# 3 Symmetric and self-adjoint operators

In this chapter $\mathcal{H}$ is always a Hilbert space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with scalar product $(\cdot, \cdot)$ and induced norm $\| \cdot \|$.

**Definition 3.1.** Let $S$ be a densely defined operator in $\mathcal{H}$, i.e. $\overline{\operatorname{dom} S} = \mathcal{H}$. Then the *adjoint operator* $S^*$ of $S$ is defined by

$$\begin{aligned} \operatorname{dom} S^* &= \{g \in \mathcal{H} : \exists g' \in \mathcal{H} : (Sf, g) = (f, g') \,\forall f \in \operatorname{dom} S\}, \\ S^* g &= g'. \end{aligned}$$

In the following $\mathcal{H} \times \mathcal{H}$ is endowed with the inner product

$$\big((f, f'), (g, g')\big) := (f, g) + (f', g'), \quad (f, f'), (g, g') \in \mathcal{H} \times \mathcal{H},$$

and we denote by $()^{\perp}$ the orthogonal complement in $\mathcal{H} \times \mathcal{H}$ w.r.t. the above inner product.

**Lemma 3.2.** Define the operator $\mathcal{U} : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$,

$$\mathcal{U}(h, h') := (h', -h), \qquad (h, h') \in \mathcal{H} \times \mathcal{H}.$$

Then for any densely defined operator $S$ in $\mathcal{H}$ one has $\mathcal{G}(S^*) = (\mathcal{U}\mathcal{G}(S))^{\perp} = \mathcal{U}(\mathcal{G}(S))^{\perp}$.

**Proposition 3.3.** Let $S$ be a densely defined operator in $\mathcal{H}$. Then the following holds:

(i) $S^* \in \mathcal{C}(\mathcal{H})$.

(ii) S is closable $\Leftrightarrow$ $\operatorname{dom} S^*$ is dense in $\mathcal{H}$. In this case one has

$$(\overline{S})^* = S^* \quad \text{and} \quad \overline{S} = S^{**}.$$

(iii) $S \subset T \Rightarrow T^* \subset S^*$.

**Lemma 3.4.** Let $S$ be a densely defined operator in $\mathcal{H}$. Then one has for any $\lambda \in \mathbb{K}$

(i) $\big(\operatorname{ran}(S - \lambda)\big)^{\perp} = \ker(S^* - \overline{\lambda})$ and

(ii) $\overline{\operatorname{ran}(S - \lambda)} = \big(\ker(S^* - \overline{\lambda})\big)^{\perp}$.

**Definition 3.5.** A densely defined operator $S$ is called

(i) *symmetric*, if $S \subset S^*$;

(ii) *self adjoint*, if $S = S^*$;

(iii) *essentially self adjoint*, if $\overline{S}$ is self adjoint, i.e. if $\overline{S} = S^*$.

**Lemma 3.6.** Let $S$ be a densely defined operator in $\mathcal{H}$. Then the following are equivalent:

(i) $S$ is symmetric.

(ii) $(Sf, g) = (f, Sg)$ for all $f, g \in \operatorname{dom} S$.

If $\mathbb{K} = \mathbb{C}$, then (i) and (ii) are equivalent to

(iii) $(Sf, f) \in \mathbb{R}$ for all $f \in \operatorname{dom} S$.

**Lemma 3.7.** (i) Each symmetric operator $S$ is closable and $\overline{S}$ is also symmetric.

(ii) Each self adjoint operator is closed.

**Proposition 3.8.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let $S$ be symmetric and closed. Then the following holds:

(i) $\mathbb{C} \setminus \mathbb{R} \subset r(S)$ and $\operatorname{ran}(S - \lambda)$ is closed for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(ii) $\sigma_p(S) \cup \sigma_c(S) \subset \mathbb{R}$.

(iii) For all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ one has $\|(S - \lambda)^1\| \leq \frac{1}{|\operatorname{Im} \lambda|}$.

**Lemma 3.9.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let $S \subset S^*$. If $\operatorname{ran}(S - \lambda) = \mathcal{H}$ for a $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $S \in \mathcal{C}(\mathcal{H})$.

**Theorem 3.10.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$, let $S$ be a symmetric operator in $\mathcal{H}$, and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the following are equivalent:

(i) $S$ is self adjoint.

(ii) $S \in \mathcal{C}(\mathcal{H})$ and $\ker(S^* - \lambda) = \{0\} = \ker(S^* - \overline{\lambda})$.

