FMM based solution of electrostatic and magnetostatic field problems

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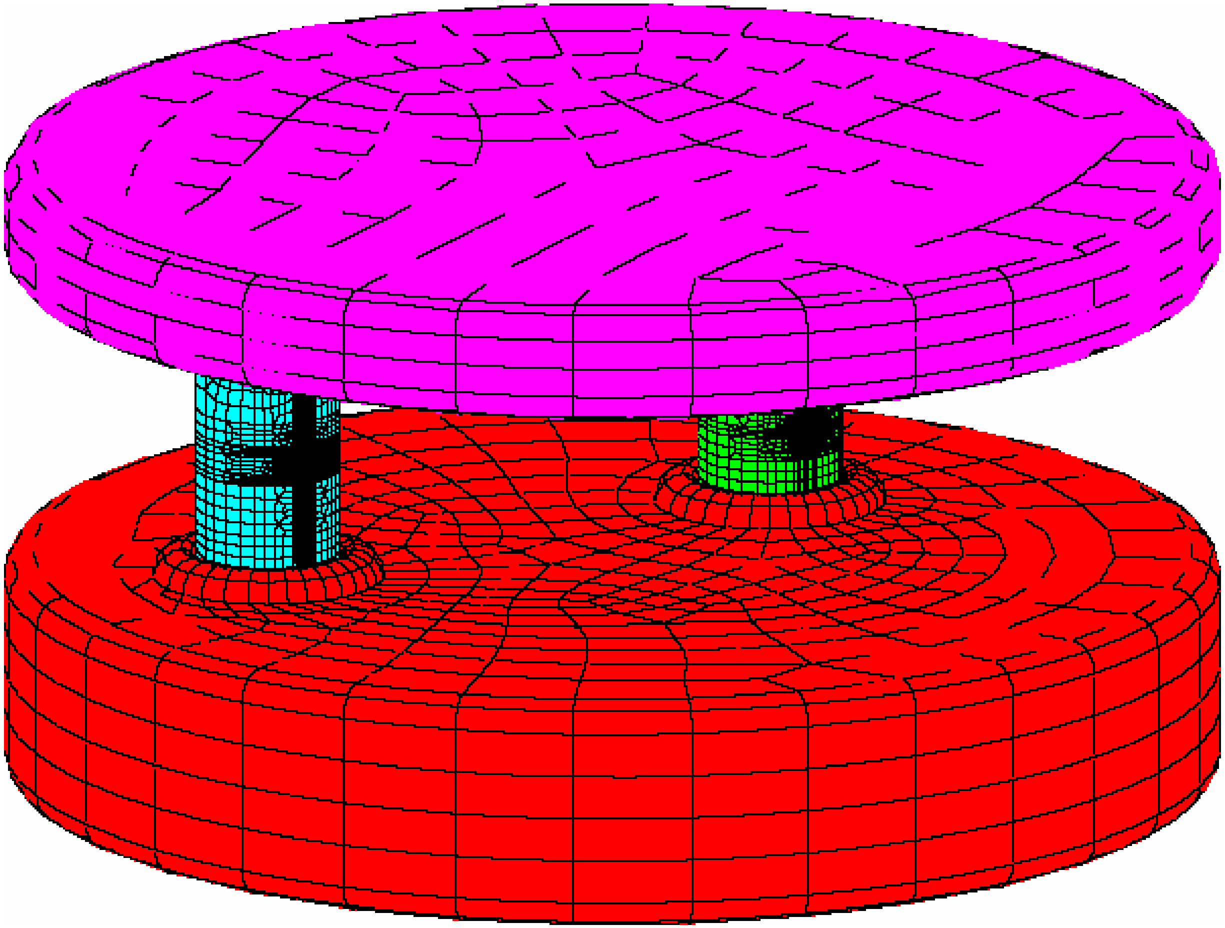
Outline

- Introduction
- Octree in practice
- FMM for direct and indirect BEM formulations
- Fast series expansion transformations
- Postprocessing
- Numerical results
- Conclusion

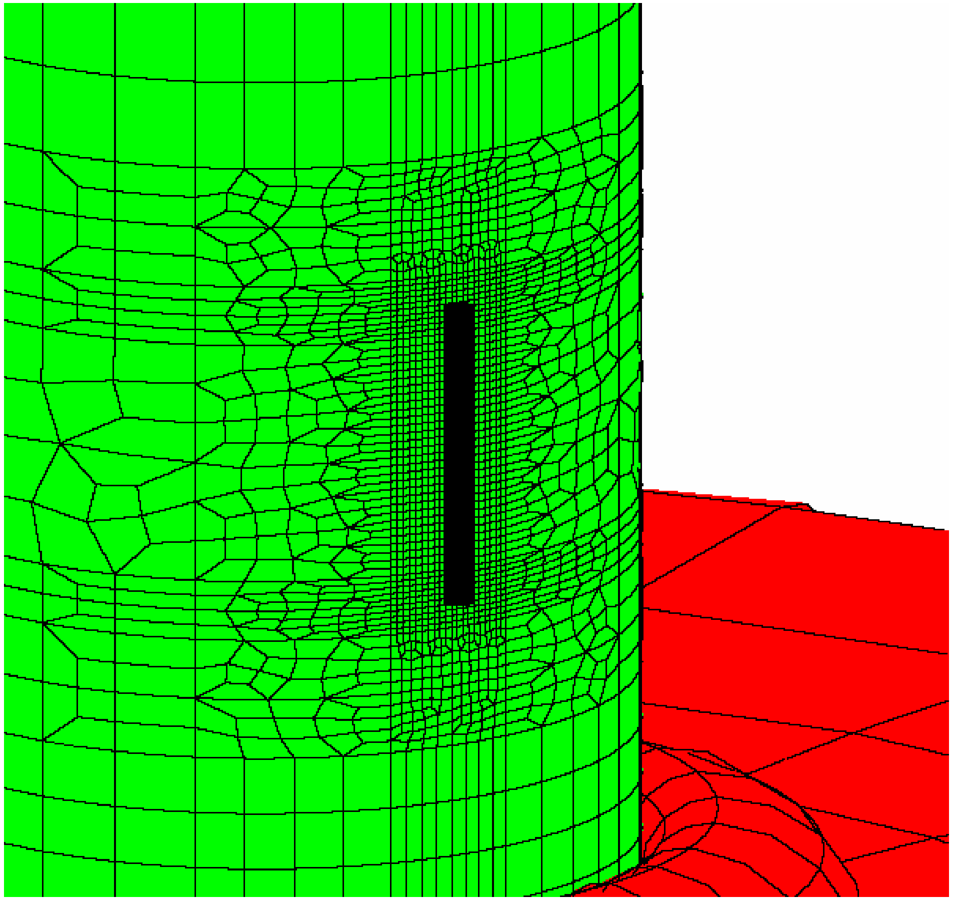
Fast adaptive multipole boundary element method (FAM-BEM)

- Electrostatic, magnetostatic, and steady current flow field problems
- Direct and indirect BEM formulations
- Dirichlet and Neumann boundary conditions
- 8-noded, second order quadrilateral elements
- 20-noded, second order hexahedral elements
- GMRES with Jacobi preconditioner
- Fast multipole method

