Chapter 1

Deformations and Transformations

Let $\Omega = \Omega(t_0) \subset \mathbb{R}^n$, $n = 2, 3$, be some reference configuration at time t_0, i.e. Ω is an open, bounded and connected domain with a piecewise smooth boundary $\Gamma = \partial\Omega$. For a material point $\boldsymbol{x} \in \Omega$ let

$$\boldsymbol{y}(t) := \boldsymbol{\varphi}(t, \boldsymbol{x}) \tag{1.1}$$

denote the deformation where we assume the following:

i. $\boldsymbol{x} = \boldsymbol{\varphi}(t_0, \boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$;

ii. $\boldsymbol{\varphi}(t, \cdot) : \Omega \to \mathbb{R}^n$ is bijective for all $t \geq t_0$;

iii. $\boldsymbol{\varphi}(t, \cdot) \in C^2(\overline{\Omega})$ for all $t \geq t_0$;

iv. $J(t) := \det D_x\boldsymbol{\varphi}(t, \boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in \Omega$ and $t \geq t_0$ where

$$\boldsymbol{F} := D_x\boldsymbol{\varphi}(t, \boldsymbol{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1}\varphi_1(t, \boldsymbol{x}) & \dots & \frac{\partial}{\partial x_n}\varphi_1(t, \boldsymbol{x}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1}\varphi_n(t, \boldsymbol{x}) & \dots & \frac{\partial}{\partial x_n}\varphi_n(t, \boldsymbol{x}) \end{pmatrix} \in \mathbb{R}^{n \times n}. \tag{1.2}$$

Note that $\boldsymbol{x} \in \Omega$ is called Lagrangian coordinate, while $\boldsymbol{y}(t)$ is called Eulerian coordinate. The displacement of a material point $\boldsymbol{x} \in \Omega$ is then defined as

$$\boldsymbol{u}(t, \boldsymbol{x}) := \boldsymbol{\varphi}(t, \boldsymbol{x}) - \boldsymbol{x}\,. \tag{1.3}$$

Hence we have

$$\boldsymbol{\varphi}(t, \boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(t, \boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \Omega,$$

and therefore

$$D_x\boldsymbol{\varphi}(t, \boldsymbol{x}) = \boldsymbol{I} + D_x\boldsymbol{u}(t, \boldsymbol{x}) \tag{1.4}$$

follows, where

$$D_x\boldsymbol{u}(t,\boldsymbol{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1}u_1(t,\boldsymbol{x}) & \dots & \frac{\partial}{\partial x_n}u_1(t,\boldsymbol{x}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1}u_n(t,\boldsymbol{x}) & \dots & \frac{\partial}{\partial x_n}u_n(t,\boldsymbol{x}) \end{pmatrix}.$$

A particle starting in $\boldsymbol{x} \in \Omega$ moves along the trajectory

$$\boldsymbol{y}(t) = \boldsymbol{\varphi}(t,\boldsymbol{x})$$

with the velocity

$$\frac{d}{dt}\boldsymbol{y}(t) = \frac{d}{dt}\boldsymbol{\varphi}(t,\boldsymbol{x}) \;=\; \boldsymbol{w}(t,\boldsymbol{x}(\boldsymbol{y})) \;=\; \boldsymbol{v}(t,\boldsymbol{y}) \tag{1.5}$$

where $\boldsymbol{x}(\boldsymbol{y}) = \boldsymbol{\varphi}^{-1}(t,\boldsymbol{y})$ is the inverse mapping of $\boldsymbol{y} = \boldsymbol{\varphi}(t,\boldsymbol{x})$. Note that $\boldsymbol{v}(t,\boldsymbol{y})$ is the velocity in Eulerian coordinates, while $\boldsymbol{w}(t,\boldsymbol{x})$ is considered in Lagrangian coordinates.

For the time derivative of a scalar density funtion $f(t,\boldsymbol{y}) : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ we conclude, by the chain rule,

$$\begin{aligned} \frac{d}{dt}f(t,\boldsymbol{y}(t)) &= \frac{\partial}{\partial t}f(t,\boldsymbol{y}(t)) + \sum_{k=1}^{n}\frac{\partial}{\partial y_k}f(t,\boldsymbol{y})\,\frac{d}{dt}y_k(t) \\ &= \frac{\partial}{\partial t}f(t,\boldsymbol{y}(t)) + \nabla_y f(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\,. \end{aligned} \tag{1.6}$$

With respect to the space variables we may write a scalar function u as

$$u(t,\boldsymbol{y}) = u(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) = \widetilde{u}(t,\boldsymbol{x})$$

and by the chain rule we find

$$\frac{\partial}{\partial x_i}\widetilde{u}(t,\boldsymbol{x}) = \frac{\partial}{\partial x_i}u(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) = \sum_{k=1}^{n}\frac{\partial}{\partial y_k}u(t,\boldsymbol{y})_{|\boldsymbol{y}=\boldsymbol{\varphi}(t,\boldsymbol{x})}\frac{\partial}{\partial x_i}\varphi_k(t,\boldsymbol{x})\,.$$

In particular for $n = 3$ we have

$$\begin{pmatrix} \frac{\partial}{\partial x_1}\widetilde{u}(t,\boldsymbol{x}) \\ \frac{\partial}{\partial x_2}\widetilde{u}(t,\boldsymbol{x}) \\ \frac{\partial}{\partial x_3}\widetilde{u}(t,\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x}) & \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x}) & \frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) \\ \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x}) & \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x}) & \frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) \\ \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x}) & \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x}) & \frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1}u(t,\boldsymbol{y}) \\ \frac{\partial}{\partial y_2}u(t,\boldsymbol{y}) \\ \frac{\partial}{\partial y_3}u(t,\boldsymbol{y}) \end{pmatrix},$$

i.e.

$$\nabla_x\widetilde{u}(t,\boldsymbol{x}) = \boldsymbol{F}^{\top}\nabla_y u(t,\boldsymbol{y}), \tag{1.7}$$

or

$$\nabla_y u(t,\boldsymbol{y}) = \boldsymbol{F}^{-\top} \nabla_x \widetilde{u}(t,\boldsymbol{x})\,. \tag{1.8}$$

For a vector valued function

$$\boldsymbol{u}(t,\boldsymbol{y}) = \boldsymbol{u}(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))$$

we define

$$\widetilde{\boldsymbol{u}}(t,\boldsymbol{x}) = \det\boldsymbol{F}\,\boldsymbol{F}^{-1}\boldsymbol{u}(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \tag{1.9}$$

for which the following result holds.

Lemma 1.1 *For the divergence of the transformed vector field* $\widetilde{\boldsymbol{u}}$ *as defined in* (1.9) *we have the representation*

$$\mathrm{div}_x \widetilde{\boldsymbol{u}}(t,\boldsymbol{x}) \;=\; \det\boldsymbol{F}\,\mathrm{div}_y \boldsymbol{u}(t,\boldsymbol{y})\,. \tag{1.10}$$

Proof: Let us first consider the two–dimensional case $n = 2$ where the inverse matrix $\boldsymbol{F}^{-1}$ is given by

$$\boldsymbol{F}^{-1} = \frac{1}{\det\boldsymbol{F}} \left(\begin{array}{cc} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{array} \right)$$

and therefore (1.9) is equivalent to

$$\left(\begin{array}{c} \widetilde{u}_1(t,\boldsymbol{x}) \\ \widetilde{u}_2(t,\boldsymbol{x}) \end{array} \right) = \left(\begin{array}{cc} F_{22} & -F_{12} \\ -F_{21} & F_{11} \end{array} \right) \left(\begin{array}{c} u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \\ u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \end{array} \right).$$