(iii) $\operatorname{ran}(S - \lambda) = \mathcal{H} = \operatorname{ran}(S - \overline{\lambda})$.

(iv) $S \in \mathcal{C}(\mathcal{H})$ and $\lambda, \overline{\lambda} \in \rho(S)$.

**Remark:** If one of the assertions (ii), (iii) or (iv) from Theorem 3.10 hold for one $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then due to their equivalence to (i) these assertions hold for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, see also the proof.

**Proposition 3.11.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ and let $S$ be a self adjoint operator in $\mathcal{H}$. Then the following holds:

(i) $\sigma(S) \subset \mathbb{R}$ and $\sigma_{\mathrm{r}}(S) = \emptyset$.

(ii) $\lambda \in \sigma(S) \Leftrightarrow$ there exists a sequence $(x_n)_n \subset \operatorname{dom} S$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ such that $\|(S - \lambda)x_n\| \to 0$ for $n \to \infty$.

**Example 3.12.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and define the operator $T : L^2(\mathbb{R}) \supset \operatorname{dom} T \to L^2(\mathbb{R})$ by

$$(Tg)(x) = f(x)g(x) \quad \text{for } x \in \mathbb{R}, \quad g \in \operatorname{dom} T := \{g \in L^2(\mathbb{R}) : fg \in L^2(\mathbb{R})\}.$$

Then one has:

(i) $T = T^*$.

(ii) $\sigma(T) = \overline{\{f(x) : x \in \mathbb{R}\}}$ and $T$ is bounded, if and only if $f$ is bounded.

(iii) $\sigma_{\mathrm{p}}(T) = \{\mu \in \mathbb{R} : |f^{-1}(\{\mu\})| > 0\}$.

# 4 Spectral theorem for self adjoint operators

Throughout the following section $\mathcal{H}$ is always a Hilbert space over $\mathbb{K} = \mathbb{C}$.

## 4.1 Motivation and preliminaries

Let

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \in \mathbb{C}^{n\times n}$$

be a self adjoint matrix with eigenvalues $\lambda_1 < \lambda_2 \cdots < \lambda_n$. The orthogonal projections onto the corresponding eigenspaces are given by

$$E(\{\lambda_1\}) := \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad \dots, \quad E(\{\lambda_n\}) := \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix}.$$

With these projections one can write

$$A = \sum_{k=1}^{n} \lambda_k E(\{\lambda_k\}) = \int_{\mathbb{R}} \mu dE(\mu),$$

where the integral is with respect to the measure $E$ which has point masses at $\lambda_1, \dots, \lambda_n$. With the help of this measure one gets for any open interval $\Delta \subset \mathbb{R}$

$$E(\Delta) = \sum_{\lambda_k \in \Delta} E(\{\lambda_k\}) = \mathbb{1}_{\Delta}(A).$$

**Goal:** We want to show that for any $A = A^* \in \mathcal{L}(\mathcal{H})$ there exists a *spectral measure* $E$ (which will be an orthogonal projection for each Borel set) such that

$$A = \int_{\mathbb{R}} \mu dE(\mu) = \int_{\sigma(A)} \mu dE(\mu).$$

**Idea:** Set $E(\Delta) = \mathbb{1}_{\Delta}(A)$ for any interval $\Delta \subset \mathbb{R}$. But how can $\mathbb{1}_{\Delta}(A)$ be understood and introduced? A function of an operator can be defined, if the function is a polynomial:

**Definition 4.1.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and $p(t) = \sum_{k=0}^{n} a_k t^k$ be a polynomial on $\mathbb{R}$ with complex coefficients $a_0, \dots, a_n$. Then $p(A)$ is defined by

$$p(A) = \sum_{k=0}^{n} a_k A^k.$$

**Lemma 4.2.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $p : \mathbb{R} \to \mathbb{C}$ be a polynomial. Then one has

$$\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.$$

## 4.2 The continuous functional calculus for self adjoint operators

Throughout this section we assume that $A$ is a bounded and self-adjoint operator, i.e. $A = A^* \in \mathcal{L}(\mathcal{H})$. The goal is to define the operator $f(A)$ for any continuous function $f$. Denote by $C(\sigma(A))$ the set of all continuous functions $f : \sigma(A) \to \mathbb{C}$ equipped with the norm

$$\|f\|_\infty := \sup_{x \in \sigma(A)} |f(x)|, \qquad f \in C(\sigma(A)).$$

By $P(\sigma(A))$ we denote the space of all polynomials defined on $\sigma(A)$. By the Weierstrass approximation theorem (see e.g. [Werner, Satz VIII.4.7]) we have that $P(\sigma(A))$ is dense in $C(\sigma(A))$.