Adaptive meshes



Adaptive meshes



Series expansions

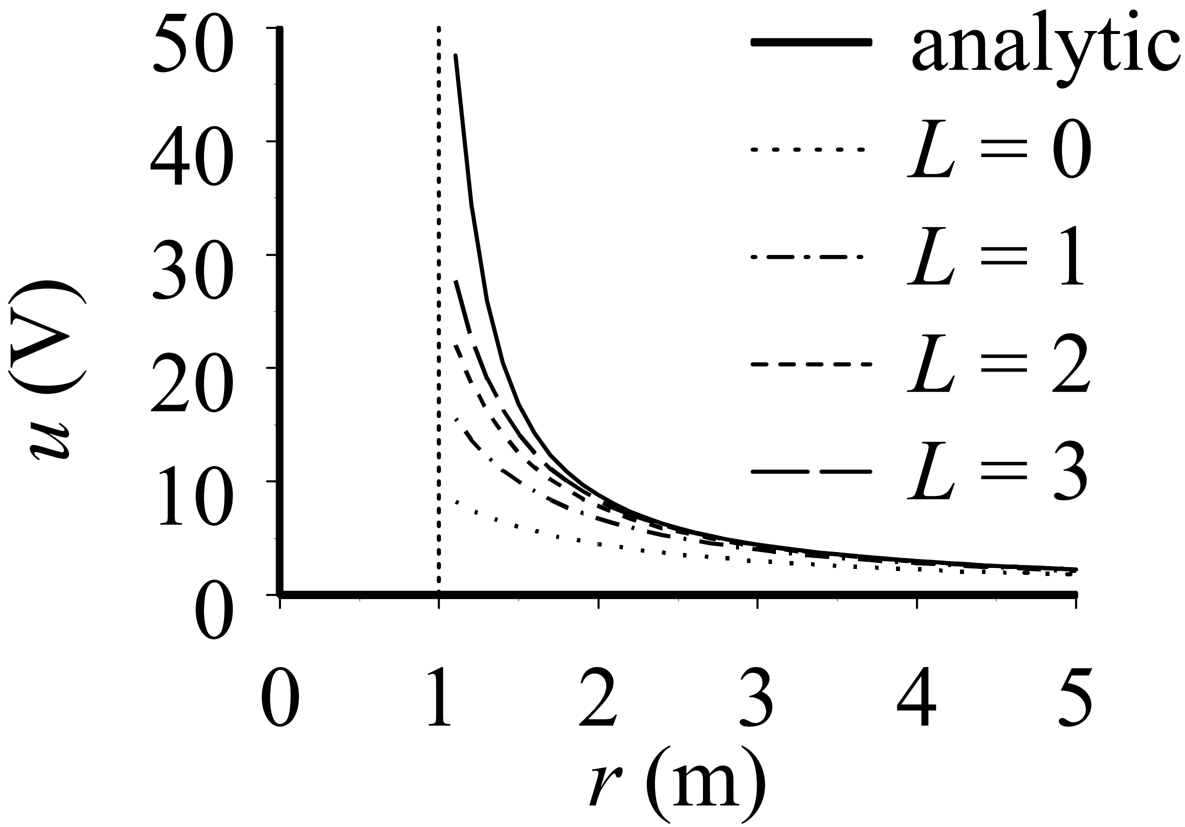
- Multipole expansion

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{L}\sum_{m=-n}^{n}\frac{1}{r^{n+1}}Y_n^m(\theta,\varphi)M_n^m$$

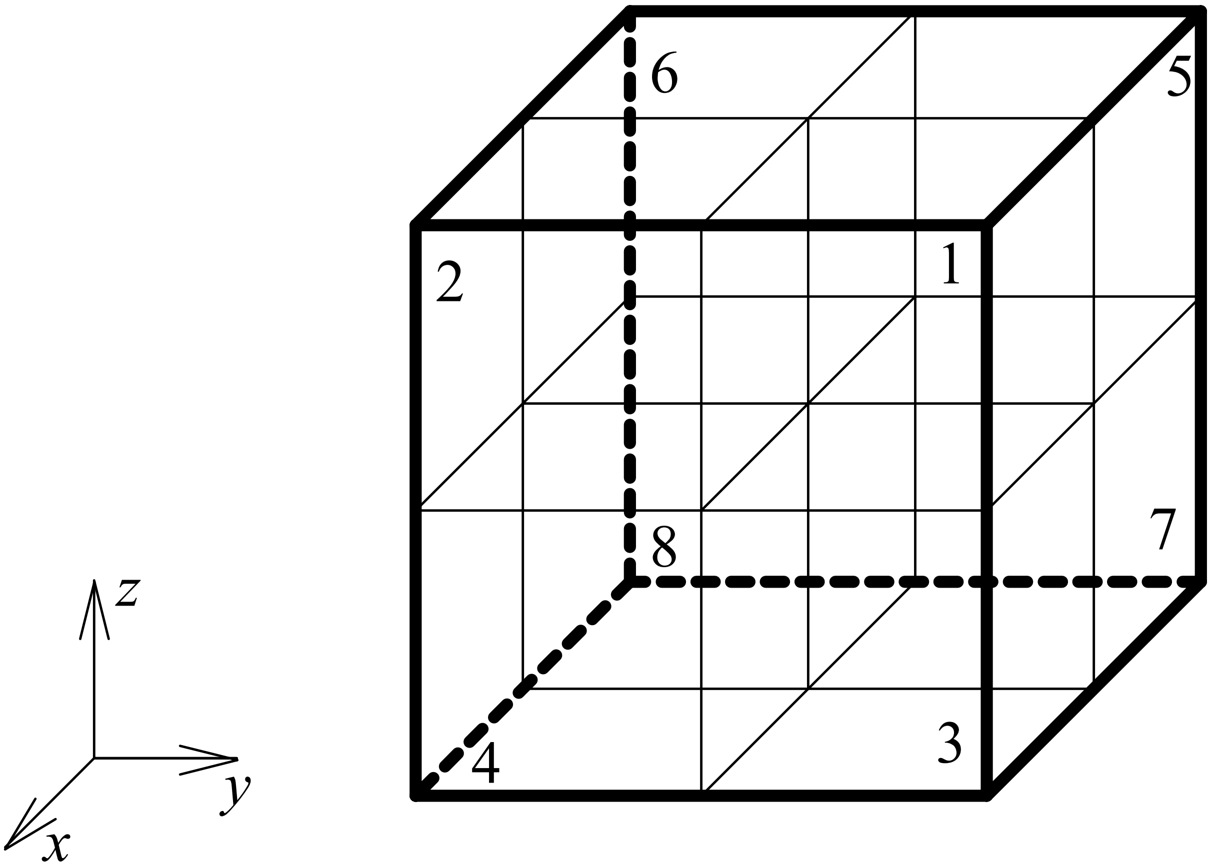
- Local expansion

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{L}\sum_{m=-n}^{n}r^n Y_n^m(\theta,\varphi)L_n^m$$