In particular we have

$$\begin{aligned}
\widetilde{u}_1(t,\boldsymbol{x}) &= \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\, u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\, u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x})), \\
\widetilde{u}_2(t,\boldsymbol{x}) &= -\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\, u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) + \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\, u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))
\end{aligned}$$

and hence we conclude

$$\begin{aligned}
\mathrm{div}_x \widetilde{\boldsymbol{u}}(t,\boldsymbol{x}) &= \frac{\partial}{\partial x_1}\widetilde{u}_1(t,\boldsymbol{x}) + \frac{\partial}{\partial x_2}\widetilde{u}_2(t,\boldsymbol{x}) \\
&= \frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\, u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\, u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right] \\
&\quad + \frac{\partial}{\partial x_2}\left[-\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\, u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) + \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\, u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]
\end{aligned}$$

$$\begin{aligned}
&= F_{22}\frac{\partial}{\partial x_1}u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) - F_{12}\frac{\partial}{\partial x_1}u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \\
&\quad -F_{21}\frac{\partial}{\partial x_2}u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) + F_{11}\frac{\partial}{\partial x_1}u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \\
&= F_{22}\left[\frac{\partial}{\partial y_1}u_1(t,\boldsymbol{y})\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x}) + \frac{\partial}{\partial y_2}u_1(t,\boldsymbol{y})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\right] \\
&\quad -F_{12}\left[\frac{\partial}{\partial y_1}u_2(t,\boldsymbol{y})\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x}) + \frac{\partial}{\partial y_2}u_2(t,\boldsymbol{y})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\right] \\
&\quad -F_{21}\left[\frac{\partial}{\partial y_1}u_1(t,\boldsymbol{y})\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x}) + \frac{\partial}{\partial y_2}u_1(t,\boldsymbol{y})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\right] \\
&\quad +F_{11}\left[\frac{\partial}{\partial y_1}u_2(t,\boldsymbol{y})\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x}) + \frac{\partial}{\partial y_2}u_2(t,\boldsymbol{y})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\right] \\
&= \Big[F_{22}F_{11} - F_{21}F_{12}\Big]\left[\frac{\partial}{\partial y_1}u_1(t,\boldsymbol{y}) + \frac{\partial}{\partial y_2}u_2(t,\boldsymbol{y})\right] \\
&= \det \boldsymbol{F}\,\mathrm{div}_y \boldsymbol{u}(t,\boldsymbol{y}).
\end{aligned}$$

For $n = 3$, the inverse matrix $\boldsymbol{F}^{-1}$ is given by

$$\boldsymbol{F}^{-1} = \frac{1}{\det \boldsymbol{F}}\begin{pmatrix} F_{22}F_{33}-F_{23}F_{32} & F_{13}F_{32}-F_{12}F_{33} & F_{12}F_{23}-F_{13}F_{22} \\ F_{23}F_{31}-F_{21}F_{33} & F_{11}F_{33}-F_{13}F_{31} & F_{13}F_{21}-F_{11}F_{23} \\ F_{21}F_{32}-F_{22}F_{31} & F_{12}F_{31}-F_{11}F_{32} & F_{11}F_{22}-F_{12}F_{21} \end{pmatrix},$$

and therefore (1.9) is equivalent to

$$\begin{aligned}
\begin{pmatrix} \widetilde{u}_1(t,\boldsymbol{x}) \\ \widetilde{u}_2(t,\boldsymbol{x}) \\ \widetilde{u}_3(t,\boldsymbol{x}) \end{pmatrix} &= \\
&= \begin{pmatrix} F_{22}F_{33}-F_{23}F_{32} & F_{13}F_{32}-F_{12}F_{33} & F_{12}F_{23}-F_{13}F_{22} \\ F_{23}F_{31}-F_{21}F_{33} & F_{11}F_{33}-F_{13}F_{31} & F_{13}F_{21}-F_{11}F_{23} \\ F_{21}F_{32}-F_{22}F_{31} & F_{12}F_{31}-F_{11}F_{32} & F_{11}F_{22}-F_{12}F_{21} \end{pmatrix}\begin{pmatrix} u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \\ u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \\ u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \end{pmatrix}.
\end{aligned}$$

In particular we have

$$\begin{aligned}
\widetilde{u}_1(t,\boldsymbol{x}) &= \Big(\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\Big)u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \\
&\quad +\Big(\frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\Big)u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) \\
&\quad +\Big(\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\Big)u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x})),
\end{aligned}$$

$$\begin{aligned}
\widetilde{u}_2(t,\boldsymbol{x}) &= \Big(\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\Big)u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad+\Big(\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\Big)u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad+\Big(\frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\Big)u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x})),
\end{aligned}$$

$$\begin{aligned}
\widetilde{u}_3(t,\boldsymbol{x}) &= \Big(\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\Big)u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad+\Big(\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\Big)u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad+\Big(\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\Big)u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x})).
\end{aligned}$$

Hence we conclude

$$\begin{aligned}
\mathrm{div}_x\widetilde{\boldsymbol{u}}(t,\boldsymbol{x}) &= \frac{\partial}{\partial x_1}\widetilde{u}_1(t,\boldsymbol{x}) + \frac{\partial}{\partial x_2}\widetilde{u}_2(t,\boldsymbol{x}) + \frac{\partial}{\partial x_3}\widetilde{u}_3(t,\boldsymbol{x})\\
&= \frac{\partial}{\partial x_1}\left[\Big(\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\Big)u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_1}\left[\Big(\frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\Big)u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_1}\left[\Big(\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\Big)u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_2}\left[\Big(\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\Big)u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_2}\left[\Big(\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\Big)u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_2}\left[\Big(\frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\Big)u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_3}\left[\Big(\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\Big)u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_3}\left[\Big(\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\Big)u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right]\\
&\quad+\frac{\partial}{\partial x_3}\left[\Big(\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\Big)u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\right].
\end{aligned}$$

When applying the product rule it is easy to check that all terms involving second order derivatives of the deformation will disappear. As an example, let us consider all expressions in front of $u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))$:

$$
\begin{aligned}
\mathrm{Coeff}\ (u_1) &= \frac{\partial}{\partial x_1}\Big(\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\Big)\\
&\quad + \frac{\partial}{\partial x_2}\Big(\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\Big)\\
&\quad + \frac{\partial}{\partial x_3}\Big(\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\Big)\\
&= \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) + \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\\
&\quad - \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\\
&\quad + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) + \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\\
&\quad - \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\\
&\quad + \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) + \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\\
&\quad - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\\
&= 0\,.
\end{aligned}
$$

Hence it remains to consider

$$
\begin{aligned}
\mathrm{div}_x\widetilde{\boldsymbol{u}}(t,\boldsymbol{x}) &= \Big(\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_1}u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_1}u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_1}u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_2}u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_2}u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_2}u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_3}u_1(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) - \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_3}u_2(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\\
&\quad + \Big(\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x}) - \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\Big)\frac{\partial}{\partial x_3}u_3(t,\boldsymbol{\varphi}(t,\boldsymbol{x})).
\end{aligned}
$$

With

$$\frac{\partial}{\partial x_j} u_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) = \sum_{k=1}^{3} \frac{\partial}{\partial y_k} u_i(t, \boldsymbol{y}) \frac{\partial}{\partial x_j} \varphi_k(t, \boldsymbol{x}) = \sum_{k=1}^{3} \frac{\partial}{\partial y_k} u_i(t, \boldsymbol{y}) F_{kj}$$