**Theorem 4.3.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$. Then the map

$$P(\sigma(A)) \ni p \mapsto p(A) \in \mathcal{L}(\mathcal{H})$$

is linear and isometric, i.e. $\|p(A)\| = \|p\|_\infty$ for all $p \in P(\sigma(A))$, and hence it has a unique isometric (and thus bounded) linear extension $\Phi : C(\sigma(A)) \to \mathcal{L}(\mathcal{H})$, which has the following properties:

(a) $\Phi$ is multiplicative, i.e. $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in C(\sigma(A))$.

(b) $\Phi$ is an involution, i.e. $\Phi(\overline{f}) = \Phi(f)^*$ for all $f \in C(\sigma(A))$.

Of course, the map $\Phi$ depends on the initially given operator $A$. We write

$$f(A) := \Phi(f), \qquad f \in C(\sigma(A)).$$

Due to the previous theorem we have $\|f(a)\| = \|f\|_\infty$ for all $f \in C(\sigma(A))$. The map $\Phi$ is called *continuous functional calculus* for $A$ (the word continuous is associated to the fact that it applies to continuous functions).

**Proposition 4.4.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$. Then the following holds for all $f, g \in C(\sigma(A))$.

(i) $f(A)g(A) = g(A)f(A)$.

(ii) If $f(t) \geq 0$ for all $t \in \sigma(A)$, then $f(A) \geq 0$ in the sense of self adjoint operators (i.e. $(f(A)x, x) \geq 0$ for all $x \in \mathcal{H}$).

(iii) $f(A)$ is a normal operator and $f(A) = f(A)^*$ if and only if $f$ is real-valued.

(iv) $Ax = \lambda x$ implies $f(A)x = f(\lambda)x$.

*Beweis.* See exercises. $\square$

**Theorem 4.5** (Spectral mapping theorem)**.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$. Then one has for all $f \in C(\sigma(A))$

$$\sigma(f(A)) = f(\sigma(A)).$$

## 4.3 The measurable functional calculus

Again, we assume throughout this section that $A = A^* \in \mathcal{L}(\mathcal{H})$. The goal in this section is to extend the continuous functional calculus from the last section for bounded and measurable functions, i.e. to define $f(A)$ for any bounded and measurable function $f : \sigma(A) \to \mathbb{C}$. We set for any compact set $K \subset \mathbb{C}$

$$B(K) := \{f : K \to \mathbb{C} : f \text{ is measurable and bounded}\},$$

which is endowed with the norm $\|\cdot\|_\infty$ a Banach space. The following elementary lemma, which can be found e.g. in [Werner, Lemma VII.1.5], will be very useful in our constructions:

**Lemma 4.6.** Let $V \subset B(K)$ such that the following holds:

(i) $C(K) \subset V$.

(ii) For any sequence $(f_n) \subset V$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in K$ imply that $f \in V$.

Then $V = B(K)$.

The previous lemma means, roughly speaking, that $B(K)$ is the smallest set of functions, which contains all continuous functions and which is closed with respect to pointwise limits of uniformyl bounded sequences.

In order to formulate the next result, recall that a *complex Borel measure* over $\sigma(A)$ is a map $\mu : \Sigma(\sigma(A)) \to \mathbb{C}$, which is $\sigma$-additive (here $\Sigma(\sigma(A))$ is the Borel-$\sigma$-algebra over $\sigma(A)$).

**Lemma 4.7.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. Then there exists a complex Borel measure $\mu_{x,y}$ such that

$$\big(f(A)x, y\big) = \int_{\sigma(A)} f d\mu_{x,y} \qquad \forall f \in C(\sigma(A)).$$

For any $f \in C(\sigma(A))$ one has

$$\left| \int_{\sigma(A)} f d\mu_{x,y} \right| \leq \|f\|_\infty \|x\| \cdot \|y\|.$$

**Theorem 4.8.** For $A = A^* \in \mathcal{L}(\mathcal{H})$ there exists a unique linear and bounded mapping $\widehat{\Phi} : B(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ with the following properties:

(a) $\widehat{\Phi}(p) = p(A)$ for all $p \in P(\sigma(A))$.