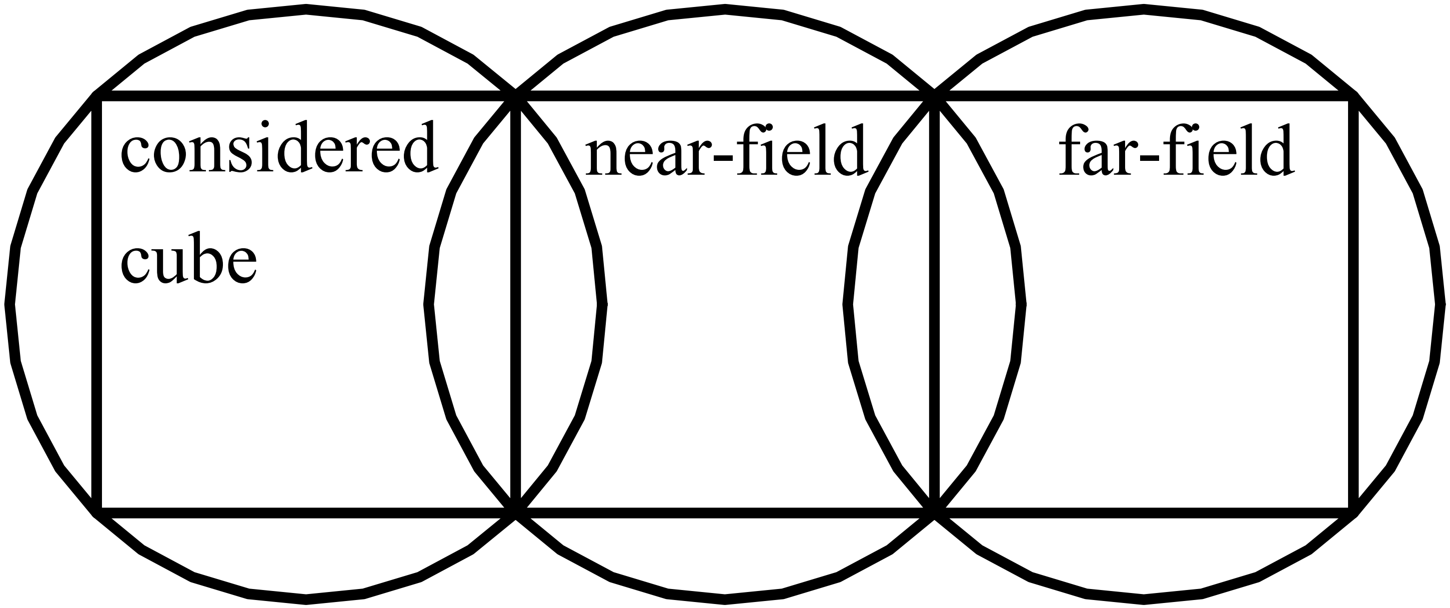
Convergence of the multipole expansion



Octree



Classical near- and far-field definition



Octree in practice

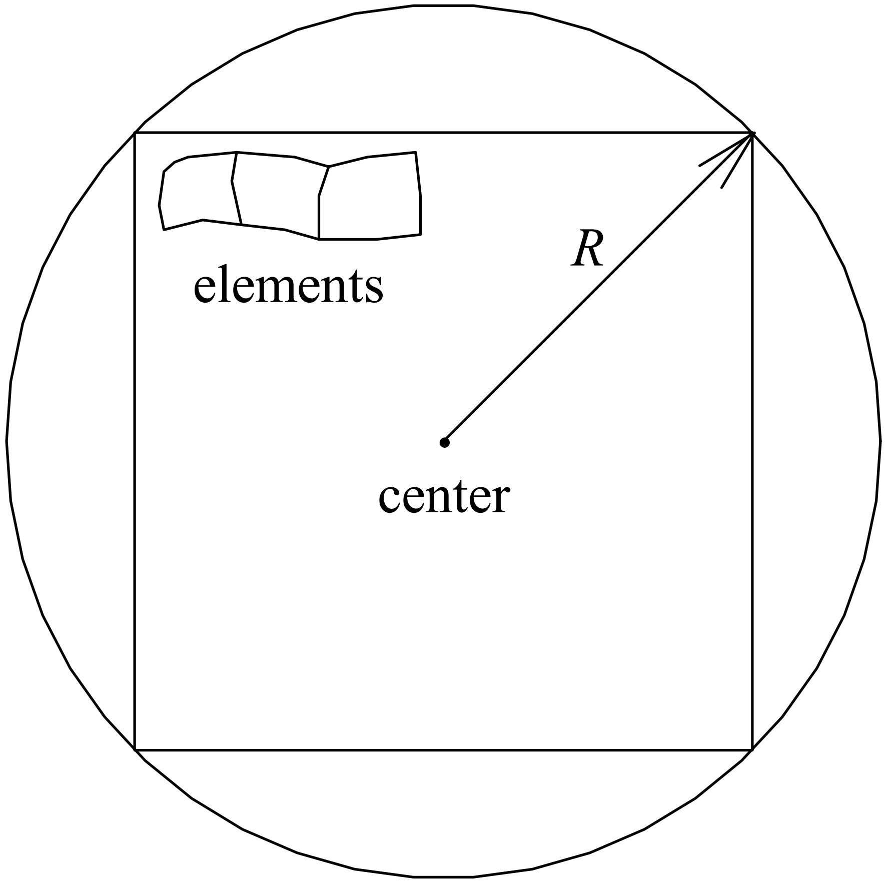
Problems caused by higher order elements

- Extremely varying size of the elements
- Inhomogeneous distribution of elements
- Elements can jut out of a cube

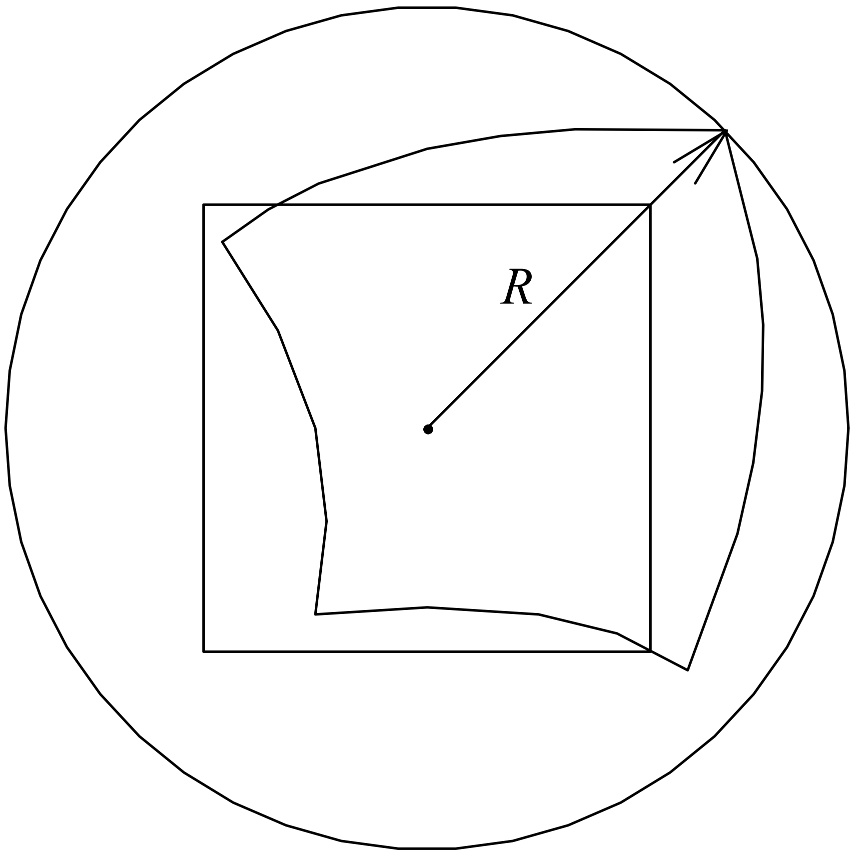
Possible solutions

- Ignore the problems
- Cut the elements at the boundaries of the cubes
- Consider real convergence radii of the cubes