we further conclude

$$\begin{aligned}
\mathrm{div}_x \widetilde{\boldsymbol{u}}(t, \boldsymbol{x}) \;\; = \;\; & \Big(F_{22}F_{33} - F_{23}F_{32}\Big)\Big(F_{11}\frac{\partial}{\partial y_1}u_1(t,\boldsymbol{y}) + F_{21}\frac{\partial}{\partial y_2}u_1(t,\boldsymbol{y}) + F_{31}\frac{\partial}{\partial y_3}u_1(t,\boldsymbol{y})\Big) \\
& +\Big(F_{13}F_{32} - F_{12}F_{33}\Big)\Big(F_{11}\frac{\partial}{\partial y_1}u_2(t,\boldsymbol{y}) + F_{21}\frac{\partial}{\partial y_2}u_2(t,\boldsymbol{y}) + F_{31}\frac{\partial}{\partial y_3}u_2(t,\boldsymbol{y})\Big) \\
& +\Big(F_{12}F_{23} - F_{13}F_{22}\Big)\Big(F_{11}\frac{\partial}{\partial y_1}u_3(t,\boldsymbol{y}) + F_{21}\frac{\partial}{\partial y_2}u_3(t,\boldsymbol{y}) + F_{31}\frac{\partial}{\partial y_3}u_3(t,\boldsymbol{y})\Big) \\
& +\Big(F_{23}F_{31} - F_{21}F_{33}\Big)\Big(F_{12}\frac{\partial}{\partial y_1}u_1(t,\boldsymbol{y}) + F_{22}\frac{\partial}{\partial y_2}u_1(t,\boldsymbol{y}) + F_{32}\frac{\partial}{\partial y_3}u_1(t,\boldsymbol{y})\Big) \\
& +\Big(F_{11}F_{33} - F_{13}F_{31}\Big)\Big(F_{12}\frac{\partial}{\partial y_1}u_2(t,\boldsymbol{y}) + F_{22}\frac{\partial}{\partial y_2}u_2(t,\boldsymbol{y}) + F_{32}\frac{\partial}{\partial y_3}u_2(t,\boldsymbol{y})\Big) \\
& +\Big(F_{13}F_{21} - F_{11}F_{23}\Big)\Big(F_{12}\frac{\partial}{\partial y_1}u_3(t,\boldsymbol{y}) + F_{22}\frac{\partial}{\partial y_2}u_3(t,\boldsymbol{y}) + F_{32}\frac{\partial}{\partial y_3}u_3(t,\boldsymbol{y})\Big) \\
& +\Big(F_{21}F_{32} - F_{22}F_{31}\Big)\Big(F_{13}\frac{\partial}{\partial y_1}u_1(t,\boldsymbol{y}) + F_{23}\frac{\partial}{\partial y_2}u_1(t,\boldsymbol{y}) + F_{33}\frac{\partial}{\partial y_3}u_1(t,\boldsymbol{y})\Big) \\
& +\Big(F_{12}F_{31} - F_{11}F_{32}\Big)\Big(F_{13}\frac{\partial}{\partial y_1}u_2(t,\boldsymbol{y}) + F_{23}\frac{\partial}{\partial y_2}u_2(t,\boldsymbol{y}) + F_{33}\frac{\partial}{\partial y_3}u_2(t,\boldsymbol{y})\Big) \\
& +\Big(F_{11}F_{22} - F_{12}F_{21}\Big)\Big(F_{13}\frac{\partial}{\partial y_1}u_3(t,\boldsymbol{y}) + F_{23}\frac{\partial}{\partial y_2}u_3(t,\boldsymbol{y}) + F_{33}\frac{\partial}{\partial y_3}u_3(t,\boldsymbol{y})\Big)
\end{aligned}$$

$$\begin{aligned}
&= \Big[F_{11}\Big(F_{22}F_{33} - F_{23}F_{32}\Big) + F_{12}\Big(F_{23}F_{31} - F_{21}F_{33}\Big) + F_{13}\Big(F_{21}F_{32} - F_{22}F_{31}\Big)\Big]\frac{\partial}{\partial y_1}u_1(t,\boldsymbol{y}) \\
&\;\; +\Big[F_{21}\Big(F_{13}F_{32} - F_{12}F_{33}\Big) + F_{22}\Big(F_{11}F_{33} - F_{13}F_{31}\Big) + F_{23}\Big(F_{12}F_{31} - F_{11}F_{32}\Big)\Big]\frac{\partial}{\partial y_2}u_2(t,\boldsymbol{y}) \\
&\;\; +\Big[F_{31}\Big(F_{12}F_{23} - F_{13}F_{22}\Big) + F_{32}\Big(F_{13}F_{21} - F_{11}F_{23}\Big) + F_{33}\Big(F_{11}F_{22} - F_{12}F_{21}\Big)\Big]\frac{\partial}{\partial y_3}u_3(t,\boldsymbol{y}) \\
&= \det\boldsymbol{F}\,\mathrm{div}_y \boldsymbol{u}(t,\boldsymbol{y})\,.
\end{aligned}$$

■

For the transformation of time–depending integrals we will need the following result.

Lemma 1.2 *Let*

$$J(t) = \det D_x \boldsymbol{\varphi}(t, \boldsymbol{x}) = \det \begin{pmatrix} \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x}) & \dots & \frac{\partial}{\partial x_n}\varphi_1(t,\boldsymbol{x}) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_1}\varphi_n(t,\boldsymbol{x}) & \dots & \frac{\partial}{\partial x_n}\varphi_n(t,\boldsymbol{x}) \end{pmatrix}$$

be the Jacobian of the transformation $\boldsymbol{y} = \boldsymbol{\varphi}(t, \boldsymbol{x})$. *We then have*

$$\frac{d}{dt} J(t) \;=\; J(t)\,\mathrm{div}_y \boldsymbol{v}(t, \boldsymbol{y}). \tag{1.11}$$

Proof: We first consider the two–dimensional case $n = 2$. In general, a time dependent matrix $A(t)$ is given by

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix},$$

and the determinant is

$$\det A(t) = a_{11}(t) a_{22}(t) - a_{12}(t) a_{21}(t)\,.$$

Since the inverse matrix of $A(t)$ is given by

$$B(t) = [A(t)]^{-1} = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} = \frac{1}{\det\, A} \begin{pmatrix} a_{22}(t) & -a_{12}(t) \\ -a_{21}(t) & a_{11}(t) \end{pmatrix},$$

we have

$$\begin{aligned} a_{22}(t) &= b_{11}(t)\det A(t), \\ a_{12}(t) &= -b_{12}(t)\det A(t), \\ a_{21}(t) &= -b_{21}(t)\det A(t), \\ a_{11}(t) &= b_{22}(t)\det A(t), \end{aligned}$$

and therefore we obtain, by applying the chain rule,

$$\begin{aligned} \frac{d}{dt}\det A(t) &= \frac{d}{dt}\Big[a_{11}(t)a_{22}(t) - a_{12}(t)a_{21}(t)\Big] \\ &= a_{11}(t)\frac{d}{dt}a_{22}(t) + a_{22}(t)\frac{d}{dt}a_{11}(t) - a_{12}(t)\frac{d}{dt}a_{21}(t) - a_{21}(t)\frac{d}{dt}a_{12}(t) \\ &= \det A(t)\left[b_{22}(t)\frac{d}{dt}a_{22}(t) + b_{11}(t)\frac{d}{dt}a_{11}(t) + b_{12}(t)\frac{d}{dt}a_{21}(t) + b_{21}(t)\frac{d}{dt}a_{12}(t)\right]. \end{aligned}$$

In particular for $A(t) = D_x\boldsymbol{\varphi}(t, \boldsymbol{x})$ and $J(t) = \det D_x\boldsymbol{\varphi}(t, \boldsymbol{x})$ this gives, by exchanging the order of differentiation,

$$\begin{aligned} \frac{d}{dt}J(t) &= J(t)\,\Big[b_{22}(t)\frac{d}{dt}\frac{\partial}{\partial x_2}\varphi_2(t, \boldsymbol{x}) + b_{11}(t)\frac{d}{dt}\frac{\partial}{\partial x_1}\varphi_1(t, \boldsymbol{x}) \\ &\qquad + b_{12}(t)\frac{d}{dt}\frac{\partial}{\partial x_1}\varphi_2(t, \boldsymbol{x}) + b_{21}(t)\frac{d}{dt}\frac{\partial}{\partial x_2}\varphi_1(t, \boldsymbol{x})\Big] \\ &= J(t)\,\Big[b_{22}(t)\frac{\partial}{\partial x_2}w_2(t, \boldsymbol{x}) + b_{11}(t)\frac{\partial}{\partial x_1}w_1(t, \boldsymbol{x}) \\ &\qquad + b_{12}(t)\frac{\partial}{\partial x_1}w_2(t, \boldsymbol{x}) + b_{21}(t)\frac{\partial}{\partial x_2}w_1(t, \boldsymbol{x})\Big] \end{aligned}$$

where

$$w_1 = \frac{d}{dt}\varphi_1(t,\boldsymbol{x}), \quad w_2 = \frac{d}{dt}\varphi_2(t,\boldsymbol{x})$$

are the components of the velocity vector in Lagrangian coordinates. By using the inverse transformation

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{y}) = \boldsymbol{\varphi}^{-1}(t,\boldsymbol{y})$$

we have

$$y_1 = \varphi_1(t,\boldsymbol{x}(\boldsymbol{y})), \quad y_2 = \varphi_2(t,\boldsymbol{x}(\boldsymbol{y}))\,.$$

