Algebraic Multigrid Preconditioners for Adaptive-Cross-Approximated Boundary Element Matrices

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Outline of the Talk

- Motivation
- Problem Formulation
- Boundary Element Method

  Boundary Integral Operators
  Sparse Representation (ACA)
- Algebraic Multigrid Methods

  Components
  Design for BE-Matrices
- Numerical Results

Motivation

Objective: Solving a partial differential equation (second order) with solvers of optimal complexity $\mathcal{O}(N_h)$

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idea $\rightarrow$ preconditioner C_h^{-1} with

$$\kappa(C_h^{-1}K_h) = \mathcal{O}(1)$$
$$ops(C_h^{-1} * \underline{v}_h) = \mathcal{O}(N_h)$$

Problem Formulation

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\Gamma = \partial\Omega$.

1. **Interior Dirichlet Problem**:

$$\begin{array}{rcll} -\Delta u & = & 0 & x \in \Omega \\ u(x) & = & g(x) & x \in \Gamma \end{array}$$

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$$\begin{aligned} -\Delta u &= 0 \quad & x \in \Omega \\ u(x) &= g(x) \quad & x \in \Gamma \end{aligned}$$

2. **Representation Formula**:

$$\sigma(y)u(y) = -\int_\Gamma u(x)\frac{\partial E}{\partial n_x}(x,y)ds_x + \int_\Gamma \frac{\partial u}{\partial n_x}(x)E(x,y)ds_x$$

$$\begin{aligned} \sigma(y) &= 0 \quad & y \notin \bar{\Omega} \\ \sigma(y) &= \tfrac{\Theta}{2\Pi} \quad & y \in \Gamma \\ \sigma(y) &= 1 \quad & y \in \Omega \end{aligned}$$

$$\sigma(y)\frac{\partial u}{\partial n_y}(y) \quad = \quad -\frac{\partial}{\partial n_y}\int_\Gamma u(x)\frac{\partial E}{\partial n_x}(x,y)ds_x + \int_\Gamma v(x)\frac{\partial E}{\partial n_y}(x,y)ds_x$$

with $v(x) = \frac{\partial u}{\partial n_x}(x)$.

we obtain the compact operator-form:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Boundary Integral Operators

Hypersingular Operator: $(Du)(y) = -\frac{\partial}{\partial n_y} \int_\Gamma \frac{\partial E}{\partial n_x}(x,y)u(x)ds_x$

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large eigenvalues $\leftrightarrow$ high frequency eigenfunctions
small eigenvalues $\leftrightarrow$ low frequency eigenfunctions

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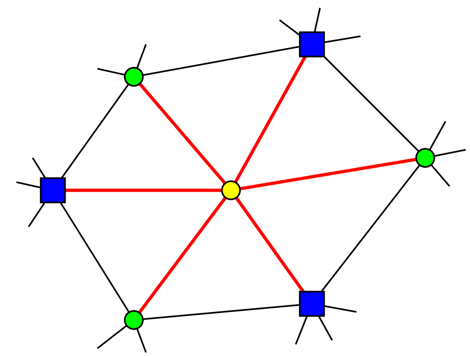
Single Layer Potential: $(Vv)(y) = \int_\Gamma E(x,y)v(x)ds_x$

pseudo differential operator of order -1
eigenvalues, eigenvectors act conversly compared to
the hypersingular operator

BEM - Linear System

Interior Dirichlet problem reduces to $Vv = (\frac{1}{2}I + K)u$

Find $\underline{v}_h \in \mathbb{R}^{N_h}$ such that $K_h \underline{v}_h = \underline{f}_h$



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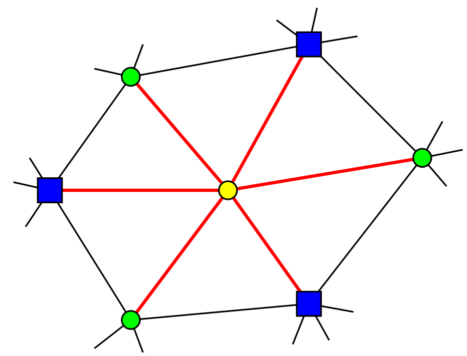

