Advanced functional analysis
1. Exercise sheet
(Closed operators)

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Exercise 1

In the Banach space $(C([0,1]), \|\cdot\|_\infty)$ let an operator $T : C([0,1]) \supseteq \operatorname{dom} T \to C([0,1])$ be given by

$$Tf = f', \quad \operatorname{dom} T = C^1([0,1]).$$

Show that T is not continuous, but closed.

Exercise 2

Let X be a Banach space and let $T \in \mathcal{L}(X)$ be such that

$$\sum_{n=0}^{\infty} T^n \quad \text{converges in } \mathcal{L}(X) \text{ (w.r.t. the operator norm).} \tag{1}$$

Prove that $I - T$ is bijective and that

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

Hint: Show that (1) implies that T^n converges to 0 in $\mathcal{L}(X)$.

Exercise 3

Let X and Y be Banach spaces and $T : X \supseteq \operatorname{dom} T \to Y$ be a linear operator. Show that two of the following statements always imply the third one:

1. T is closed.
2. T is continuous.
3. $\operatorname{dom} T$ is closed.

Moreover, find an example of a continuous linear operator which is not closed.[1]

Exercise 4

Let $T_i : \ell^2 \supseteq \operatorname{dom} T_i \to \ell^2$, $i = 1, 2$, be given by $T_i(x_n)_{n\in\mathbb{N}} = (nx_n)_{n\in\mathbb{N}}$ on domains

$$\operatorname{dom} T_1 = \left\{(x_n)_{n\in\mathbb{N}} : (nx_n)_{n\in\mathbb{N}} \in \ell^2\right\}$$

and

$$\operatorname{dom} T_2 = \left\{(x_n)_{n\in\mathbb{N}} : \exists\, N \in \mathbb{N} \,\text{s.t.}\, x_n = 0 \,\forall\, n \geq N\right\}.$$

Examine, whether T_i is closed, $i = 1, 2$.

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[1] So keep in mind: A closed operator is not necessarily bounded and also the converse does, in general, not hold.

Exercise 5

Let X be a Banach space and S, T be closed operators in X. **Prove or disprove** the following statements.

1. $S + T$ is closed.
2. For $S \in \mathcal{L}(X)$, ST is closed.
3. For $T \in \mathcal{L}(X)$, ST is closed.
4. If $S - \lambda$ is injective, then $(S - \lambda)^{-1}$ is closed.
5. For each $\lambda \in \mathbb{C}$, λS is closed.

Exercise 6

1. Let X and Y be Banach spaces and let $T : X \supseteq \operatorname{dom} T \to Y$ be a closed operator. Are the kernel $\ker T$ of T or the range $\operatorname{ran} T$ of T closed subspaces of X or Y, respectively?
2. Show that in the statement of the Closed Graph Theorem the condition of completeness of the space X cannot be dropped.