From the chain rule we therefore find

$$\begin{aligned}
\frac{d}{dy_1}y_1 = \frac{d}{dy_1}\varphi_1(t,\boldsymbol{x}(\boldsymbol{y})) &= \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) = 1,\\
\frac{d}{dy_2}y_1 = \frac{d}{dy_2}\varphi_1(t,\boldsymbol{x}(\boldsymbol{y})) &= \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) = 0,\\
\frac{d}{dy_1}y_2 = \frac{d}{dy_1}\varphi_2(t,\boldsymbol{x}(\boldsymbol{y})) &= \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) = 0,\\
\frac{d}{dy_2}y_2 = \frac{d}{dy_2}\varphi_2(t,\boldsymbol{x}(\boldsymbol{y})) &= \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) = 1.
\end{aligned}$$

In particular we have

$$\begin{pmatrix} \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x}) & \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x}) \\ \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x}) & \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x}) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) \\ \frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and since the inverse matrix $B(t) = [A(t)]^{-1}$ of $A(t) = D_x\boldsymbol{\varphi}(t,\boldsymbol{x})$ is uniquely determined, we therefore conclude

$$b_{11}(t) = \frac{\partial}{\partial y_1}x_1(\boldsymbol{y}), \quad b_{21}(t) = \frac{\partial}{\partial y_1}x_2(\boldsymbol{y}), \quad b_{12}(t) = \frac{\partial}{\partial y_2}x_1(\boldsymbol{y}), \quad b_{22}(t) = \frac{\partial}{\partial y_2}x_2(\boldsymbol{y}).$$

Hence we obtain

$$\begin{aligned}
\frac{d}{dt}J(t) &= J(t)\left[b_{22}(t)\frac{\partial}{\partial x_2}w_2(t,\boldsymbol{x}) + b_{11}(t)\frac{\partial}{\partial x_1}w_1(t,\boldsymbol{x})\right.\\
&\qquad\left.+b_{12}(t)\frac{\partial}{\partial x_1}w_2(t,\boldsymbol{x}) + b_{21}(t)\frac{\partial}{\partial x_2}w_1(t,\boldsymbol{x})\right]\\
&= J(t)\left[\frac{\partial}{\partial x_2}w_2(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_1}w_1(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_1(\boldsymbol{y})\right.\\
&\qquad\left.+\frac{\partial}{\partial x_1}w_2(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}w_1(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_2(\boldsymbol{y})\right]\\
&= J(t)\left[\frac{\partial}{\partial y_1}w_1(t,\boldsymbol{x}(\boldsymbol{y})) + \frac{\partial}{\partial y_2}w_2(t,\boldsymbol{x}(\boldsymbol{y}))\right]
\end{aligned}$$

$$
\begin{aligned}
&= J(t)\,\mathrm{div}_y \boldsymbol{w}(t, \boldsymbol{x}(\boldsymbol{y})) \\
&= J(t)\,\mathrm{div}_y \boldsymbol{v}(t, \boldsymbol{y})\,.
\end{aligned}
$$

For $n = 3$ the proof follows in the same way as for $n = 2$, i.e. we have

$$
A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix}
$$

and

$$
\det A(t) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.
$$

By applying the chain rule we obtain

$$
\begin{aligned}
\frac{d}{dt}\det A(t) = {}& \Big(a_{22}a_{33} - a_{32}a_{23}\Big)\frac{d}{dt}a_{11} + \Big(a_{31}a_{23} - a_{33}a_{21}\Big)\frac{d}{dt}a_{12} + \Big(a_{32}a_{21} - a_{31}a_{22}\Big)\frac{d}{dt}a_{13} \\
&+\Big(a_{13}a_{32} - a_{12}a_{33}\Big)\frac{d}{dt}a_{21} + \Big(a_{11}a_{33} - a_{13}a_{31}\Big)\frac{d}{dt}a_{22} + \Big(a_{12}a_{31} - a_{11}a_{32}\Big)\frac{d}{dt}a_{23} \\
&+\Big(a_{12}a_{23} - a_{13}a_{22}\Big)\frac{d}{dt}a_{31} + \Big(a_{13}a_{21} - a_{11}a_{23}\Big)\frac{d}{dt}a_{32} + \Big(a_{11}a_{22} - a_{12}a_{21}\Big)\frac{d}{dt}a_{33}\,.
\end{aligned}
$$

The inverse of $A(t)$ is, on the other hand, given by, by using Cramer's rule,

$$
B(t) = \frac{1}{\det A(t)} \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.
$$

Hence we conclude

$$
\begin{aligned}
\frac{d}{dt}\det A(t) &= \det A(t)\,\Big[b_{11}\frac{d}{dt}a_{11} + b_{21}\frac{d}{dt}a_{12} + b_{31}\frac{d}{dt}a_{13} + b_{12}\frac{d}{dt}a_{21} \\
&\qquad + b_{22}\frac{d}{dt}a_{22} + b_{32}\frac{d}{dt}a_{23} + b_{13}\frac{d}{dt}a_{31} + b_{23}\frac{d}{dt}a_{32} + b_{33}\frac{d}{dt}a_{33}\Big]\,.
\end{aligned}
$$

In particular for $A(t) = D_x\boldsymbol{\varphi}(t, \boldsymbol{x})$ and $J(t) = \det D_x\boldsymbol{\varphi}(t, \boldsymbol{x})$ this gives

$$
\begin{aligned}
\frac{d}{dt}J(t) &= J(t)\,\Big[b_{11}\frac{d}{dt}\frac{\partial}{\partial x_1}\varphi_1(t, \boldsymbol{x}) + b_{21}\frac{d}{dt}\frac{\partial}{\partial x_2}\varphi_1(t, \boldsymbol{x}) + b_{31}\frac{d}{dt}\frac{\partial}{\partial x_3}\varphi_1(t, \boldsymbol{x}) \\
&\qquad + b_{12}\frac{d}{dt}\frac{\partial}{\partial x_1}\varphi_2(t, \boldsymbol{x}) + b_{22}\frac{d}{dt}\frac{\partial}{\partial x_2}\varphi_2(t, \boldsymbol{x}) + b_{32}\frac{d}{dt}\frac{\partial}{\partial x_3}\varphi_2(t, \boldsymbol{x}) \\
&\qquad + b_{13}\frac{d}{dt}\frac{\partial}{\partial x_1}\varphi_3(t, \boldsymbol{x}) + b_{23}\frac{d}{dt}\frac{\partial}{\partial x_2}\varphi_3(t, \boldsymbol{x}) + b_{33}\frac{d}{dt}\frac{\partial}{\partial x_3}\varphi_3(t, \boldsymbol{x})\Big] \\
&= J(t)\,\Big[b_{11}\frac{\partial}{\partial x_1}w_1(t, \boldsymbol{x}) + b_{21}\frac{\partial}{\partial x_2}w_1(t, \boldsymbol{x}) + b_{31}\frac{\partial}{\partial x_3}w_1(t, \boldsymbol{x}) \\
&\qquad + b_{12}\frac{\partial}{\partial x_1}w_2(t, \boldsymbol{x}) + b_{22}\frac{\partial}{\partial x_2}w_2(t, \boldsymbol{x}) + b_{32}\frac{\partial}{\partial x_3}w_2(t, \boldsymbol{x}) \\
&\qquad + b_{13}\frac{\partial}{\partial x_1}w_3(t, \boldsymbol{x}) + b_{23}\frac{\partial}{\partial x_2}w_3(t, \boldsymbol{x}) + b_{33}\frac{\partial}{\partial x_3}w_3(t, \boldsymbol{x})\Big]\,.
\end{aligned}
$$