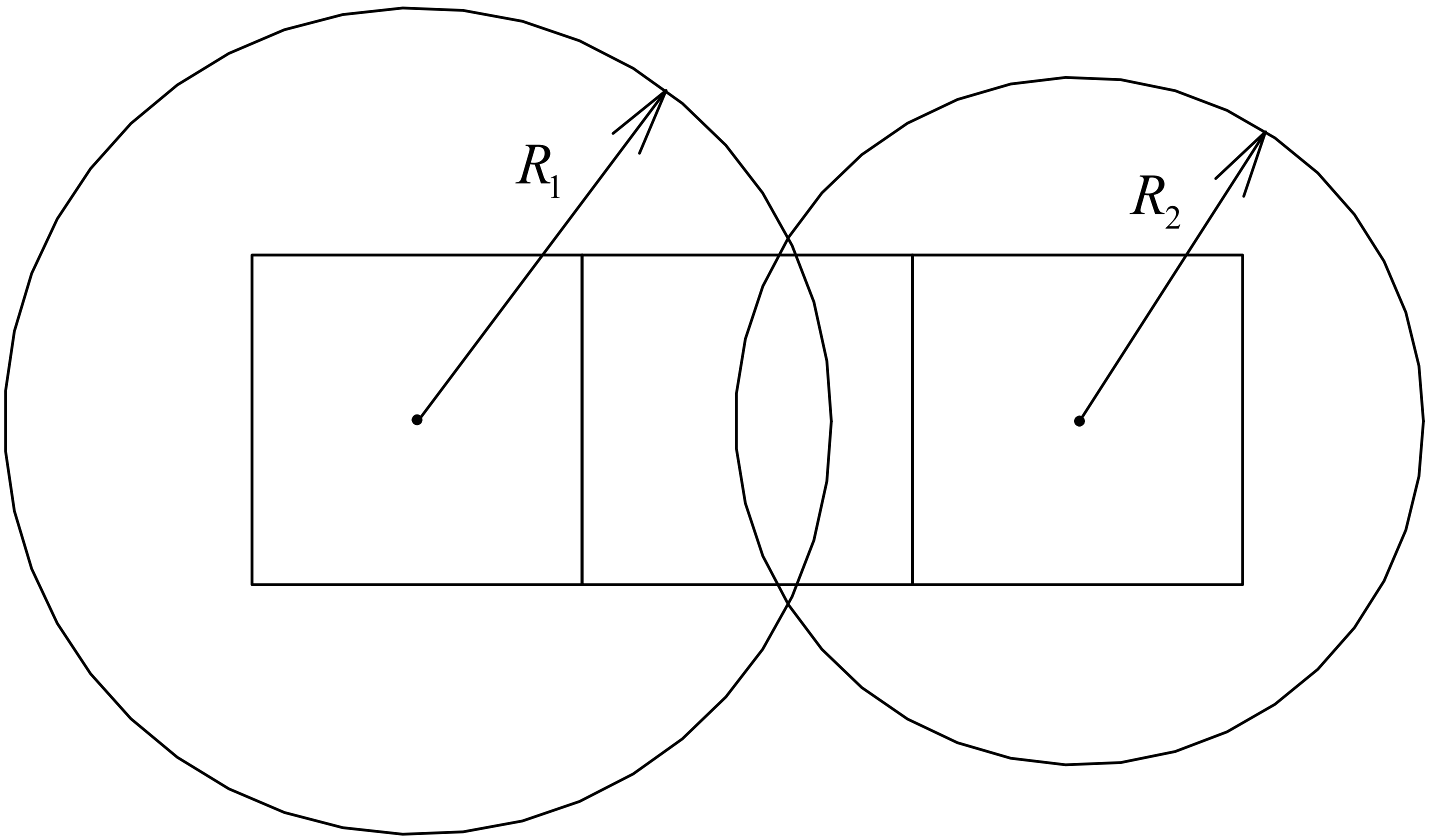
Convergence radius of a cube



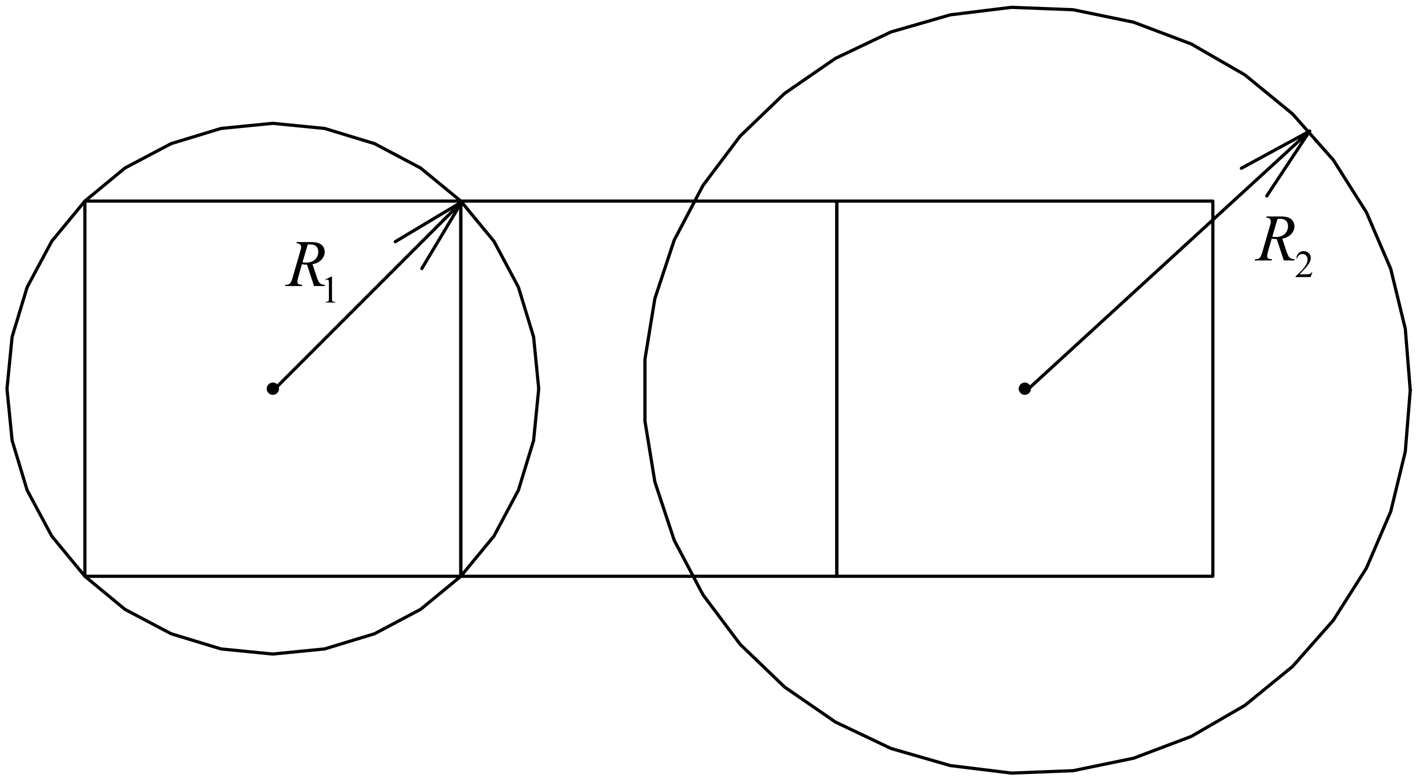
Convergence radius of a cube



Near-field interactions



Far-field interactions



Direct BEM formulation

- Electrostatics
- Steady current flow fields
- Green's theorem

$$c(\boldsymbol{r})u(\boldsymbol{r}) = \oint \frac{\partial u(\boldsymbol{r}')}{\partial n'} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \,\mathrm{d}A' - \oint u(\boldsymbol{r}') \frac{\partial}{\partial n'} \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \,\mathrm{d}A'$$

- Dirichlet boundary conditions
- Neumann boundary conditions

Indirect BEM formulation

- Electrostatics
- Magnetostatics
- Charge densities

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_A \frac{\sigma(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}A'$$

- Dirichlet boundary conditions
- Neumann boundary conditions

Classical multipole expansion

- Classical integral

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_A \frac{\sigma(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}A'$$

- Multipole expansion

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{L} \sum_{m=-n}^{n} \frac{1}{r^{n+1}} Y_n^m(\theta,\varphi) M_n^m$$

$$M_n^m = \int_A \sigma(\boldsymbol{r}') r'^n Y_n^{-m}(\theta',\varphi') \mathrm{d}A'$$

Fast multipole method for double-layer potentials

- Integral

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int_A \tau(\boldsymbol{r}')\nabla_{\boldsymbol{r}'}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\boldsymbol{n}'\,\mathrm{d}A'$$

- Multipole expansion

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{L}\sum_{m=-n}^{n}\frac{1}{r^{n+1}}Y_n^m(\theta,\varphi)M_n^m$$

$$M_n^m = \int_A \tau(\boldsymbol{r}')\boldsymbol{n}(\boldsymbol{r}')\cdot\nabla_{\boldsymbol{r}'}\left(r'^n Y_n^{-m}(\theta',\varphi')\right)\mathrm{d}A'$$

Series expansion transformations

- Multipole-to-multipole transformation
- Multipole-to-local transformation $\leftarrow$ large CPU-time
- Local-to-local-transformation