(b) $\widehat{\Phi}$ is multiplicative and an involution.

(c) For any sequence $(f_n) \subset B(\sigma(A))$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in \sigma(A)$ imply that

$$\big(\widehat{\Phi}(f_n)x, y\big) \to \big(\widehat{\Phi}(f)x, y\big) \qquad \forall x, y \in \mathcal{H}.$$

Moreover, for all $f \in C(\sigma(A))$ one has $\widehat{\Phi}(f) = \Phi(f)$.

As for the continuous functional calculus we set for $f \in B(\sigma(A))$

$$f(A) := \widehat{\Phi}(f).$$

The map $\widehat{\Phi}$ is called *measurable functional calculus*.

**Remark 4.9.** Condition (c) in Theorem 4.8 can be improved in the following way:

(c') For any sequence $(f_n) \subset B(\sigma(A))$ the conditions $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ and $f(t) := \lim_{n\to\infty} f_n(t)$ exist for all $t \in \sigma(A)$ imply that

$$\widehat{\Phi}(f_n)x \to \widehat{\Phi}(f)x \qquad \forall x \in \mathcal{H}.$$

**Recall:** $P \in \mathcal{L}(\mathcal{H})$ is called *orthogonal projection*, if $P = P^* = P^2$. This is equivalent to the fact that $\operatorname{ran} P$ is closed and that $P$ is the orthogonal projection in $\mathcal{H}$ onto $\operatorname{ran} P$.

**Lemma 4.10.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$. Then the following is true:

(i) $\mathbb{1}_B(A)(:= \widehat{\Phi}(\mathbb{1}_B))$ is an orthogonal projection for any Borel set $B \subset \sigma(A)$.

(ii) $\mathbb{1}_\emptyset(A) = 0$ and $\mathbb{1}_{\sigma(A)}(A) = I$.

(iii) For any family of pairwise disjoint Borel sets $B_1, B_2, \dots \subset \sigma(A)$ and all $x \in \mathcal{H}$ one has

$$\sum_{k=1}^{\infty} \mathbb{1}_{B_k}(A)x = \mathbb{1}_{\cup_{k=1}^{\infty} B_k}(A)x.$$

(iv) For any two Borel sets $B_1, B_2 \subset \sigma(A)$ one has $\mathbb{1}_{B_1}(A)\mathbb{1}_{B_2}(A) = \mathbb{1}_{B_1 \cap B_2}(A)$.

## 4.4 Spectral measures and integration

Throughout this section $\Sigma$ is the Borel $\sigma$-algebra on $\mathbb{R}$.

**Definition 4.11.** A map $E : \Sigma \to \mathcal{L}(\mathcal{H})$, $B \mapsto E_B$, is called *spectral measure*, if $E_B$ is an orthogonal projection for all $B \in \Sigma$ and if the following holds:

(i) $E_\emptyset = 0$ and $E_\mathbb{R} = I$.

(ii) For all pairwise disjoint sets $B_1, B_2, \dots \in \Sigma$ and all $x \in \mathcal{H}$ one has the following $\sigma$-additivity:

$$\sum_{k=1}^{\infty} E_{B_k} x = E_{\cup_{k=1}^{\infty} B_k} x.$$

A spectral measure $E$ has *compact support*, if there exists a compact set $K \subset \mathbb{R}$ such that $E_K = I$.

**Properties of spectral measures:**

(a) Finite additivity: $E_{B_1} + E_{B_2} = E_{B_1 \cup B_2}$ for all disjoint $B_1, B_2 \in \Sigma$.

(b) $E_{B_1} E_{B_2} = E_{B_1 \cap B_2}$ for all Borel sets $B_1, B_2$.

**Corollary 4.12.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $\widehat{\Phi}$ be the associated measurable calculus. Then the map

$$E : \Sigma \to \mathcal{L}(\mathcal{H}), \quad B \mapsto E_B := \mathbb{1}_{B\cap\sigma(A)}(A) = \widehat{\Phi}(\mathbb{1}_{B\cap\sigma(A)})$$

is a spectral measure with compact support. The above map $E$ is called *spectral measure associated to $A$.*

**Integration with respect to spectral measures**

In the following let $E$ be a fixed spectral measure.

