Fast evaluation of boundary integral operators arising from an eddy current problem

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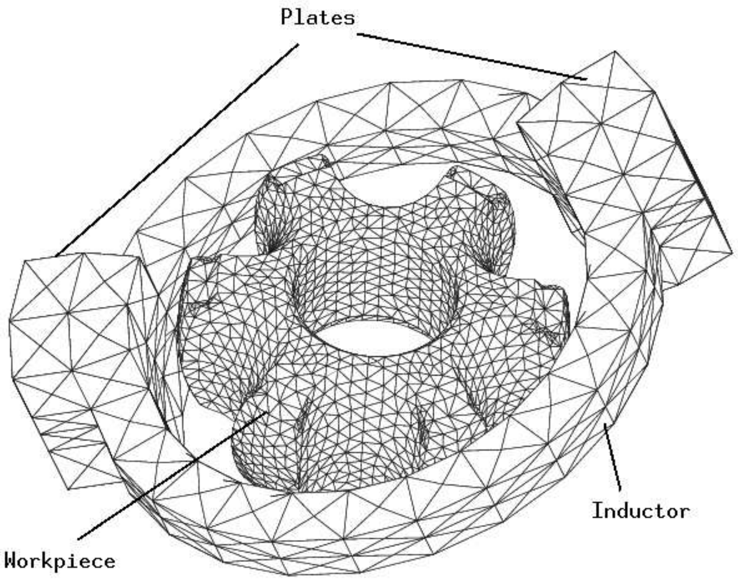


Induction heating: Time harmonic eddy current problem

$$
\begin{aligned}
\operatorname{div} \mathbf{E} &= 0, \quad \text{in } \Omega^+ \\
\mathbf{curl} \frac{1}{\mu} \mathbf{curl} \, \mathbf{E} &= -i\omega(\sigma \mathbf{E} + \mathbf{j}_0) \quad \text{in } \mathbb{R}^3, \\
[\mathbf{n} \times \mathbf{E}] &= [\mathbf{n} \times \frac{1}{\mu} \mathbf{curl} \, \mathbf{E}] = 0 \quad \text{in } \partial\Omega^-
\end{aligned}
$$

R. Hiptmair "Symmetric coupling f. eddy current problems" SIAM

Representation formula of Stratton-Chu kind $\forall \mathbf{x} \in \Omega^+$:

$$\mathbf{E}(\mathbf{x}) = -\int_\Gamma \frac{\mathbf{curl}\,\mathbf{E}(\mathbf{y}) \times \mathbf{n}_y}{4\pi|\mathbf{x}-\mathbf{y}|}\,dS(\mathbf{y}) + \mathbf{curl}\int_\Gamma \frac{\mathbf{n}_y \times \gamma_D \mathbf{E}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\,dS(\mathbf{y})$$

(Remark: $\mathbf{E}\cdot\mathbf{n} = 0$)

Apply traces

$$\begin{aligned} \gamma_D \mathbf{E} &:= \mathbf{n} \times (\mathbf{E} \times \mathbf{n}) \\ \gamma_N \mathbf{E} &:= \mathbf{curl}\,\mathbf{E} \times \mathbf{n} = \mathbf{grad}\,\phi \times \mathbf{n} = \mathbf{curl}_\Gamma\,\phi \end{aligned}$$

from outside (Ω^+) on representation formula and test equations by using **impedance boundary conditions**

$$\gamma_D \mathbf{E} = (1-i)\sqrt{\frac{1}{2\sigma\mu\omega}}\,\mathbf{curl}_\Gamma\,\phi$$

