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Numerical Mathematics 4

Exercise sheet 1, October 17, 2024

Exercise 1: Consider the Dirichlet boundary value problem of the Yukawa equation

$$-\Delta p + \alpha p = f \qquad \text{in } \Omega, \qquad p = 0 \qquad \text{on } \Gamma,$$

where $\alpha > 0$. Derive mixed variational formulations in the spirit of (1.7) and (1.8) in suitable function spaces. How does the structure of the mixed system change?

Exercise 2: Consider a non-overlapping domain decomposition $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ where $\Omega_1 \cap \Omega_2 = \emptyset$. The interface is denote by $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$.

Derive a mixed formulation in the spirit of (1.10) for the coupled problem

$$\begin{aligned} -\Delta u_i &= f_i && \text{in } \Omega_i,\ i = 1,2, \\ u_i &= 0 && \text{on } \partial\Omega_i \cap \partial\Omega,\ i = 1,2, \\ u_1 &= u_2 && \text{on } \Sigma, \\ \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} &= 0 && \text{on } \Sigma, \end{aligned}$$

where n_i denotes the exterior unit normal vector on Ω_i. Hint: Start with Green's first formulae on the subdomains, define $\lambda = \frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2}$ and enforce the continuity $u_1 = u_2$ on Σ in a weak sense.

Exercise 3: For a subspace $Z \subset \mathbb{R}^M$ the orthogonal subspace is defined by

$$Z^\perp = \left\{ \underline{v} \in \mathbb{R}^M : \underline{v} \cdot \underline{z} = 0 \quad \forall \underline{z} \in Z \right\}.$$

a) Prove the following statement by basic means:

The restriction of a $N \times M$ matrix A to $(\ker A)^\perp$ is a bijective mapping between $(\ker A)^\perp$ and $\operatorname{Im} A$.

b) Prove:

$$\dim(\ker A)^\perp = \dim(\operatorname{Im} A),$$
$$\dim(\ker A) + \dim(\operatorname{Im} A) = M.$$

Exercise 4: Let A be a $N \times M$ matrix. Prove by basic means that:

i) $\ker A^\top = (\operatorname{Im} A)^\perp$
ii) $\operatorname{Im} A = (\ker A^\top)^\perp$
iii) $\ker A = (\operatorname{Im} A^\top)^\perp$
iv) $\operatorname{Im} A^\top = (\ker A)^\perp$

Exercise 5: Prove Lemma 2.1.6 (without utilizing a reflexive space).