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Exponential integrators for DG finite element discretizations of parabolic problems

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Exponential integrators constitute a well-studied class of numerical methods for the time integration of stiff differential equations. For time-dependent partial differential equations, however, their combination with conforming finite element methods is not efficient because of the mass matrices of the spatial discretization.
In this talk we propose and analyze exponential integrators for *Discontinuous Galerkin methods*. For linear parabolic problems $u_t = \mathcal{L}u + g$, with $\mathcal{L}$ elliptic, the error analysis relates to the well-studied numerical analysis of the semidiscrete problem in space. We establish convergence and stability results, in particular, optimal order error estimates

$$\|u(t_n) - u_h^n\|_{L^2(\Omega)} = \mathcal{O}(h^{k+1} + \tau^p),$$

in terms of the mesh size h and the time step τ. Here, $k+1$ and p are the orders of the DG formulation, resp. the exponential integrator. The proposed method allows to combine exponential integrators with a wide range of space discretizations, e.g. graded or adaptively generated meshes to recover the optimal order of convergence in the presence of singularities. Because the stiff linear part is integrated exactly, the method admits large time steps without parabolic CFL restrictions. Algorithmically, the semi-discrete DG formulation leads to a system

$$M\dot{u}(t) = Au(t) + f(t),$$

where M is the mass matrix and A the stiffness operator arising from an interior penalty DG discretization. The time integration involves the action of the matrix exponential $\exp(\tau M^{-1}A)$ and related matrix functions on certain vectors. These actions are computed efficiently using interpolation methods or Krylov subspace methods, thereby avoiding explicit matrix exponentials and retaining scalability. The combination of exponential integrators and DG finite elements leads to high-order, stable and scalable methods for parabolic problems, also in the presence of singularities. The time integration allows large steps without stability constraints, and the block-diagonal DG mass matrix ensures local and parallelizable computations.

References

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