Goal: Find $\mathbf{E} \in \mathcal{V}$ and $\phi \in \mathcal{W}$ solving

$$\begin{aligned} a(\mathbf{v},\mathbf{E}) \quad & -b(\mathbf{v},\phi) && = f(\mathbf{v}) && \text{for all } \mathbf{v} \in \mathcal{V} \\ -b(\mathbf{E},\psi) \quad & -q(\psi,\phi) && = \zeta(\psi) && \text{for all } \psi \in \mathcal{W} \end{aligned}$$

with the bilinear forms

$$\begin{aligned} a(\mathbf{v},\mathbf{E}) &= \int_\Gamma\int_\Gamma \langle \operatorname{curl}_\Gamma \mathbf{v}(\mathbf{x}), \operatorname{curl}_\Gamma \mathbf{E}(\mathbf{y})\rangle \Phi(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} + \text{sparse}, \\ q(\psi,\phi) &= \int_\Gamma\int_\Gamma \langle \mathbf{curl}_\Gamma\, \psi(\mathbf{x}), \mathbf{curl}_\Gamma\, \phi(\mathbf{y})\rangle \Phi(\mathbf{x},\mathbf{y})\, d\mathbf{y}\, d\mathbf{x}, \\ b(\mathbf{v},\phi) &= \int_\Gamma\int_\Gamma \langle \mathbf{curl}_\Gamma\, \phi(\mathbf{y}), \mathbf{v}(\mathbf{x})\rangle \langle \mathbf{grad}_x\, \Phi(\mathbf{x},\mathbf{y}), \mathbf{n}(\mathbf{x})\rangle\, d\mathbf{y}\, d\mathbf{x} \\ &\quad - \int_\Gamma\int_\Gamma \langle \mathbf{curl}_\Gamma\, \phi(\mathbf{y}), \mathbf{n}(\mathbf{x})\rangle \langle \mathbf{grad}_x\, \Phi(\mathbf{x},\mathbf{y}), \mathbf{v}(\mathbf{x})\rangle\, d\mathbf{y}\, d\mathbf{x} \\ &\quad + \text{sparse}, \end{aligned}$$

where $\Phi(\mathbf{x},\mathbf{y}) = 1/(4\pi\|\mathbf{x}-\mathbf{y}\|)$ is the Laplace singularity function.

Surface edge elements: Let $\mathcal{V}_h = \text{span}\{\mathbf{b}_i \ : \ i \in \mathcal{E}\}$ be the set of first order surface edge elements.

Surface nodal elements: Let $\mathcal{W}_h = \text{span}\{\psi_i \ : \ i \in \mathcal{N}\}$ be the set of surface nodal elements.

Galerkin discretization: Leads to

$$\begin{pmatrix} \mathsf{A} & -\mathsf{B} \\ -\mathsf{B}^\top & -\mathsf{Q} \end{pmatrix} = \text{rhs}$$

with

$$\mathsf{A}_{ij} = a(\mathbf{b}_i, \mathbf{b}_j), \quad \mathsf{Q}_{ij} = q(\psi_i, \psi_j) \quad \text{and} \quad \mathsf{B}_{ij} = b(\mathbf{b}_i, \psi_j).$$

Problem: All matrices dense. 10000 Triangles $\rightarrow$ 4 Gbyte

Split bilinear forms into their components

$$\mathsf{M}_{ij} := \int_\Gamma \int_\Gamma \chi_i(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{y}) \lambda_j(\mathbf{y}) \, d\mathbf{y} \, d\mathbf{x}$$

$$\mathsf{G}_{ij} := \int_\Gamma \int_\Gamma \chi_i(\mathbf{x}) \, \mathbf{grad}_i \, \Phi(\mathbf{x}, \mathbf{y}) \lambda_j(\mathbf{y}) \, d\mathbf{y} \, d\mathbf{x}$$

Problem: The singularity function Φ is not local,

so M, G are not sparse. $\Rightarrow$ High complexity.

Still expensive matrix-vector multiplication $\mathbf{y} = \mathsf{M}\mathbf{x}$ or $\mathbf{y} = \mathsf{G}\mathbf{x}$.