Multipole-to-local transformation

- Classical approach

$$L_n^m = \sum_{k=0}^{L} \sum_{l=-k}^{k} \frac{M_k^l \, j^{|m-l|-|m|-|l|} \, A_k^l A_n^m Y_{k+n}^{l-m}(\mu, \nu)}{(-1)^k \, \rho^{k+n+1} A_{n+k}^{l-m}}$$

$$O\left(L^4\right)$$

Multipole-to-local transformation

- Transformation in z-direction: $O(L^3)$
- Rotation about the z-axis

$$M'^{m}_{n} = M^{m}_{n}\,\mathrm{e}^{\mathrm{j}m\beta}$$

- Rotation about the y-axis

$$M'^{m'}_{n} = \sum_{m=-n}^{-1} R(n,m,m',\alpha)(-1)^{m}\left(M^{m}_{n}\right)^{*} + \sum_{m=0}^{n} R(n,m,m',\alpha)M^{m}_{n}$$

- Transformation in z-direction

$$L^{m}_{n} = \sum_{k=m}^{L} M^{m}_{k}\,\frac{Y^{0}_{k+n}(0,0)(-1)^{k+m}(n+k)!}{\rho^{k+n+1}\sqrt{(k-m)!(k+m)!(n-m)!(n+m)!}}$$

Multipole-to-local transformation

- "Plane waves": $O(L^2)$
- Definition of main-directions: up, down, north, south, east, west
- Rotation of the coordinate system
- Outgoing wave

$$W(k,l) = \frac{w_k}{dM_k} \sum_{m=-L}^{L} \mathrm{j}^{|m|} \, \mathrm{e}^{\mathrm{j} m \alpha_{l,k}} \sum_{n=|m|}^{L} \frac{M_n^m}{\sqrt{(n-m)!(n+m)!}} \left(\frac{\lambda_k}{d} \right)^n$$

Multipole-to-local transformation

- "Plane waves": $O(L^2)$
- Incoming wave

$$V(k,l) = W(k,l)\,\mathrm{e}^{-\lambda_k z_0}\,\mathrm{e}^{\mathrm{j}\lambda_k\left(x_0\cos\left(\alpha_{l,k}\right)+y_0\sin\left(\alpha_{l,k}\right)\right)}$$

- Local expansion

$$L_n^m = \frac{\mathrm{j}^{|m|}}{\sqrt{(n-m)!(n+m)!}}\sum_{k=1}^{s(\varepsilon)}\left(-\frac{\lambda_k}{d}\right)^n\sum_{l=1}^{M_k}V(k,l)\,\mathrm{e}^{-\mathrm{j}m\alpha_{l,k}}$$

Fast series expansion transformations

In practice

- $L = 9$
- Multipole-to-multipole transformation in z-direction
- Local-to-local transformation in z-direction
- Multipole-to-local transformation in z-direction

Classical

- Potential

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_A \frac{\sigma(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{d}A'$$

- Field strength

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_A \sigma(\boldsymbol{r}') \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \mathrm{d}A'$$

FMM

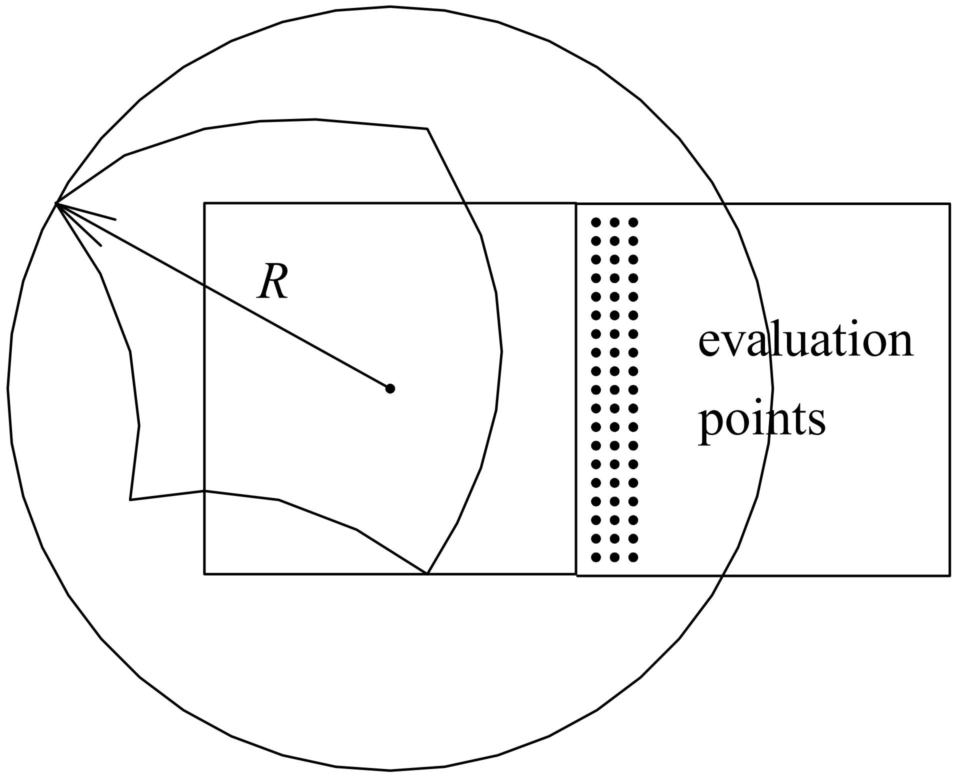
- Octree for elements and evaluation points
- Same FMM algorithm as for matrix-by-vector-product
- Local expansion

$$u(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{L}\sum_{m=-n}^{n} r^n Y_n^m(\theta,\varphi) L_n^m$$

$$\boldsymbol{E} = -\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{L}\sum_{m=-n}^{n} \nabla\left(r^n Y_n^m(\theta,\varphi)\right) L_n^m$$