**Step 1: integration of simple functions:** Let $f = \sum_{k=1}^n \alpha_k \mathbb{1}_{B_K}$ for $\alpha_k \in \mathbb{C}$ and pairwise disjoint sets $B_k \in \Sigma$, $k \in \{1, \dots, n\}$. Then we define

$$\int_{\mathbb{R}} f dE := \sum_{k=1}^n \alpha_k E_{B_k}.$$

One verifes that the above definition of the integral is independent of the representation of $f$.

**Step 2: bounded and measurable functions**

**Lemma 4.13.** Let $E$ be a spectral measure. Then one has

$$\left\| \int_{\mathbb{R}} f dE \right\| \leq \|f\|_\infty$$

for all simple functions $f$. Inparticular, the map $f \mapsto \int_{\mathbb{R}} f dE$, defined on the set of all simple functions, is a bounded, densely defined, and linear operator from $B(\mathbb{R})$ to $\mathcal{L}(\mathcal{H})$.

**Consequence:** There exists a unique continuation of the integral with respect to $E$ to the space $B(\mathbb{R})$, which is again a bounded linear map. For an arbitrary $f \in B(\mathbb{R})$ we denote this extension applied to $f$ by $\int_{\mathbb{R}} f dE \in \mathcal{L}(\mathcal{H})$ (or sometimes $\int_{\mathbb{R}} f(t) dE(t)$) and this operator is defined by

$$\int_{\mathbb{R}} f dE := \lim_{n\to\infty} \int_{\mathbb{R}} f_n dE,$$

where $(f_n)$ is any sequence of simple functions with $\|f_n - f\|_\infty$, as $n \to \infty$.

If $f$ is defined on $\sigma(A)$, then we identify this function with its zero continuation.

## 4.5 Spectral theorem for bounded self adjoint operators

**Theorem 4.14** (Spectral theorem for bounded self-adjoint operators)**.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$, let $\widehat{\Phi}$ be the measurable functional calculus, and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to $A$. Then

$$\widehat{\Phi}(f) = \int_{\mathbb{R}} f dE$$

holds for all $f \in B(\sigma(A))$. In particular, one has

$$A = \int_{\mathbb{R}} t dE(t).$$

**Theorem 4.15** (Inversion of the spectral theorem)**.** Let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be a spectral measure with compact support. Then

$$A := \int_{\mathbb{R}} t dE(t)$$

defines a self adjoint operator in $\mathcal{L}(\mathcal{H})$ and the measurable functional calculus $\widehat{\Phi}$ associated to $A$ satisfies

$$\widehat{\Phi}(f) = \int_{\mathbb{R}} f dE \qquad \forall f \in B(\mathbb{R}).$$

Without proof.

**Theorem 4.16.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to $A$. Then the following holds for any $\lambda \in \mathbb{R}$:

(i) $\lambda \in \rho(A) \Leftrightarrow \exists$ an open neighborhood $B \subset \mathbb{R}$ of $\lambda$ with $E_B = 0$.

(ii) $\operatorname{ran} E_{\{\lambda\}} = \ker(A - \lambda)$. In particular, $\lambda \in \sigma_p(A) \Leftrightarrow E_{\{\lambda\}} \neq 0$.

(iii) If $\lambda$ is an isolated point in $\sigma(A)$, i.e. there exists an open neighborhood $B \subset \mathbb{R}$ of $\lambda$ with $B \cap \sigma(A) = \{\lambda\}$, then $\lambda$ is an eigenvalue of $A$.

**Theorem 4.17.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $E : \Sigma \to \mathcal{L}(\mathcal{H})$ be the spectral measure associated to $A$. Moreover, let $B \in \Sigma$ and set $\mathcal{H}_B := \operatorname{ran} E_B$. Then the following holds:

(i) $A\mathcal{H}_B \subset \mathcal{H}_B$, $\mathcal{H}_B^\perp = \operatorname{ran} E_{\mathbb{R}\setminus B}$ and $A\mathcal{H}_B^\perp \subset \mathcal{H}_B^\perp$.

(ii) $A_B := A \restriction \mathcal{H}_B$ is bounded and self-adjoint in $\mathcal{H}_B$.

(iii) $(\sigma(A) \cap B^\circ) \subset \sigma(A_B) \subset (\sigma(A) \cap \overline{B})$.

In particular, $A = A_B \oplus A_{\mathbb{R}\setminus B}$.