Idea: Replace Φ and **grad** Φ on a sub-domain $\tau \times \sigma \subseteq \Gamma \times \Gamma$ by

$$\tilde{\Phi}^{\tau,\sigma}(x,y) \quad := \quad \sum_{\nu=1}^{m}\sum_{\mu=1}^{m} \Phi(x_\nu^\tau, x_\mu^\sigma)\mathcal{L}_\nu^\tau(x)\mathcal{L}_\mu^\sigma(y) \quad \text{and}$$

$$\widetilde{\mathbf{grad}\ \Phi}^{\tau,\sigma}(x,y) \quad := \quad \mathbf{grad}\ \tilde{\Phi}^{\tau,\sigma}\ .$$

$\Longrightarrow$ **ONLY POINTWISE EVALUATION OF KERNEL**

Result: Local matrices M and G, for example

$$\mathsf{M}_{ij}^{\tau,\sigma} := \int_\tau\int_\sigma \chi_i(x)\Phi(x,y)\lambda_j(y)\,dy\,dx \approx \int_\tau\int_\sigma \chi_i(x)\tilde{\Phi}^{\tau,\sigma}(x,y)\lambda_j(y)\,dy\,dx$$

$$= \sum_{\nu=1}^{m}\sum_{\mu=1}^{m} \underbrace{\Phi(x_\nu^\tau, x_\mu^\sigma)}_{=:\mathsf{S}_{\nu\mu}^{\tau,\sigma}} \underbrace{\int_\tau \chi_i(x)\mathcal{L}_\nu^\tau(x)\,dx}_{=:\mathsf{V}_{i\nu}^{\tau}} \underbrace{\int_\sigma \lambda_j(y)\mathcal{L}_\mu^\sigma(y)\,dy}_{=:\mathsf{W}_{j\mu}^{\sigma}}$$

$$= (\mathsf{V}^\tau \mathsf{S}^{\tau,\sigma}(\mathsf{W}^\sigma)^\top)_{ij}$$

Problem: The function Φ is **not globally smooth** $\Longrightarrow m$ too large.

Idea: It is asymptotically smooth, i.e., **locally smooth** far from the diagonal $x = y \Longrightarrow m \ll N$.

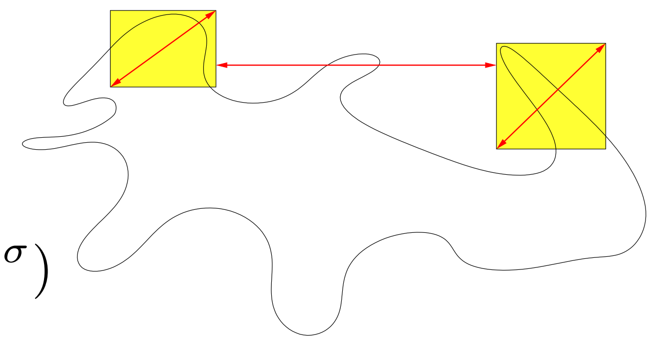
Error estimate: For the tensor-product interpolation of order m on axis-parallel boxes $B^\tau \supseteq \tau$ and $B^\sigma \supseteq \sigma$ we have

$$\|\tilde{\Phi}^{\tau,\sigma} - \Phi\|_{L^\infty(B^\tau \times B^\sigma)} \leq \frac{C}{\operatorname{dist}(B^\tau, B^\sigma)} 3^{-m} \left(\frac{\operatorname{dist}(B^\tau, B^\sigma)}{\operatorname{diam}(B^\tau \times B^\sigma)} \right)^{-(m+1)} .$$

Admissibility condition:

Convergence for $\eta \in]0, \frac{3}{2\sqrt{2}}[$ if

$$max\{\operatorname{diam}(B^\tau), \operatorname{diam}(B^\sigma)\} \leq 2\eta \operatorname{dist}(B^\tau, B^\sigma)$$



Cluster tree: Split Γ into a hierarchy of subdomains.

Block partition: Partition P of $\Gamma \times \Gamma$ consisting of admissible subdomains in the **farfield** P_{far} and non-admissible subdomains in **nearfield** P_{near}.