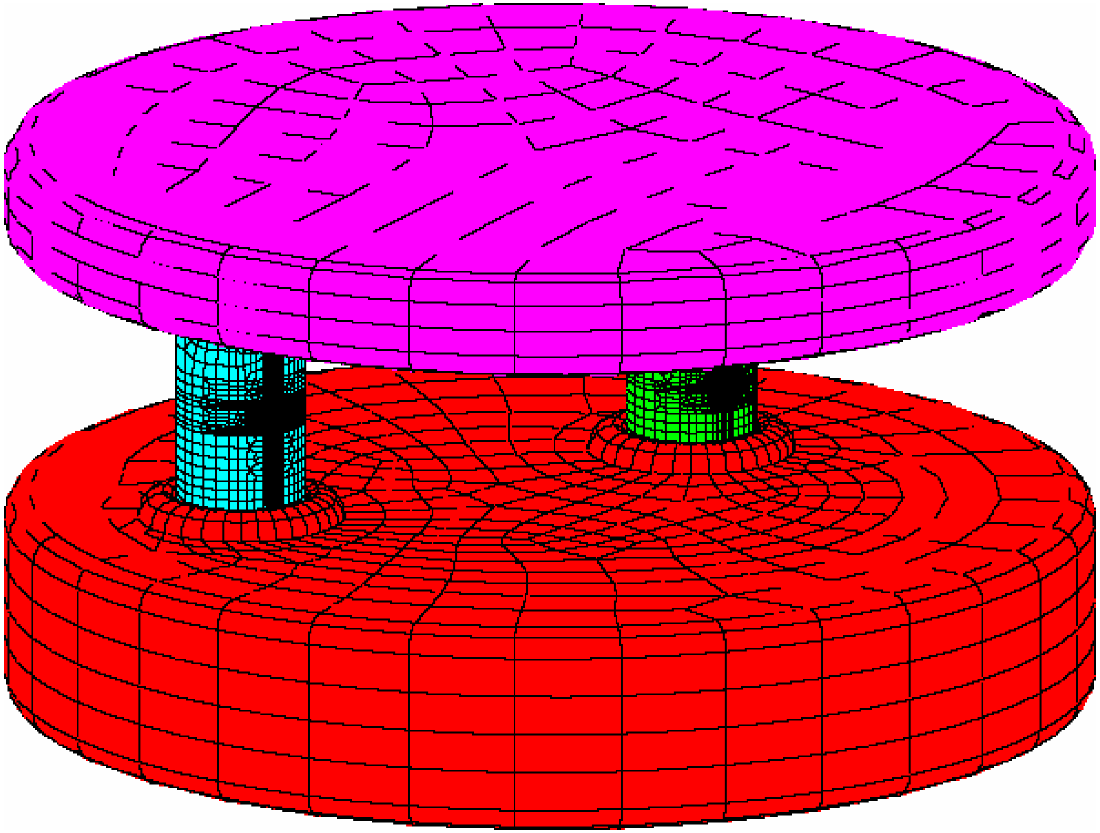
Meshing strategies

- Element size near evaluation points



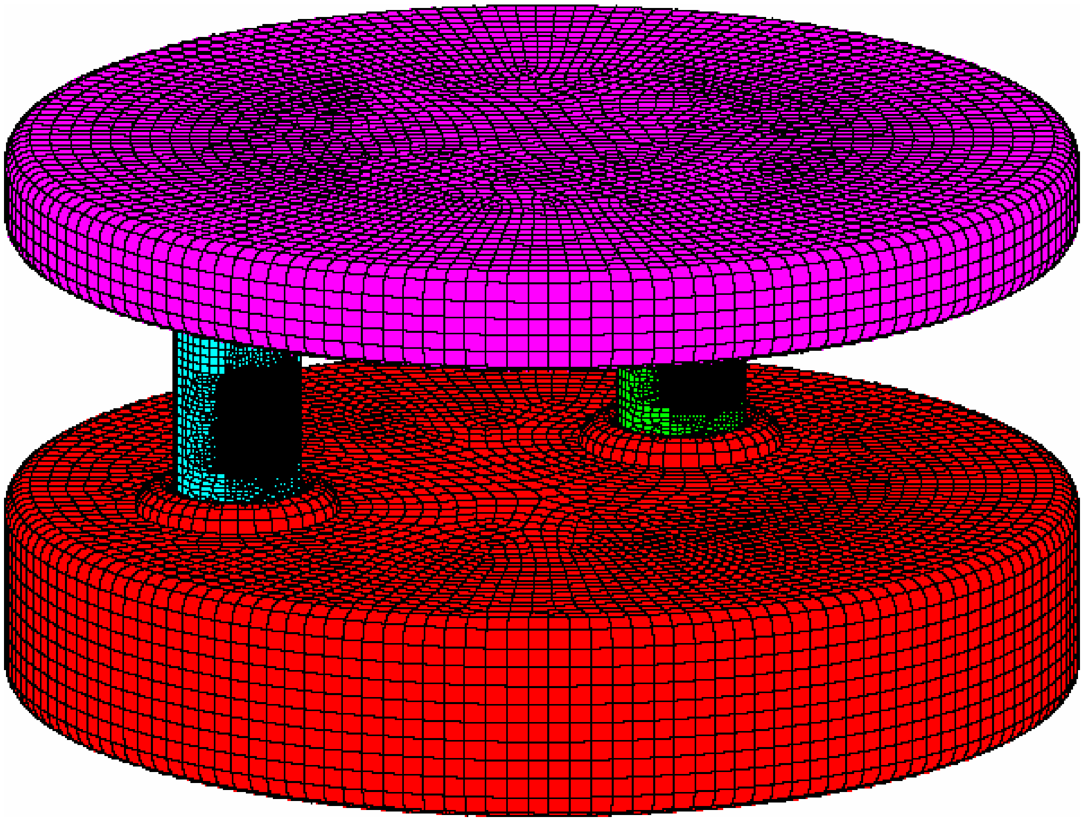
Experiment in high voltage technique

- Geometrical configuration (adaptive mesh)



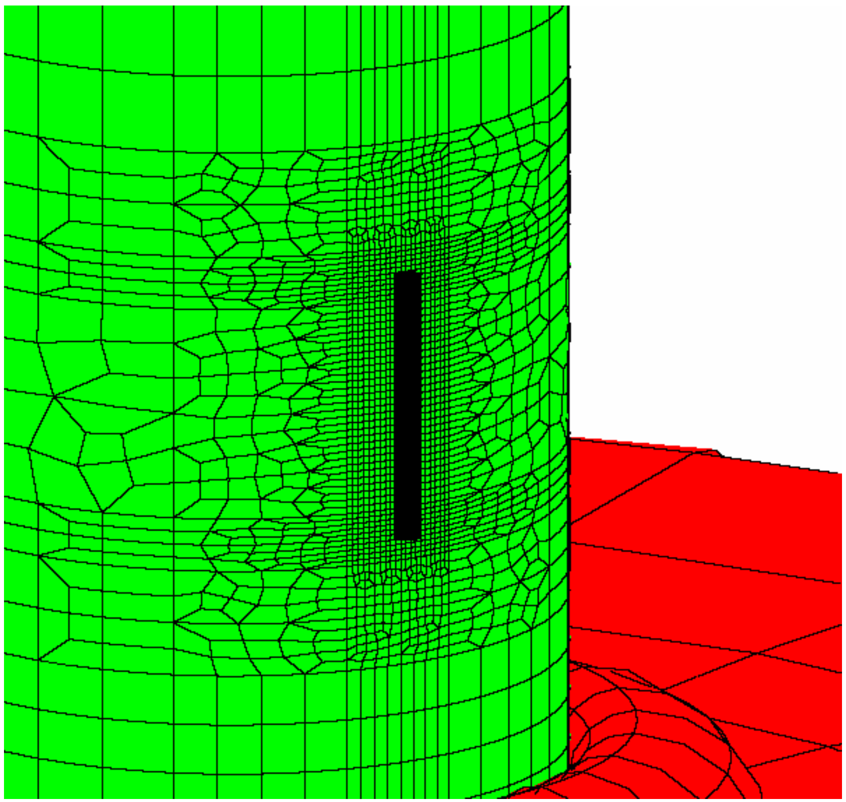
Experiment in high voltage technique

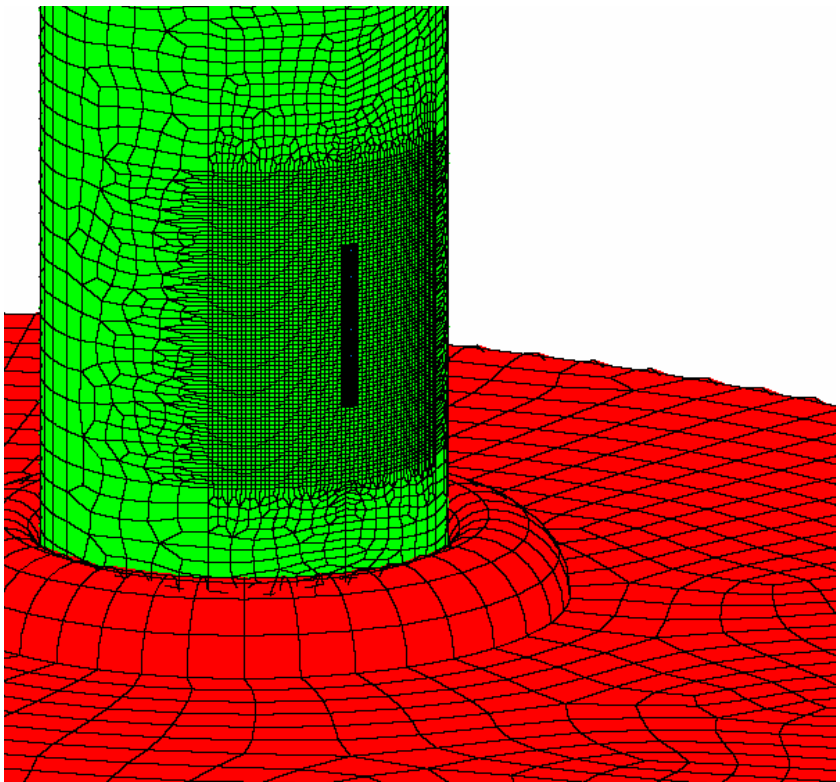
- Geometrical configuration (fine mesh)



