+b|\gamma_A| < \gamma_A - \lambda$$

$$\Rightarrow \underline{\frac{a+b|\gamma_A|}{\gamma_A-\lambda} < 1} \qquad (2)$$

From (1), (2), $(\square)$ we conclude $\|V(A-\lambda)^{-1}\| < 1$

$\Rightarrow$ By Lemma 5.3 that $\lambda \in \rho(A+V)$ $\forall \lambda < \gamma$
$\Rightarrow (-\infty, \gamma) \subset \rho(A+V)$ $\square$

5.2 Compact and finite dimensional perturbations

Definition 5.9. Let $A = A^*$ in $\mathcal{H}$. The *discrete spectrum* of A is defined by

$$\sigma_d(A) := \big\{\lambda \in \sigma_p(A) : \dim\ker(A-\lambda) < \infty \text{ and } \exists\varepsilon > 0 : (\lambda-\varepsilon, \lambda+\varepsilon) \cap \sigma(A) = \{\lambda\}\big\}.$$

The *essential spectrum* of A is

$$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A).$$

The discrete spectrum of A consists of all isolated eigenvalues with finite multiplicity and the essential spectrum of all eigenvalues with infinite multiplicity and all accumulation points of $\sigma(A)$. In particular, we have $\sigma_c(A) \subset \sigma_{ess}(A)$.

In the following we characterize points in the essential spectrum. For that we repeat two facts from basic functional analysis:

(i) A sequence $(x_n) \subset \mathcal{H}$ is called *weakly convergent* to $x \in \mathcal{H}$ (notation: $x_n \rightharpoonup x$), if for all $y \in \mathcal{H}$ the relation $(x_n, y) \to (x, y)$ holds for $n \to \infty$. E.g. by the Bessel inequality each infinite orthonormal system converges weakly to zero.

(ii) An operator $K \in \mathcal{L}(\mathcal{H})$ is called *compact* (notation $K \in \mathfrak{S}_\infty$), if it maps bounded sets onto relativly compact sets. This is equivalent to the fact that for any bounded sequence $(x_n) \subset \mathcal{H}$ there exists a subsequence (x_{n_k}) such that (Kx_{n_k}) is convergent in $\mathcal{H}$. Another equivalent condition is that $x_n \rightharpoonup x$ implies $Kx_n \to Kx$ in $\mathcal{H}$.

Recall that any operator with $\dim\operatorname{ran} K < \infty$ is compact. Moreover, if $K \in \mathfrak{S}_\infty$ and $A \in \mathcal{L}(\mathcal{H})$, then $AK \in \mathfrak{S}_\infty$ and $KA \in \mathfrak{S}_\infty$.

Proposition 5.10. Let $A = A^*$ in $\mathcal{H}$ and let $\lambda \in \mathbb{R}$. Then the following is equivalent:

(i) $\lambda \in \sigma_{ess}(A)$;

(ii) $\exists (x_n) \subset \operatorname{dom} A$ with $\|x_n\| = 1$, $x_n \rightharpoonup 0$ and $(A-\lambda)x_n \to 0$ (such a sequence (x_n) is called singular sequence);

(iii) $\dim\operatorname{ran} E_{(\lambda-\varepsilon,\lambda+\varepsilon)} = \infty$ for all $\varepsilon > 0$.