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Numerical Mathematics 4

Exercise sheet 4, November 28, 2024

Exercise 13: Consider the Neumann boundary value problem of the Poisson problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma$$

for $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma)$ and the solvability condition

$$\int_\Omega f(x)dx + \int_\Gamma g(x)ds_x = 0.$$

Use the constraint

$$\int_\Omega u\,dx = 0$$

to define a unique solution. Formulate this problem as mixed problem by using a Lagrangian multiplier and investigate the unique solvability of the continuous problem and of asuitably chosen discrete problem.

Exercise 15: Consider the boundary value problem of the biharmonic equation

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma$$

for sufficiently smooth u, f and Ω.

a) Derive a variational formulation in $H_0^2(\Omega) := \{v \in H^2(\Omega) : \gamma_0^{int} u = 0, \gamma_1^{int} u = 0\}$ informally. Which kind of conforming discrete ansatz functions are suitable at first glance?

b) Introduce $w := \Delta u$ and derive a mixed problem in u and w by integration by parts. Which kind of conforming discrete ansatz functions are suitable at first glance? Check if the formulation satisfies the assumptions of Remark 3.1.5.

Exercise 16: Consider the primal mixed problem (4.1.2) of the Dirichlet boundary value problem (4.1.1) in a domain which is decomposed into paraaxial rectangles. How can you choose conforming discretization space X_h and Y_h (based on tensor products of 1d form functions) such that X_h is relatively small compared to Y_h and such that the related discrete inf-sup condition of the bilinear form $b(\cdot,\cdot)$ is satisfied?

Exercise 17: We consider the lowest order local Raviart–Thomas space for 2D

$$\mathrm{RT}(T) = \left\{ \vec{v}_h(x) = \begin{pmatrix} a \\ b \end{pmatrix} + c \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, a, b, c \in \mathbb{R} \right\}.$$

Prove the following statements:

a) $|\mathrm{RT}(T)| = 3$, i.e. 3 degrees of freedom,

b) $[P_0(T)]^2 \subset \mathrm{RT}(T) \subset [P_1(T)]^2$,

c) $\mathrm{div}(\mathrm{RT}(T)) = P_0(T)$ and for $\vec{v}_h \in \mathrm{RT}(T) : \vec{v}_h \in H(\mathrm{div}, T)$,

d) $\vec{v}_h \in \mathrm{RT}(T) \Rightarrow \vec{v}_h \in [H^1(T)]^2$,

e) $\vec{v}_h \in \mathrm{RT}(T)$ has constant normal traces along lines $\Sigma = \left\{ \vec{x} \in \overline{T} : (\vec{x} - \vec{p}) \cdot \vec{m} = 0 \right\}$, hence constant normal traces $\vec{v}_h \cdot \vec{n}_{E_{T,i}}$ on the edges $E_{T,1}, E_{T,2}, E_{T,3}$ of T.

f) vice versa, $\vec{v}_h \in \mathrm{RT}(T)$ is uniquely defined be the constant normal traces $\gamma_{n,E_i} \vec{v}_h = \vec{v}_h \cdot \vec{n}_{E_i} = d_i$, $i = 1, 2, 3$, on the edges.

Exercise 18: Consider the Neumann boundary value problem of the Poisson equation

$$\begin{aligned} -\Delta p &= f \quad \text{in } \Omega, \\ \frac{\partial p}{\partial n} &= g \quad \text{on } \Gamma \end{aligned}$$

for $f \in L_2(\Omega)$ and $g \in H^{-1/2}(\Gamma)$ and the solvability condition

$$\int_\Omega f(x)dx + \langle g, 1 \rangle_\Gamma = 0.$$

Derive a dual mixed formulation in appropriated function spaces. Consider the constraint $\int_\Omega p(x)dx = 0$ and the space $L_{2,0} = \{v \in L_2(\Omega) : \int_\Omega v(x)dx = 0\}$ to define a unique solution.

Hint: Note that the Neumann condition is an essential boundary condition for the dual formulation and use Green's formula (4.2.4).