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   (a) (non-)symmetric, positive definite
   (b) dense

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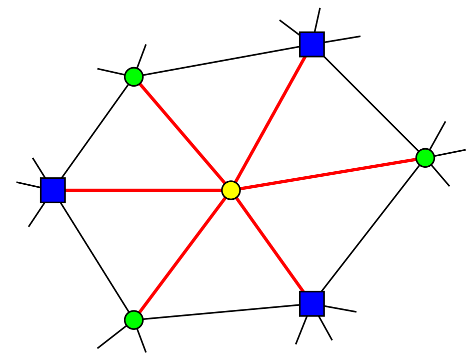

1. System matrix $K_h (= V_h) \in \mathbb{R}^{N_h \times N_h}$ is
   (a) (non-)symmetric, positive definite
   (b) dense
2. Right-hand side $\underline{f}_h \in \mathbb{R}^{N_h}$
3. Construction of an iterative solver

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Problem: Dense matrix, vector multiplication is of order $\mathcal{O}(N_h^2)$

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M. Bebendorf, S. Rjasanow [2001]

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Split the dense BE-matrix into clusters D^i
with corresponding indices $I = \{1, \ldots, N_h\}$

$$I \times I = \bigcup_{i=1}^{k} t_1^i \times t_2^i$$

BEM - Sparse Representation (ACA)

Let D_1, D_2 clusters with

$$\text{diam } D_2 < \eta \, \text{dist}(D_1, D_2)$$

$$\rightarrow \eta\text{-admissible}$$

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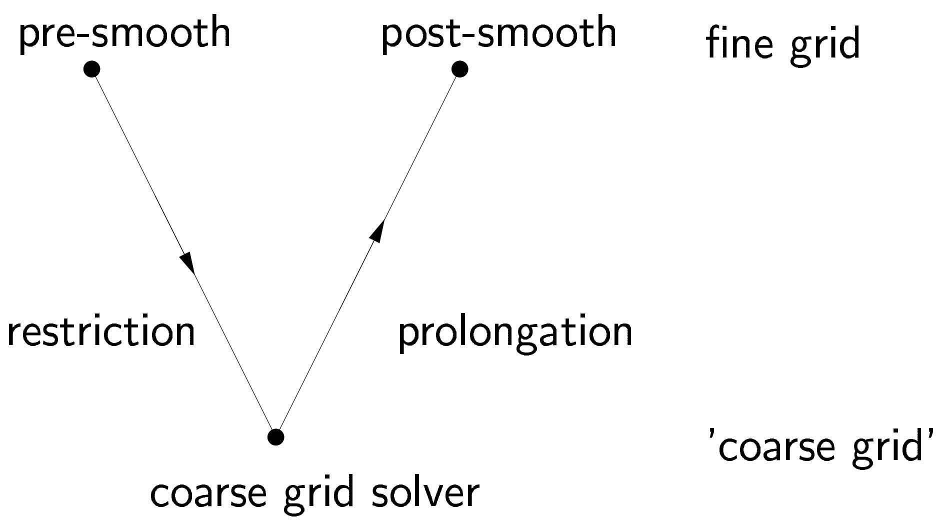
$$\to \tilde{K}_h = K^{near} + \sum_{i=1}^{N_B} \sum_{j=1}^{r_i} u_j^i v_j^{i\top}$$

Algebraic Multigrid Methods - General



Objective: Construction of a multigrid cycle by using the system matrix and the right-hand side