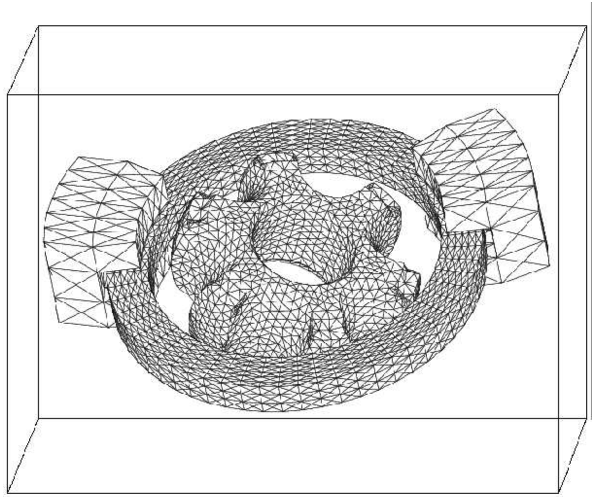
$$P_{\text{far}} := \{\tau \times \sigma \in P \ : \ \tau \times \sigma \text{ is admissible}\}, \qquad P_{\text{near}} := P \setminus P_{\text{far}}.$$

Matrix: M is approximated by

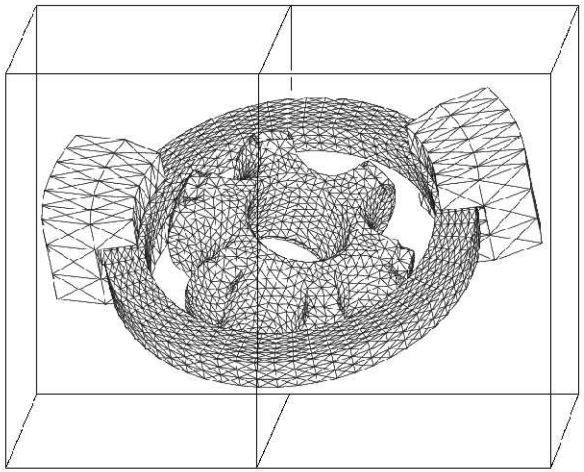
$$\widetilde{\mathsf{M}} := \sum_{\tau \times \sigma \in P_{\text{far}}} \mathsf{V}^{\tau} \mathsf{S}^{\tau,\sigma} \mathsf{W}^{\sigma\top} + \sum_{\tau \times \sigma \in P_{\text{near}}} \mathsf{M}^{\tau,\sigma}.$$

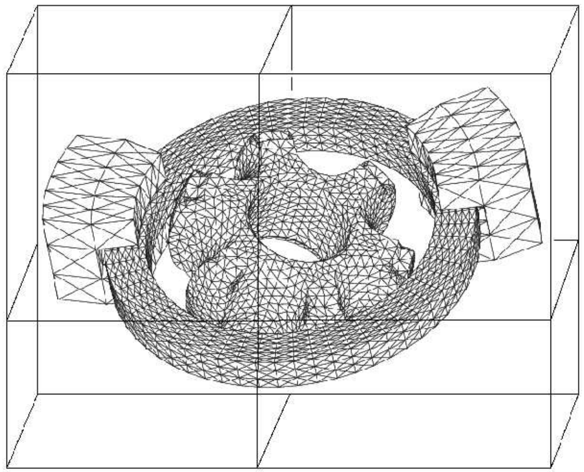
Error decreases exponentially in m, **complexity** $\mathcal{O}(Nm^3 \log N)$.

1. **Preparation:** Build cluster tree + partition, calculate V, S, W
2. **Multiplications**



Cluster tree by **bisection:**





Block partition by recursive algorithm.