where
$$w_1 = \frac{d}{dt}\varphi_1(t,\boldsymbol{x}), \quad w_2 = \frac{d}{dt}\varphi_2(t,\boldsymbol{x}), \quad w_3 = \frac{d}{dt}\varphi_3(t,\boldsymbol{x})$$
are the components of the velocity. By using
$$y_1 = \varphi_1(t,\boldsymbol{x}(\boldsymbol{y})), \quad y_2 = \varphi_2(t,\boldsymbol{x}(\boldsymbol{y})), \quad y_3 = \varphi_3(t,\boldsymbol{x}(\boldsymbol{y}))$$
we find, by the chain rule,
$$\begin{aligned}
\frac{d}{dy_1}y_1 &= \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_3(\boldsymbol{y}) = 1,\\
\frac{d}{dy_2}y_1 &= \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_3(\boldsymbol{y}) = 0,\\
\frac{d}{dy_3}y_1 &= \frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_3(\boldsymbol{y}) = 0,\\
\frac{d}{dy_1}y_2 &= \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_3(\boldsymbol{y}) = 0,\\
\frac{d}{dy_2}y_2 &= \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_3(\boldsymbol{y}) = 1,\\
\frac{d}{dy_3}y_2 &= \frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_3(\boldsymbol{y}) = 0,\\
\frac{d}{dy_1}y_3 &= \frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_1}x_3(\boldsymbol{y}) = 0,\\
\frac{d}{dy_2}y_3 &= \frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_2}x_3(\boldsymbol{y}) = 0,\\
\frac{d}{dy_3}y_3 &= \frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_1(\boldsymbol{y}) + \frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_2(\boldsymbol{y}) + \frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})\frac{\partial}{\partial y_3}x_3(\boldsymbol{y}) = 1.
\end{aligned}$$
In particular we have
$$\begin{pmatrix}
\frac{\partial}{\partial x_1}\varphi_1(t,\boldsymbol{x}) & \frac{\partial}{\partial x_2}\varphi_1(t,\boldsymbol{x}) & \frac{\partial}{\partial x_3}\varphi_1(t,\boldsymbol{x})\\
\frac{\partial}{\partial x_1}\varphi_2(t,\boldsymbol{x}) & \frac{\partial}{\partial x_2}\varphi_2(t,\boldsymbol{x}) & \frac{\partial}{\partial x_3}\varphi_2(t,\boldsymbol{x})\\
\frac{\partial}{\partial x_1}\varphi_3(t,\boldsymbol{x}) & \frac{\partial}{\partial x_2}\varphi_3(t,\boldsymbol{x}) & \frac{\partial}{\partial x_3}\varphi_3(t,\boldsymbol{x})
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) & \frac{\partial}{\partial y_3}x_1(\boldsymbol{y})\\
\frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) & \frac{\partial}{\partial y_3}x_2(\boldsymbol{y})\\
\frac{\partial}{\partial y_1}x_3(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_3(\boldsymbol{y}) & \frac{\partial}{\partial y_3}x_3(\boldsymbol{y})
\end{pmatrix} = I,$$
and since the inverse matrix $B(t) = [A(t)]^{-1}$ of $A(t) = D_x\boldsymbol{\varphi}(t,\boldsymbol{x})$ is uniquely determined, we therefore conclude
$$B(t) = \begin{pmatrix}
\frac{\partial}{\partial y_1}x_1(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_1(\boldsymbol{y}) & \frac{\partial}{\partial y_3}x_1(\boldsymbol{y})\\
\frac{\partial}{\partial y_1}x_2(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_2(\boldsymbol{y}) & \frac{\partial}{\partial y_3}x_2(\boldsymbol{y})\\
\frac{\partial}{\partial y_1}x_3(\boldsymbol{y}) & \frac{\partial}{\partial y_2}x_3(\boldsymbol{y}) & \frac{\partial}{\partial y_3}x_3(\boldsymbol{y})
\end{pmatrix}.$$

Hence we obtain

$$\begin{aligned}
\frac{d}{dt}J(t) &= J(t)\Big[\frac{\partial}{\partial y_1}x_1(\boldsymbol{y})\frac{\partial}{\partial x_1}w_1(t,\boldsymbol{x}) + \frac{\partial}{\partial y_1}x_2(\boldsymbol{y})\frac{\partial}{\partial x_2}w_1(t,\boldsymbol{x}) + \frac{\partial}{\partial y_1}x_3(\boldsymbol{y})\frac{\partial}{\partial x_3}w_1(t,\boldsymbol{x}) \\
&\qquad + \frac{\partial}{\partial y_2}x_1(\boldsymbol{y})\frac{\partial}{\partial x_1}w_2(t,\boldsymbol{x}) + \frac{\partial}{\partial y_2}x_2(\boldsymbol{y})\frac{\partial}{\partial x_2}w_2(t,\boldsymbol{x}) + \frac{\partial}{\partial y_2}x_3(\boldsymbol{y})\frac{\partial}{\partial x_3}w_2(t,\boldsymbol{x}) \\
&\qquad + \frac{\partial}{\partial y_3}x_1(\boldsymbol{y})\frac{\partial}{\partial x_1}w_3(t,\boldsymbol{x}) + \frac{\partial}{\partial y_3}x_2(\boldsymbol{y})\frac{\partial}{\partial x_2}w_3(t,\boldsymbol{x}) + \frac{\partial}{\partial y_3}x_3(\boldsymbol{y})\frac{\partial}{\partial x_3}w_3(t,\boldsymbol{x})\Big] \\
&= J(t)\left[\frac{\partial}{\partial y_1}w_1(t,\boldsymbol{x}(\boldsymbol{y})) + \frac{\partial}{\partial y_2}w_2(t,\boldsymbol{x}(\boldsymbol{y})) + \frac{\partial}{\partial y_3}w_3(t,\boldsymbol{x}(\boldsymbol{y}))\right] \\
&= J(t)\,\mathrm{div}_y\boldsymbol{w}(t,\boldsymbol{x}(\boldsymbol{y})) \\
&= J(t)\,\mathrm{div}_y\boldsymbol{v}(t,\boldsymbol{y}).
\end{aligned}$$

■

In what follows we will consider the rate of change of integrals which are considered in the actual configuration $\Omega(t)$. This is required for the reformulation of conservation or balance equations, which are given in integral form, by means of partial differential equations.

Theorem 1.1 (Reynold's transport theorem) *Let $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ be a continuously differentiable density function, and let $\varphi(t,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ be continuously differentiable for all $t\in\mathbb{R}$. For any bounded control volumina $\omega(t)\subset\Omega(t)$ we then have*

$$\frac{d}{dt}\int_{\omega(t)} f(t,\boldsymbol{y})\,d\boldsymbol{y} = \int_{\omega(t)}\left[\frac{\partial}{\partial t}f(t,\boldsymbol{y}) + \nabla_y\cdot[f(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})]\right]d\boldsymbol{y}. \tag{1.12}$$

Proof: We first note that

$$\omega(t) = \Big\{\boldsymbol{y} = \boldsymbol{\varphi}(t,\boldsymbol{x}) \text{ for all } \boldsymbol{x}\in\omega(t_0)\Big\},$$

and, by using (1.2),

$$J(t) = \det D_x\boldsymbol{\varphi}(t,\boldsymbol{x})\,.$$

Hence we obtain, due to (1.11),

$$\begin{aligned}
\frac{d}{dt}\int_{\omega(t)} f(t,\boldsymbol{y})\,d\boldsymbol{y} &= \frac{d}{dt}\int_{\omega(t_0)} f(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\,J(t)\,d\boldsymbol{x} \\
&= \int_{\omega(t_0)}\frac{d}{dt}\Big[f(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\,J(t)\Big]d\boldsymbol{x} \\
&= \int_{\omega(t_0)}\left[J(t)\frac{d}{dt}f(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) + f(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\,\frac{d}{dt}J(t)\right]d\boldsymbol{x} \\
&= \int_{\omega(t_0)}\left[\frac{d}{dt}f(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) + f(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\,\mathrm{div}_y\boldsymbol{w}(t,\boldsymbol{x}(\boldsymbol{y}))\right]J(t)\,d\boldsymbol{x}.
\end{aligned}$$

By using

$$\frac{d}{dt}f(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) = \frac{\partial}{\partial t}f(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) + \boldsymbol{w}(t,\boldsymbol{x})\cdot\nabla_y f(t,\boldsymbol{y})_{|\boldsymbol{y}=\boldsymbol{\varphi}(t,\boldsymbol{x})}$$

we therefore obtain, with $\boldsymbol{y} = \varphi(t,\boldsymbol{x})$,

$$\begin{aligned}
\frac{d}{dt}\int_{\omega(t)} f(t,\boldsymbol{y})\,d\boldsymbol{y} \quad &= \\
&= \int_{\omega(t_0)} \left[\frac{\partial}{\partial t} f(t,\boldsymbol{\varphi}(t,\boldsymbol{x})) + \boldsymbol{w}(t,\boldsymbol{x})\cdot\nabla_y f(t,\boldsymbol{y})_{|\boldsymbol{y}=\boldsymbol{\varphi}(t,\boldsymbol{x})} + f(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\,\mathrm{div}_y \boldsymbol{w}(t,\boldsymbol{x}(\boldsymbol{y}))\right] J(t) d\boldsymbol{x} \\
&= \int_{\omega(t)} \left[\frac{\partial}{\partial t} f(t,\boldsymbol{y}) + \boldsymbol{v}(t,\boldsymbol{y})\cdot\nabla_y f(t,\boldsymbol{y}) + f(t,\boldsymbol{y})\,\mathrm{div}_y \boldsymbol{v}(t,\boldsymbol{y})\right] d\boldsymbol{y} \\
&= \int\limits_{\omega(t)} \left[\frac{\partial}{\partial t} f(t,\boldsymbol{y}) + \nabla_y\cdot[\boldsymbol{v}(t,\boldsymbol{y}) f(t,\boldsymbol{y})]\right] d\boldsymbol{y}\,.
\end{aligned}$$

■

While Reynold's transport theorem (Theorem 1.1) is on the time derivative of moving volume integrals, we will also consider the time derivative of moving surface integrals which will be used in electrodynamics.