Experiment in high voltage technique

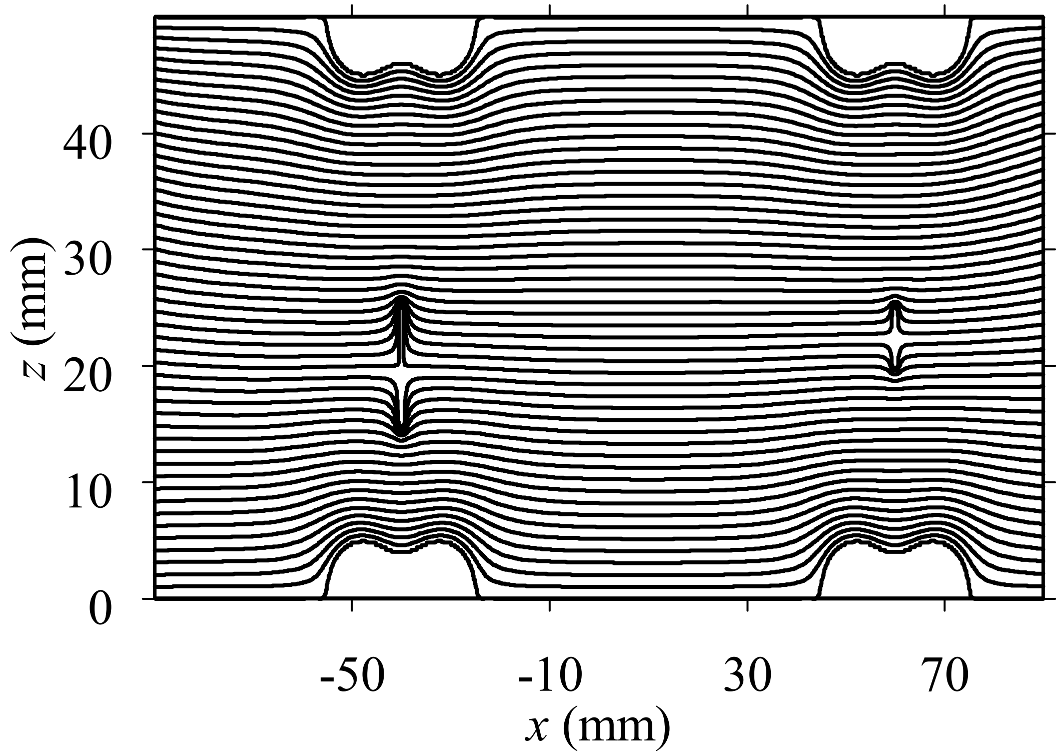
- Geometrical configuration (particle on the right spacer)





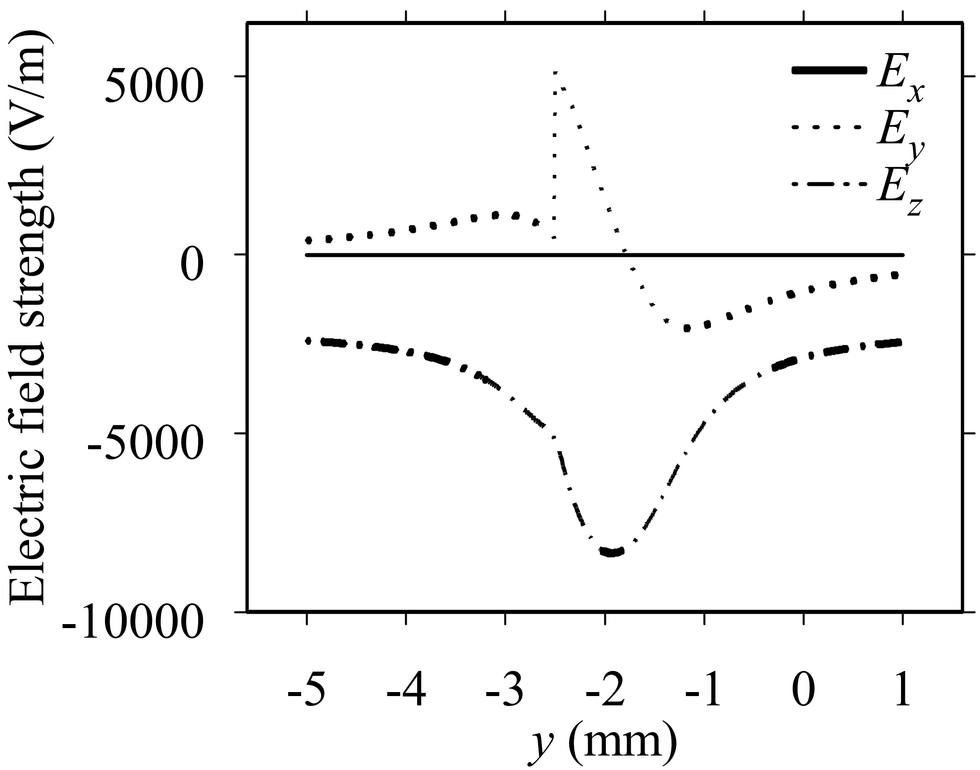
Experiment in high voltage technique

- Potential between the electrodes



Experiment in high voltage technique

- Electric field strength above the particle



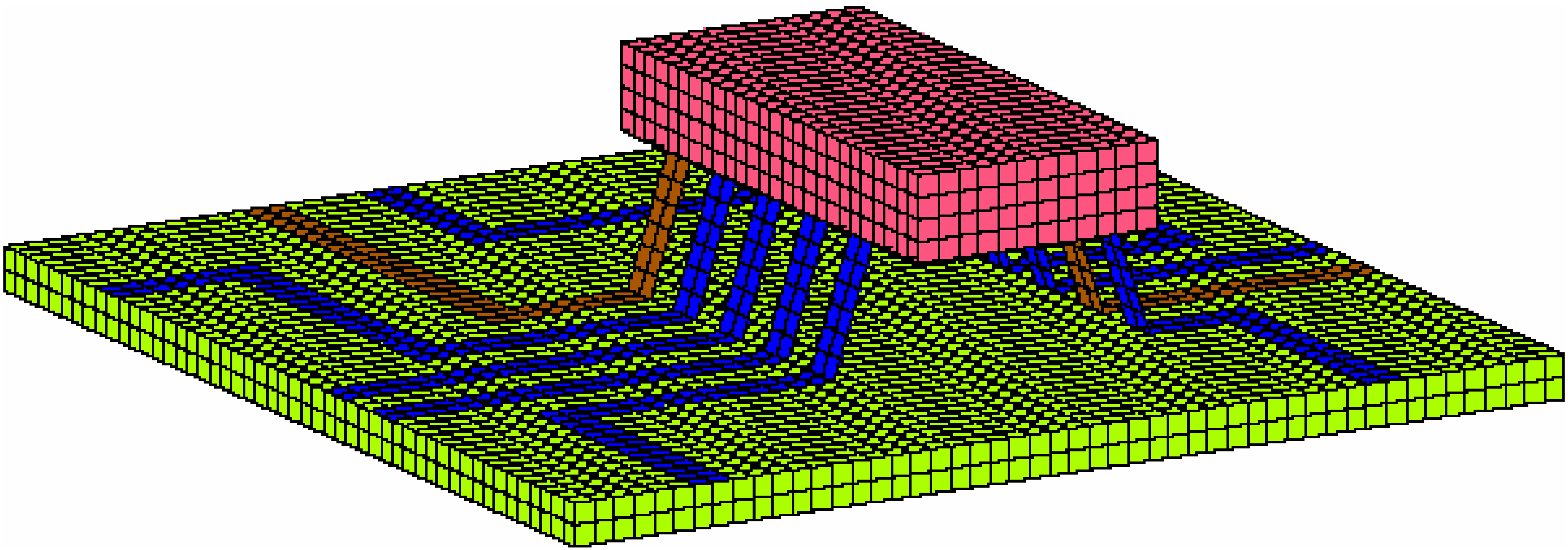
Experiment in high voltage technique

- Computer resources

	Coarse mesh	Fine mesh
Unknowns	28857	93409
Memory	932 MByte	1.2 GByte
CPU-time	41662 s	86385 s
Postprocessing	4324 s	1062 s
Compression rate	85 %	98 %