Calculation of V and W for each cluster τ

$$\mathsf{V}^{\tau}_{i\nu} := \int_{\tau} \chi_i(x)\mathcal{L}^{\tau}_{\nu}(x)\,dx\,, \qquad \mathsf{W}^{\tau}_{i\nu} := \int_{\tau} \lambda_i(x)\mathcal{L}^{\tau}_{\nu}(x)\,dx$$

and S for admissible blocks $\tau \times \sigma$

$$\mathsf{S}^{\tau,\sigma}_{\nu\mu} := \Phi(x^{\tau}_{\nu}, x^{\sigma}_{\mu}).$$

Idea of nested basis: For cluster τ with sons τ' holds

$$\mathcal{L}^{\tau}_{\nu} = \sum_{\nu'=1}^{k} \underbrace{\mathcal{L}^{\tau}_{\nu}(x^{\tau'}_{\nu'})}_{=:\mathsf{T}^{\tau',\tau}_{\nu'\nu}} \mathcal{L}^{\tau'}_{\nu'}.$$

Store V^{τ}, W^{τ}, only for leaf clusters. Others: $\mathsf{V}^{\tau} = \sum_{\tau' \in \mathrm{sons}(\tau)} \mathsf{V}^{\tau'} \mathsf{T}^{\tau',\tau}$

$$\mathbf{y} \approx \widetilde{\mathsf{M}}\mathbf{x} := \sum_{\tau\times\sigma\in P_{\text{far}}} \mathsf{V}^{\tau}\mathsf{S}^{\tau,\sigma}\mathsf{W}^{\sigma\top}\mathbf{x} + \sum_{\tau\times\sigma\in P_{\text{near}}} \mathsf{M}^{\tau,\sigma}\mathbf{x}$$

1. Forward transform: $x^{\sigma} := \mathsf{W}^{\sigma\top} x$

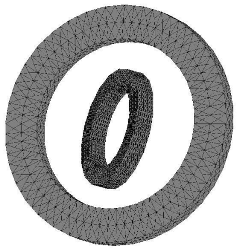
Use W^{σ} in leaves and $\mathsf{T}^{\sigma',\sigma}$ in remaining clusters.

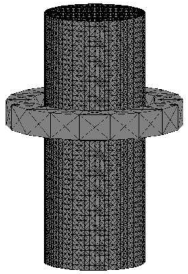
2. Multiply farfield: $y^{\tau} := \sum_{\sigma,(\tau,\sigma)\in P_{\text{far}}} \mathsf{S}^{\tau,\sigma} x^{\sigma}$

3. Backward transform: $y := \sum_{\tau} \mathsf{V}^{\tau} y^{\tau}$

Use V^{τ} in leaves and $\mathsf{T}^{\tau',\tau}$ in remaining clusters.

4. Add Nearfield





$$m = 2, \quad \eta = 1, \quad \text{Rel. Error} := \int_\Gamma \frac{\|\mathbf{j}_{H^2}(\mathbf{x}) - \mathbf{j}_{St.}(\mathbf{x})\|}{\|\mathbf{j}_{St.}(\mathbf{x})\|}\, dS_{\mathbf{x}}$$

	Standard		$\mathcal{H}^2$		
N	Mem[MB]	Time[min]	Mem	Time	Rel. Error
6916	125.3	114	35.0	43	2.5_{-3}
11420	342.0	402	71.0	96	1.5_{-3}
23840	954.1	876	93.4	144	2.2_{-3}
46724	5725.0	–	333.0	540	–

N=8836, Operator Err.= $\frac{\|K-\tilde{K}\|_2}{\|K\|_2}$

η	m	Build[min]	Mem[MB]	MVM[s]	Op. Err.
1.4	2	38.1	52	3.3	1.1_{-4}
1.1		44.5	59	3.8	9.2_{-5}
0.8		61.5	79	4.5	4.5_{-5}
0.5		70.9	90	5.0	1.7_{-5}
1.4	3	50.4	61	5.5	2.1_{-5}
	4	70.3	78	8.2	1.0_{-5}
	5	101.8	108	11.0	4.3_{-6}
Standard		127.4	205	8.6	