## 4.6 Spectral theorem for unbounded self-adjoint operators

First, we discuss, how $f(A)$ can be constructed, if $A = A^* \in \mathcal{L}(\mathcal{H})$ and $f : \mathbb{R} \to \mathbb{C}$ is measurable, but unbounded.

**Proposition 4.18.** Let $A = A^* \in \mathcal{L}(\mathcal{H})$ and let $f : \mathbb{R} \to \mathbb{C}$ be measurable. Define

$$f_n(\lambda) := f(\lambda)\mathbb{1}_{|f|\leq n}(\lambda) = \begin{cases} f(\lambda), & \text{if } |f(\lambda)| \leq n, \\ 0, & \text{if } |f(\lambda)| > n. \end{cases}$$

Then $f$ is measurable and bounded for all $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty} f_n(A)x$ exists, if and only if $x \in D_f := \left\{x \in \mathcal{H} : \int_{\mathbb{R}} |f(\lambda)|^2 d(E(\lambda)x, x) < \infty\right\}$. In particular,

$$f(A)x := \lim_{n\to\infty} f_n(A)x, \quad x \in \operatorname{dom} f(A) := D_f,$$

is a well-defined linear operator in $\mathcal{H}$. If $f$ is real-valued, then $f(A)$ is self adjoint.

**Notation:** We set

$$\int_{\mathbb{R}} f dEx := \lim_{n\to\infty} f_n(A)x = \lim_{n\to\infty} \int_{\mathbb{R}} f_n dEx,$$

where the limit is w.r.t. the norm in $\mathcal{L}(\mathcal{H})$, so that $\int_{\mathbb{R}} f dEx = f(A)x$ for all $x \in D_f$.

**Theorem 4.19.** Spectral theorem for self adjoint operators Let $A : \mathcal{H} \supset \operatorname{dom} A \to \mathcal{H}$ be self adjoint. Then there exists a spectral measure such that

$$Ax = \int_{\mathbb{R}} t dE(t)x \qquad \forall x \in \operatorname{dom} A.$$

If $h : \mathbb{R} \to \mathbb{R}$ is measurable, then

$$h(A)x := \int_{\mathbb{R}} h dEx, \qquad \operatorname{dom} h(A) = \left\{ x \in \mathcal{H} : \int_{\mathbb{R}} |h|^2 d(Ex, x) < \infty \right\},$$

defines a self adjoint operator in $\mathcal{H}$.

The integral in the definition of $h(A)$ has to be understood as in Proposition 4.18, so $\int_{\mathbb{R}} h dEx = \lim_{n\to\infty} \int_{\mathbb{R}} h_n dEx$.

We remark that versions of Theorem 4.16 and Theorem 4.17 hold for unbounded $A$ as well.

# 5 Perturbation theory for self adjoint operators

Throughout this section $\mathcal{H}$ is a complex Hilbert space with inner product $(\cdot, \cdot)$ and induced norm $\|\cdot\|$.

## 5.1 Relatively bounded perturbations

**Definition 5.1.** Let $A$ and $V$ be linear operators in $\mathcal{H}$. Then $V$ is called $A$-*bounded* (or *relatively bounded with respect to* $A$), if $\operatorname{dom} A \subset \operatorname{dom} V$ and if there exist $a, b \geq 0$ such that

$$\|Vx\| \leq a\|x\| + b\|Ax\|$$

holds for all $x \in \operatorname{dom} A$. The infimum over all $b$, such that there exists an $a$ so that the above inequality holds, is called $A$-*bound of* $V$.

**Remark:**

- If $V \in \mathcal{L}(\mathcal{H})$, then $V$ is $A$ bounded with $A$-bound zero.
- If $V$ is a bounded with $A$-bound $b$, then there exists for all $\varepsilon > 0$ a number $a_\varepsilon \geq 0$ such that
  $$\|Vx\| \leq a_\varepsilon\|x\| + (b + \varepsilon)\|Ax\|$$
  holds for all $x \in \operatorname{dom} A$. For $\varepsilon = 0$ this does not have to be the case!
- $V$ is $A$-bounded if and only if $\operatorname{dom} A \subset \operatorname{dom} V$ and there exist $\alpha, \beta \geq 0$ such that
  $$\|Vx\|^2 \leq \alpha\|x\|^2 + \beta\|Ax\|^2$$
  holds for all $x \in \operatorname{dom} A$. The infimum over all $\sqrt{\beta}$, such that there exists an $\alpha$ so that the above inequality holds, coincides with the $A$-bound of $V$ (see exercises).