Theorem 1.2 *Let $\boldsymbol{g}$ be a vector valued function where the components $g_k : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, $k = 1,2,3$, are continuously differentiable, and let A be a smooth surface which is described by a two times differentiable time dependent parametrisation $\boldsymbol{\psi}$ with velocity $\boldsymbol{v} = \frac{d}{dt}\boldsymbol{\psi}$. Then there holds*

$$\begin{aligned}
&\frac{d}{dt}\int_A \boldsymbol{g}(t,\boldsymbol{y})\cdot\boldsymbol{n}_y\,ds_y \qquad (1.13)\\
&\qquad = \int_A \left[\frac{\partial}{\partial t}\boldsymbol{g}(t,\boldsymbol{y}) + \mathrm{div}\,\boldsymbol{g}(t,\boldsymbol{y})\,\boldsymbol{v}(t,\boldsymbol{y}) + \mathrm{curl}\Big(\boldsymbol{g}(t,\boldsymbol{y})\times\boldsymbol{v}(t,\boldsymbol{y})\Big)\right]\cdot\boldsymbol{n}_y\,ds_y\,.
\end{aligned}$$

Proof: Assume that the surface A is given as

$$A = \Big\{\boldsymbol{y} = \boldsymbol{\psi}(t,u,v) \quad \text{for } (u,v)\in P\subset\mathbb{R}^2\Big\}\,.$$

By using

$$\begin{aligned}
E &= \frac{\partial}{\partial u}\boldsymbol{\psi}(t,u,v)\cdot\frac{\partial}{\partial u}\boldsymbol{\psi}(t,u,v) = \sum_{k=1}^3\left[\frac{\partial}{\partial u}\psi_k(t,u,v)\right]^2,\\
G &= \frac{\partial}{\partial v}\boldsymbol{\psi}(t,u,v)\cdot\frac{\partial}{\partial v}\boldsymbol{\psi}(t,u,v) = \sum_{k=1}^3\left[\frac{\partial}{\partial v}\psi_k(t,u,v)\right]^2,\\
F &= \frac{\partial}{\partial u}\boldsymbol{\psi}(t,u,v)\cdot\frac{\partial}{\partial v}\boldsymbol{\psi}(t,u,v) = \sum_{k=1}^3\frac{\partial}{\partial u}\psi_k(t,u,v)\frac{\partial}{\partial v}\psi_k(t,u,v)
\end{aligned}$$

we can write the surface integral as

$$\int_A \boldsymbol{g}(t,\boldsymbol{y})\cdot\boldsymbol{n}_y\,ds_y = \int_P \boldsymbol{g}(t,\boldsymbol{\psi}(t,u,v))\cdot\boldsymbol{n}_y\sqrt{EG-F^2}\,du\,dv\,.$$

To express the normal vector along A we first compute

$$\widetilde{\boldsymbol{n}}_y = \boldsymbol{\psi}_u \times \boldsymbol{\psi}_v = \begin{pmatrix} i & j & k \\ \psi_{1,u} & \psi_{2,u} & \psi_{3,u} \\ \psi_{1,v} & \psi_{2,v} & \psi_{3,v} \end{pmatrix} = \begin{pmatrix} \psi_{2,u}\psi_{3,v} - \psi_{3,u}\psi_{2,v} \\ \psi_{3,u}\psi_{1,v} - \psi_{1,u}\psi_{3,v} \\ \psi_{1,u}\psi_{2,v} - \psi_{2,u}\psi_{1,v} \end{pmatrix}$$

and its norm

$$\begin{aligned}
|\widetilde{\boldsymbol{n}}_y|^2 &= (\psi_{2,u}\psi_{3,v} - \psi_{3,u}\psi_{2,v})^2 + (\psi_{3,u}\psi_{1,v} - \psi_{1,u}\psi_{3,v})^2 + (\psi_{1,u}\psi_{2,v} - \psi_{2,u}\psi_{1,v})^2 \\
&= \psi_{2,u}^2\psi_{3,v}^2 + \psi_{3,u}^2\psi_{2,v}^2 - 2\psi_{2,u}\psi_{2,v}\psi_{3,u}\psi_{3,v} \\
&\qquad +\psi_{3,u}^2\psi_{1,v}^2 + \psi_{1,u}^2\psi_{3,v}^2 - 2\psi_{1,u}\psi_{1,v}\psi_{3,u}\psi_{3,v} \\
&\qquad\qquad +\psi_{1,u}^2\psi_{2,v}^2 + \psi_{2,u}^2\psi_{1,v}^2 - 2\psi_{1,u}\psi_{1,v}\psi_{2,u}\psi_{2,v} \\
&= \Big(\psi_{1,u}^2 + \psi_{2,u}^2 + \psi_{3,u}^2\Big)\Big(\psi_{1,v}^2 + \psi_{2,v}^2 + \psi_{3,v}^2\Big) - \psi_{1,u}^2\psi_{1,v}^2 - \psi_{2,u}^2\psi_{2,v}^2 - \psi_{3,u}^2\psi_{3,v}^2 \\
&\quad -2\psi_{2,u}\psi_{2,v}\psi_{3,u}\psi_{3,v} - 2\psi_{1,u}\psi_{1,v}\psi_{3,u}\psi_{3,v} - 2\psi_{1,u}\psi_{1,v}\psi_{2,u}\psi_{2,v} \\
&= \Big(\psi_{1,u}^2 + \psi_{2,u}^2 + \psi_{3,u}^2\Big)\Big(\psi_{1,v}^2 + \psi_{2,v}^2 + \psi_{3,v}^2\Big) - \Big(\psi_{1,u}\psi_{1,v} + \psi_{2,u}\psi_{2,v} + \psi_{3,u}\psi_{3,v}\Big)^2 \\
&= EG - F^2\,.
\end{aligned}$$

Hence we have

$$\boldsymbol{n}_y = \frac{\widetilde{\boldsymbol{n}}_y}{\sqrt{EG - F^2}}, \tag{1.14}$$

and therefore

$$\begin{aligned}
\int_A \boldsymbol{g}(t,\boldsymbol{y}) \cdot \boldsymbol{n}_y\, ds_y &= \int_P \boldsymbol{g}(t,\boldsymbol{\psi}(t,u,v)) \cdot \boldsymbol{n}_y \sqrt{EG - F^2}\, du\, dv \\
&= \int_P \boldsymbol{g}(t,\boldsymbol{\psi}(t,u,v)) \cdot \widetilde{\boldsymbol{n}}_y\, du\, dv
\end{aligned}$$

follows. Now we can interchange differentiation and integration to obtain

$$\begin{aligned}
\frac{d}{dt}\int_A \boldsymbol{g}(t,\boldsymbol{y}) \cdot \boldsymbol{n}_y\, ds_y &= \int_P \frac{d}{dt}\Big[\boldsymbol{g}(t,\boldsymbol{\psi}(t,u,v)) \cdot \widetilde{\boldsymbol{n}}_y\Big]\, du\, dv \\
&= \int_P \sum_{k=1}^{3} \frac{d}{dt}\Big[g_k(t,\boldsymbol{\psi}(t,u,v))\,\widetilde{n}_k\Big]\, du\, dv \\
&= \int_P \sum_{k=1}^{3} \left[\widetilde{n}_k\,\frac{d}{dt} g_k(t,\boldsymbol{\psi}(t,u,v)) + g_k(t,\boldsymbol{\psi}(t,u,v))\,\frac{d}{dt}\widetilde{n}_k\right]\, du\, dv \\
&= \int_P \sum_{k=1}^{3} \left[\widetilde{n}_k\left(\frac{\partial}{\partial t} g_k(t,\boldsymbol{\psi}(t,u,v)) + \nabla_y g_k(t,\boldsymbol{y})_{y=\psi(t,u,v)} \cdot \frac{d}{dt}\boldsymbol{\psi}(t,u,v)\right)\right. \\
&\qquad\qquad \left. + g_k(t,\boldsymbol{\psi}(t,u,v))\frac{d}{dt}\widetilde{n}_k\right]\, du\, dv.
\end{aligned}$$