Chip on a printed circuit board

- Geometrical configuration



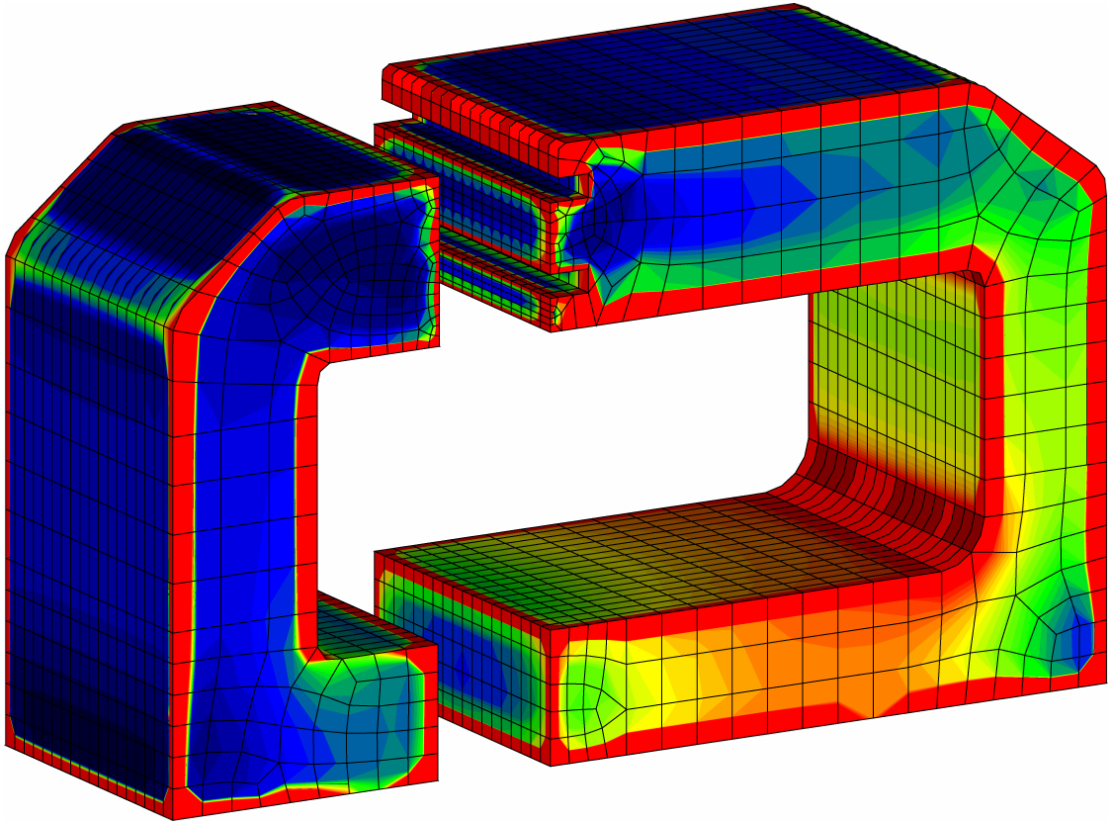
Chip on a printed circuit board

- Computer resources

	Coarse mesh	Fine mesh
Unknowns	20964	56980
Memory	195 MByte	832 MByte
CPU-time	25657 s	344877 s
Compression rate	94 %	99.6 %

Contactor

- Geometrical configuration



Contactor

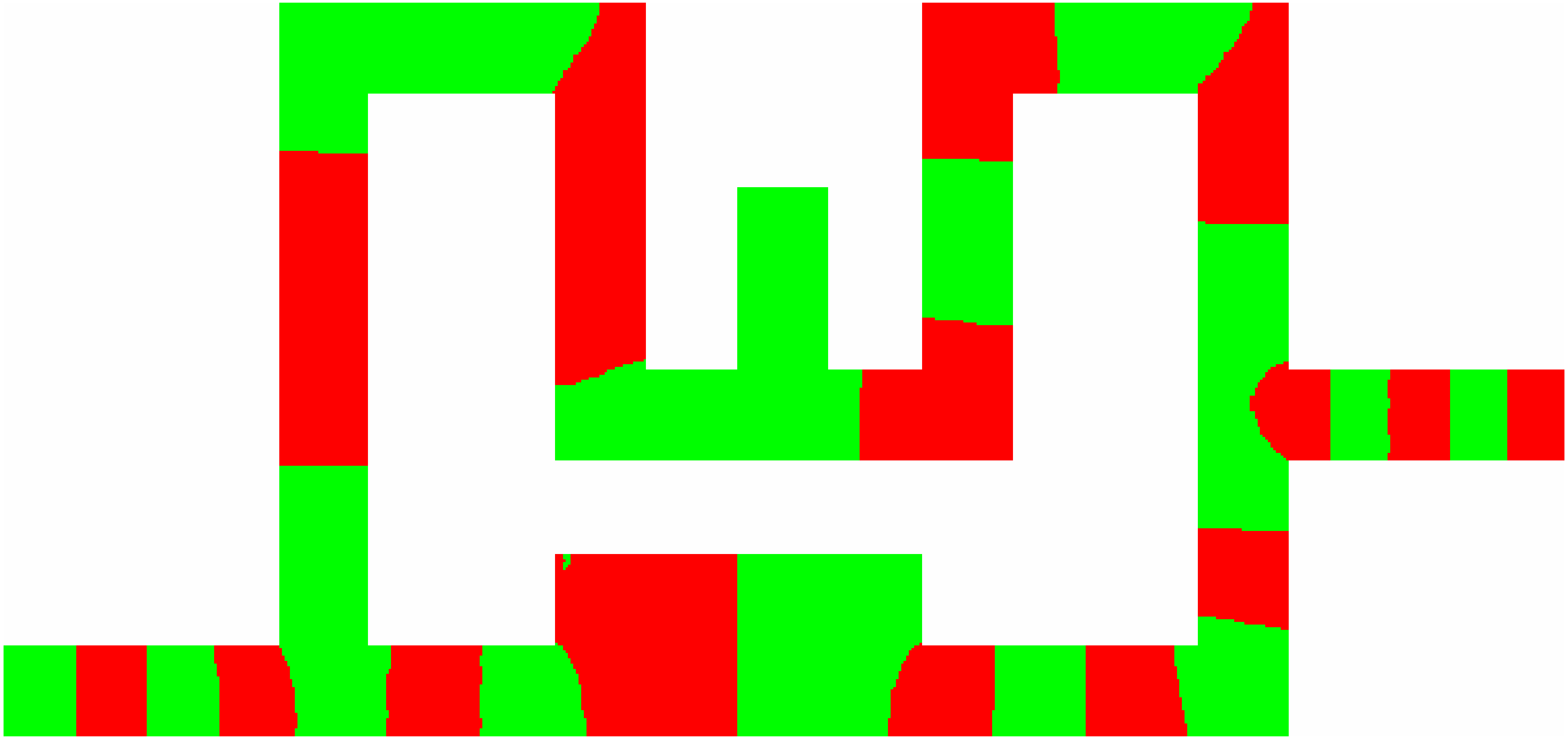
- Number of unknowns: 43949
- CPU time: 1 day
- Non-linear iterations steps: 9
- Memory requirements: 990 MByte
- Compression rate: 93 %

Steady current flow field problem

- Geometrical configuration

Steady current flow field problem

- Potential inside the conductor



Steady current flow field problem

- 3720 second order boundary elements
- 11244 unknowns
- 160 linear iteration steps
- Compression rate: 88.3 %
- CPU-time: 1 hour and 8 minutes (Pentium III 1 GHz)
- 113 MByte (instead of 965 MByte)
- Computation of the potential in 17220 evaluation points in 145 s

Conclusion

- Static electric and magnetic field problems
- Direct and indirect boundary element method
- Volume integral equations for non-linear problems
- Fast adaptive multilevel multipole method
- Adaptive meshes
- High compression rates and accuracy
- Fast postprocessing