**Proposition 5.2.** Let $A = A^*$ in $\mathcal{H}$ and let $V$ be a linear operator in $\mathcal{H}$ such that $\operatorname{dom} A \subset \operatorname{dom} V$. Set

$$c_\pm := \limsup_{\eta \to \pm\infty} \left\|V(A - i\eta)^{-1}\right\|$$

with $c_\pm = \infty$, if $V(A - i\eta)^{-1}$ is unbounded. Then

$$V \text{ is } A\text{-bounded} \qquad \Leftrightarrow \qquad c_+ < \infty \qquad \Leftrightarrow \qquad c_- < \infty.$$

In this case one has $c_+ = c_-$ is the $A$-bound of $V$ and the limit superior is a limit.

**Lemma 5.3.** Let $A = A^*$ in $\mathcal{H}$ and let $V$ be a linear operator in $\mathcal{H}$ with $\operatorname{dom} A \subset \operatorname{dom} V$ such that $A + V$ is closed. If $\|V(A - \lambda)^{-1}\| < 1$ for some $\lambda \in \rho(A)$, then $\lambda \in \rho(A + V)$.

**Theorem 5.4** (Kato-Rellich)**.** Let $A$ be a linear operator in $\mathcal{H}$ and let $V$ be a symmetric operator in $\mathcal{H}$ that is $A$-bounded with $A$-bound less than one. Then, the following is true:

(i) If $A = A^*$, then $(A + V)^* = A + V$, i.e. $A + V$ is self adjoint.

(ii) If $\overline{A} = A^*$, then $(A + V)^* = \overline{A + V}$, i.e. $A + V$ is essentially self adjoint.

**Theorem 5.5** (Wüst)**.** Let $\overline{A} = A^*$ in $\mathcal{H}$ and let $V$ be a symmetric operator in $\mathcal{H}$ such that $\operatorname{dom} A \subset \operatorname{dom} V$. If there exists an $a \geq 0$ such that $\|Vx\| \leq a\|x\| + \|Ax\|$ holds for all $x \in \operatorname{dom} A$, then $A + V$ is essentially self adjoint.

**Caution:** The condition in Wüst's theorem is not equivalent to

"$V$ is $A$-bounded with $A$-bound 1."

Under the last condition, the statement of the theorem is not true in general!

**Definition 5.6.** A self adjoint operator $A$ in $\mathcal{H}$ is called *semibounded from below*, if there exists a $\gamma \in \mathbb{R}$ such that

$$(Ax, x) \geq \gamma\|x\|^2$$

holds for all $x \in \operatorname{dom} A$. Each such $\gamma$ is called *lower bound of* $A$ and we write $A \geq \gamma$ in this case.

**Lemma 5.7.** Let $A$ be a self adjoint operator in $\mathcal{H}$. Then $A \geq \gamma$ if and only if $\sigma(A) \subset [\gamma, \infty)$.

**Theorem 5.8.** Let $A = A^*$ in $\mathcal{H}$ be bounded from below, $A \geq \gamma_A$. Assume that $V$ is a symmetric operator in $\mathcal{H}$ that is $A$-bounded with $A$-bound less than one, i.e. there exist $a \geq 0$ and $b \in (0, 1)$ such that

$$\|Vx\| \leq a\|x\| + b\|Ax\|$$

holds for all $x \in \operatorname{dom} A$. Then $A + V$ is bounded from below and

$$\gamma := \gamma_A - \max\left\{\frac{a}{1-b}, a + b|\gamma_A|\right\}$$

is a lower bound for $A + V$.

## 5.2 Compact and finite dimensional perturbations

**Definition 5.9.** Let $A = A^*$ in $\mathcal{H}$. The *discrete spectrum* of $A$ is defined by

$$\sigma_d(A) := \big\{\lambda \in \sigma_p(A) : \dim\ker(A-\lambda) < \infty \text{ and } \exists\varepsilon > 0 : (\lambda-\varepsilon, \lambda+\varepsilon) \cap \sigma(A) = \{\lambda\}\big\}.$$

The *essential spectrum* of $A$ is

$$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_d(A).$$

The discrete spectrum of $A$ consists of all isolated eigenvalues with finite multiplicity and the essential spectrum of all eigenvalues with infinite multiplicity and all accumulation points of $\sigma(A)$. In particular, we have $\sigma_c(A) \subset \sigma_{ess}(A)$.