It remains to compute the time derivative of the moving normal vector. In particular for the first component we have

$$\begin{aligned}
\frac{d}{dt}\widetilde{n}_1 &= \frac{d}{dt}\Big[\psi_{2,u}\psi_{3,v} - \psi_{3,u}\psi_{2,v}\Big] \\
&= \frac{d}{dt}\left[\frac{\partial}{\partial u}\psi_2(t,u,v)\frac{\partial}{\partial v}\psi_3(t,u,v) - \frac{\partial}{\partial u}\psi_3(t,u,v)\frac{\partial}{\partial v}\psi_2(t,u,v)\right] \\
&= \frac{\partial}{\partial u}\psi_2(t,u,v)\frac{d}{dt}\frac{\partial}{\partial v}\psi_3(t,u,v) + \frac{\partial}{\partial v}\psi_3(t,u,v)\frac{d}{dt}\frac{\partial}{\partial u}\psi_2(t,u,v) \\
&\quad -\frac{\partial}{\partial u}\psi_3(t,u,v)\frac{d}{dt}\frac{\partial}{\partial v}\psi_2(t,u,v) - \frac{\partial}{\partial v}\psi_2(t,u,v)\frac{d}{dt}\frac{\partial}{\partial u}\psi_3(t,u,v) \\
&= \frac{\partial}{\partial u}\psi_2(t,u,v)\frac{\partial}{\partial v}\frac{d}{dt}\psi_3(t,u,v) + \frac{\partial}{\partial v}\psi_3(t,u,v)\frac{\partial}{\partial u}\frac{d}{dt}\psi_2(t,u,v) \\
&\quad -\frac{\partial}{\partial u}\psi_3(t,u,v)\frac{\partial}{\partial v}\frac{d}{dt}\psi_2(t,u,v) - \frac{\partial}{\partial v}\psi_2(t,u,v)\frac{\partial}{\partial u}\frac{d}{dt}\psi_3(t,u,v) \\
&= \frac{\partial}{\partial u}\psi_2(t,u,v)\frac{\partial}{\partial v}w_3(t,u,v) + \frac{\partial}{\partial v}\psi_3(t,u,v)\frac{\partial}{\partial u}w_2(t,u,v) \\
&\quad -\frac{\partial}{\partial u}\psi_3(t,u,v)\frac{\partial}{\partial v}w_2(t,u,v) - \frac{\partial}{\partial v}\psi_2(t,u,v)\frac{\partial}{\partial u}w_3(t,u,v)
\end{aligned}$$

with the velocity

$$w_k(t,u,v) = \frac{d}{dt}\psi_k(t,u,v) = v_k(t,\boldsymbol{\psi}(t,u,v)) = v_k(t,\boldsymbol{y}), \quad k=1,2,3.$$

With the chain rule we then obtain, for $k=1,2,3$,

$$\frac{\partial}{\partial u}w_k(t,u,v) = \frac{\partial}{\partial u}v_k(t,\boldsymbol{\psi}(t,u,v)) = \sum_{\ell=1}^{3}\frac{\partial}{\partial y_\ell}v_k(t,\boldsymbol{y})\frac{\partial}{\partial u}\psi_\ell(t,u,v),$$

as well as

$$\frac{\partial}{\partial v}w_k(t,u,v) = \frac{\partial}{\partial v}v_k(t,\boldsymbol{\psi}(t,u,v)) = \sum_{\ell=1}^{3}\frac{\partial}{\partial y_\ell}v_k(t,\boldsymbol{y})\frac{\partial}{\partial v}\psi_\ell(t,u,v).$$

With this we have

$$\begin{aligned}
\frac{d}{dt}\widetilde{n}_1 &= \frac{\partial}{\partial u}\psi_2(t,u,v)\left[\frac{\partial}{\partial y_1}v_3(t,y)\frac{\partial}{\partial v}\psi_1(t,u,v)\right. \\
&\qquad\qquad \left. +\frac{\partial}{\partial y_2}v_3(t,y)\frac{\partial}{\partial v}\psi_2(t,u,v) + \frac{\partial}{\partial y_3}v_3(t,y)\frac{\partial}{\partial v}\psi_3(t,u,v)\right] \\
&\quad +\frac{\partial}{\partial v}\psi_3(t,u,v)\left[\frac{\partial}{\partial y_1}v_2(t,y)\frac{\partial}{\partial u}\psi_1(t,u,v)\right. \\
&\qquad\qquad \left. +\frac{\partial}{\partial y_2}v_2(t,y)\frac{\partial}{\partial u}\psi_2(t,u,v) + \frac{\partial}{\partial y_3}v_2(t,y)\frac{\partial}{\partial u}\psi_3(t,u,v)\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial u}\psi_3(t,u,v)\left[\frac{\partial}{\partial y_1}v_2(t,y)\frac{\partial}{\partial v}\psi_1(t,u,v)\right.\\
&\qquad\qquad\left.+\frac{\partial}{\partial y_2}v_2(t,y)\frac{\partial}{\partial v}\psi_2(t,u,v)+\frac{\partial}{\partial y_3}v_2(t,y)\frac{\partial}{\partial v}\psi_3(t,u,v)\right]\\
&-\frac{\partial}{\partial v}\psi_2(t,u,v)\left[\frac{\partial}{\partial y_1}v_3(t,y)\frac{\partial}{\partial u}\psi_1(t,u,v)\right.\\
&\qquad\qquad\left.+\frac{\partial}{\partial y_2}v_3(t,y)\frac{\partial}{\partial u}\psi_2(t,u,v)+\frac{\partial}{\partial y_3}v_3(t,y)\frac{\partial}{\partial u}\psi_3(t,u,v)\right]
\end{aligned}
$$

and by some reordering we further conclude

$$
\begin{aligned}
\frac{d}{dt}\widetilde{n}_1 &= \frac{\partial}{\partial y_1}v_2(t,y)\Big[\psi_{1,u}\psi_{3,v}-\psi_{3,u}\psi_{1,v}\Big]+\frac{\partial}{\partial y_1}v_3(t,y)\Big[\psi_{2,u}\psi_{1,v}-\psi_{1,u}\psi_{2,v}\Big]\\
&\quad+\Big[\frac{\partial}{\partial y_2}v_2(t,y)+\frac{\partial}{\partial y_3}v_3(t,y)\Big]\Big[\psi_{2,u}\psi_{3,v}-\psi_{3,u}\psi_{2,v}\Big]\\
&= \frac{\partial}{\partial y_1}v_2(t,y)\Big[\psi_{1,u}\psi_{3,v}-\psi_{3,u}\psi_{1,v}\Big]+\frac{\partial}{\partial y_1}v_3(t,y)\Big[\psi_{2,u}\psi_{1,v}-\psi_{1,u}\psi_{2,v}\Big]\\
&\quad+\Big[\frac{\partial}{\partial y_1}v_1(t,y)+\frac{\partial}{\partial y_2}v_2(t,y)+\frac{\partial}{\partial y_3}v_3(t,y)\Big]\Big[\psi_{2,u}\psi_{3,v}-\psi_{3,u}\psi_{2,v}\Big]\\
&\quad+\frac{\partial}{\partial y_1}v_1(t,y)\Big[\psi_{3,u}\psi_{2,v}-\psi_{2,u}\psi_{3,v}\Big]\\
&= \operatorname{div}_y\boldsymbol{v}(t,\boldsymbol{y})\,\widetilde{n}_1-\frac{\partial}{\partial y_1}\boldsymbol{v}(t,\boldsymbol{y})\cdot\widetilde{\boldsymbol{n}}_y\,.
\end{aligned}
$$