In the following we characterize points in the essential spectrum. For that we repeat two facts from basic functional analysis:

(i) A sequence $(x_n) \subset \mathcal{H}$ is called *weakly convergent* to $x \in \mathcal{H}$ (notation: $x_n \rightharpoonup x$), if for all $y \in \mathcal{H}$ the relation $(x_n, y) \to (x, y)$ holds for $n \to \infty$. E.g. by the Bessel inequality each infinite orthonormal system converges weakly to zero.

(ii) An operator $K \in \mathcal{L}(\mathcal{H})$ is called *compact* (notation $K \in \mathfrak{S}_\infty$), if it maps bounded sets onto relativly compact sets. This is equivalent to the fact that for any bounded sequence $(x_n) \subset \mathcal{H}$ there exists a subsequence $(x_{n_k})$ such that $(Kx_{n_k})$ is convergent in $\mathcal{H}$. Another equivalent condition is that $x_n \rightharpoonup x$ implies $Kx_n \to Kx$ in $\mathcal{H}$.

    Recall that any operator with $\dim\operatorname{ran} K < \infty$ is compact. Moreover, if $K \in \mathfrak{S}_\infty$ and $A \in \mathcal{L}(\mathcal{H})$, then $AK \in \mathfrak{S}_\infty$ and $KA \in \mathfrak{S}_\infty$.

**Proposition 5.10.** Let $A = A^*$ in $\mathcal{H}$ and let $\lambda \in \mathbb{R}$. Then the following is equivalent:

(i) $\lambda \in \sigma_{ess}(A)$;

(ii) $\exists (x_n) \subset \operatorname{dom} A$ with $\|x_n\| = 1$, $x_n \rightharpoonup 0$ and $(A-\lambda)x_n \to 0$ (such a sequence $(x_n)$ is called singular sequence);

(iii) $\dim\operatorname{ran} E_{(\lambda-\varepsilon,\lambda+\varepsilon)} = \infty$ for all $\varepsilon > 0$.

**Lemma 5.11.** Let $A = A^*$ in $\mathcal{H}$ and $\mu \in \rho(A)$. Then one has for $\lambda \neq \mu$ that $\lambda \in \sigma_{ess}(A)$ if and only if there exists a sequence $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1$, $x_n \rightharpoonup 0$ and

$$\big((A-\mu)^{-1} - (\lambda-\mu)^{-1}\big)x_n \to 0.$$

**Theorem 5.12** (Stability of the essential spectrum under compact perturbations)**.** Let $A = A^*$ and $B = B^*$ in $\mathcal{H}$. If

$$(A-\mu)^{-1} - (B-\mu)^{-1} \in \mathfrak{S}_\infty$$

holds for one (and hence for all) $\mu \in \rho(A) \cap \rho(B)$, then $\sigma_{ess}(A) = \sigma_{ess}(B)$.

**Remark 5.13.** In the above theorem $B$ is the perturbed operator (in Section 5.1 $B = H+V$). Under our assumptions, one can not find an answer to the question, if $V := B - A$ is compact (or the restriction of a compact operator), as $\operatorname{dom}(B-A) = \operatorname{dom} A \cap \operatorname{dom} B$ can be an arbitrarily small set for unbounded operators $A$ and $B$. Hence, one investigates the bounded operators $(B-\mu)^{-1}$ and $(A-\mu)^{-1}$ instead.

**Theorem 5.14** (without proof)**.** Let $A = A^*$ and $B = B^*$ in $\mathcal{H}$, denote the corresponding spectral measures by $E^A$ and $E^B$, respectively, and assume

$$\dim \operatorname{ran} \big( (A-\mu)^{-1} - (B-\mu)^{-1} \big) = n < \infty$$

for one (and hence for all) $\mu \in \rho(A) \cap \rho(B)$. Let $(\alpha, \beta)$ be an interval such that $\dim \operatorname{ran} E^A_{(\alpha,\beta)} < \infty$. Then

$$\big| \dim \operatorname{ran} E^A_{(\alpha,\beta)} - \dim \operatorname{ran} E^B_{(\alpha,\beta)} \big| \leq n.$$

If $(\alpha, \beta) \subset \rho(A)$, then $(\alpha, \beta) \cap \sigma(B)$ consists of at most $n$ eigenvalues counted with multiplicities.