Doing the same computations for the remaining components we obtain

$$
\frac{d}{dt}\widetilde{n}_k=\operatorname{div}_y\boldsymbol{v}(t,\boldsymbol{y})\,\widetilde{n}_k-\frac{\partial}{\partial y_k}\boldsymbol{v}(t,\boldsymbol{y})\cdot\widetilde{\boldsymbol{n}}_y\,,\quad k=1,2,3.
$$

Hence we have, again inserting the parametrisation $\boldsymbol{y}=\boldsymbol{\psi}(t,u,v)$,

$$
\begin{aligned}
&\frac{d}{dt}\int_A\boldsymbol{g}(t,\boldsymbol{y})\cdot\boldsymbol{n}_y\,ds_y\\
&=\int_P\sum_{k=1}^3\left[\widetilde{n}_k\left(\frac{\partial}{\partial t}g_k(t,\boldsymbol{y})+\nabla_y g_k(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})\right)+g_k(t,\boldsymbol{y})\,\frac{d}{dt}\widetilde{n}_k\right]du\,dv\\
&=\int_P\sum_{k=1}^3\left[\frac{\partial}{\partial t}g_k(t,\boldsymbol{y})+\nabla_y g_k(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})+g_k(t,\boldsymbol{y})\operatorname{div}_y\boldsymbol{v}(t,\boldsymbol{y})\right]\widetilde{n}_k\,du\,dv\\
&\qquad\qquad\qquad-\int_P\sum_{k=1}^3 g_k(t,\boldsymbol{y})\,\frac{\partial}{\partial y_k}\boldsymbol{v}(t,\boldsymbol{y})\cdot\widetilde{\boldsymbol{n}}_y\,du\,dv\\
&=\int_P\sum_{k=1}^3\left[\frac{\partial}{\partial t}g_k(t,\boldsymbol{y})+\nabla_y g_k(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y})+g_k(t,\boldsymbol{y})\operatorname{div}_y\boldsymbol{v}(t,\boldsymbol{y})\right]\widetilde{n}_k\,du\,dv
\end{aligned}
$$

$$-\int_P \sum_{\ell=1}^3 \sum_{k=1}^3 g_k(t,\boldsymbol{y})\,\frac{\partial}{\partial y_k} v_\ell(t,\boldsymbol{y})\,\widetilde{n}_\ell\,du\,dv.$$

In the second part we interchange the roles of k and ℓ to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_A \boldsymbol{g}(t,\boldsymbol{y})\cdot\boldsymbol{n}_y\,ds_y \\
&= \int_P \sum_{k=1}^3 \left[\frac{\partial}{\partial t} g_k(t,\boldsymbol{y}) + \nabla_y g_k(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y}) + g_k(t,\boldsymbol{y})\,\mathrm{div}_y\boldsymbol{v}(t,\boldsymbol{y})\right]\widetilde{n}_k\,du\,dv \\
&\qquad - \int_P \sum_{k=1}^3\sum_{\ell=1}^3 g_\ell(t,\boldsymbol{y})\frac{\partial}{\partial y_\ell} v_k(t,\boldsymbol{y})\,\widetilde{n}_k\,du\,dv \\
&= \int_A \sum_{k=1}^3 \left[\frac{\partial}{\partial t} g_k(t,\boldsymbol{y}) + \nabla_y g_k(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y}) + g_k(t,\boldsymbol{y})\,\mathrm{div}_y\boldsymbol{v}(t,\boldsymbol{y})\right] n_k\,ds_y \\
&\qquad - \int_A \sum_{k=1}^3\sum_{\ell=1}^3 g_\ell(t,\boldsymbol{y})\frac{\partial}{\partial y_\ell} v_k(t,\boldsymbol{y})\,n_k\,ds_y
\end{aligned}$$

when returning back to the surface integral over A, recall the representation of the normal vector $\boldsymbol{n}_y$, see (1.14). In what follows we will consider the vector components, $k = 1, 2, 3$,

$$f_k = \frac{\partial}{\partial t} g_k(t,\boldsymbol{y}) + \nabla_y g_k(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y}) + g_k(t,\boldsymbol{y})\,\mathrm{div}_y\boldsymbol{v}(t,\boldsymbol{y}) - \sum_{\ell=1}^3 g_\ell(t,\boldsymbol{y})\frac{\partial}{\partial y_\ell} v_k(t,\boldsymbol{y}).$$

In particular for $k = 1$ we have

$$f_1 = \frac{\partial}{\partial t} g_1(t,\boldsymbol{y}) + \nabla_y g_1(t,\boldsymbol{y})\cdot\boldsymbol{v}(t,\boldsymbol{y}) + g_1(t,\boldsymbol{y})\,\mathrm{div}_y\boldsymbol{v}(t,\boldsymbol{y}) - \sum_{\ell=1}^3 g_\ell(t,\boldsymbol{y})\frac{\partial}{\partial y_\ell} v_1(t,\boldsymbol{y}).$$

When neglecting the arguments and evaluating all operations we can write this as

$$\begin{aligned}
f_1 &= \frac{\partial}{\partial t}g_1 + v_1\frac{\partial}{\partial y_1}g_1 + v_2\frac{\partial}{\partial y_2}g_1 + v_3\frac{\partial}{\partial y_3}g_1 + g_1\left[\frac{\partial}{\partial y_1}v_1 + \frac{\partial}{\partial y_2}v_2 + \frac{\partial}{\partial y_3}v_3\right] \\
&\qquad -g_1\frac{\partial}{\partial y_1}v_1 - g_2\frac{\partial}{\partial y_2}v_1 - g_3\frac{\partial}{\partial y_3}v_1 \\
&= \frac{\partial}{\partial t}g_1 + v_1\frac{\partial}{\partial y_1}g_1 + v_2\frac{\partial}{\partial y_2}g_1 + v_3\frac{\partial}{\partial y_3}g_1 + g_1\frac{\partial}{\partial y_2}v_2 + g_1\frac{\partial}{\partial y_3}v_3 - g_2\frac{\partial}{\partial y_2}v_1 - g_3\frac{\partial}{\partial y_3}v_1.
\end{aligned}$$

When introducing $\mathrm{div}\,\boldsymbol{g}$ we further obtain

$$\begin{aligned}
f_1 &= \frac{\partial}{\partial t}g_1 + v_1\,\mathrm{div}\,\boldsymbol{g} - v_1\frac{\partial}{\partial y_2}g_2 - v_1\frac{\partial}{\partial y_3}g_3 + v_2\frac{\partial}{\partial y_2}g_1 + v_3\frac{\partial}{\partial y_3}g_1 \\
&\qquad +g_1\frac{\partial}{\partial y_2}v_2 + g_1\frac{\partial}{\partial y_3}v_3 - g_2\frac{\partial}{\partial y_2}v_1 - g_3\frac{\partial}{\partial y_3}v_1 \\
&= \frac{\partial}{\partial t}g_1 + v_1\,\mathrm{div}\,\boldsymbol{g} + \frac{\partial}{\partial y_2}\Big(g_1v_2 - g_2v_1\Big) - \frac{\partial}{\partial y_3}\Big(g_3v_1 - g_1v_3\Big).
\end{aligned}$$

In the same way we conclude

$$
\begin{aligned}
f_2 &= \frac{\partial}{\partial t} g_2 + v_2 \operatorname{div} \boldsymbol{g} + \frac{\partial}{\partial y_3}\Big(g_2 v_3 - g_3 v_2\Big) - \frac{\partial}{\partial y_1}\Big(g_1 v_2 - g_2 v_1\Big), \\
f_3 &= \frac{\partial}{\partial t} g_3 + v_3 \operatorname{div} \boldsymbol{g} + \frac{\partial}{\partial y_1}\Big(g_3 v_1 - g_1 v_3\Big) - \frac{\partial}{\partial y_2}\Big(g_2 v_3 - g_3 v_2\Big).
\end{aligned}
$$

In particular we have

$$
\boldsymbol{f} = \frac{\partial}{\partial t}\boldsymbol{g} + \operatorname{div} \boldsymbol{g}\, \boldsymbol{v} + \operatorname{curl}\left[\boldsymbol{g} \times \boldsymbol{v}\right].
$$

This concludes the proof. ■