Motivation: no hierarchical grid
(GMG) coarse grid is very large

Components of an AMG Method

1. **Coarsening strategy**

   split the degrees of freedom into fine and coarse grid nodes

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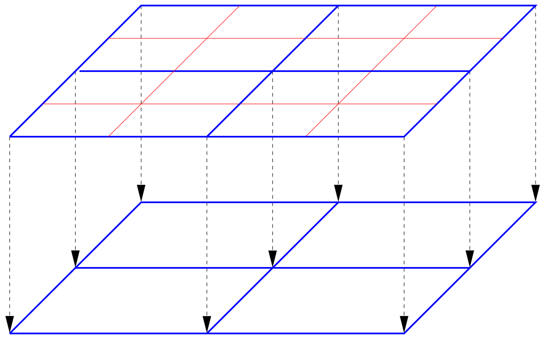
4. **Smoothing operator**

   Jacobi method
   Gauss-Seidel method
   Bramble-Leyk-Pasciak smoother

AMG designed for BEM - Coarsening

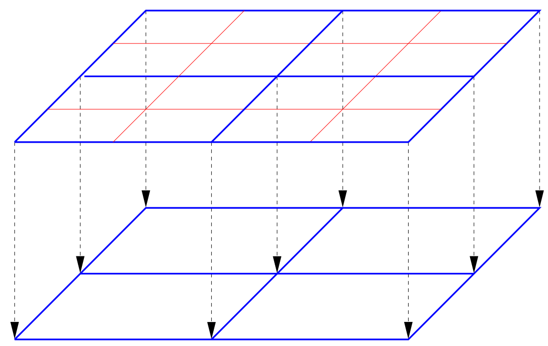
Definition of an **auxiliary matrix**

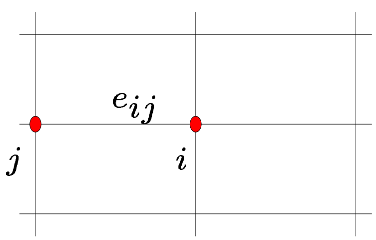


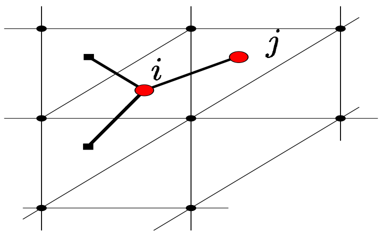
AMG designed for BEM - Coarsening

Definition of an **auxiliary matrix**

$$(B_h)_{ij} = \begin{cases} -\frac{1}{\|e_{ij}\|} & i \neq j \text{ connected} \\ d_{ii} - \sum_{i \neq j} b_{ij} \geq 0 & i = j \\ 0 & \text{otherwise} \end{cases}$$







Coarse auxiliary/system matrix (sparse, M-matrix)

$$B_H = (P_h^B)^T B_h P_h^B \qquad K_H = (P_h^B)^T K_h P_h^B$$

AMG designed for BEM - Smoothing

1. Hypersingular Operator $\rightarrow$ Gauss-Seidel method
2. Single Layer Potential $\rightarrow$ Bramble-Leyk-Pasciak (BLP) smoother

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Problem: Converse behavior of eigenvalues and eigenfunctions w.r.t. Hypersingular operator

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Idea: suggested by Bramble, Leyk and Pasciak:
The Analysis of Multigrid Algorithms for Pseudo-Differential Operators of Order Minus One (1994)

BLP - Smoother

Let A_h be some discretization of the boundary corresponding to the Laplace-Beltrami operator. $A_h =$ auxiliary matrix B_h

An appropriate BLP-smoothing sweep is

$$\underline{u}_h = \underline{u}_h + \tau_h A_h(\underline{f} - K_h \underline{u}_h)$$

with $0 < \tau_h < 1/\lambda_u$ and λ_u the largest eigenvalue of

$$A_h K_h \underline{w} = \lambda \underline{w}.$$

AMG designed for BEM - Restriction

Remember:

$$\tilde{K}_h = K_h^{near} + \tilde{K}_h^{far}$$

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Galerkin's method:

$$\begin{aligned} K_H^{near} &= P_h^\top K_h^{near} P_h \\ \tilde{K}_H^{far} &= P_h^\top \sum_{i=1}^{N_B} \sum_{j=1}^{r_i} u_j^i \, (v_j^i)^\top P_h \\ &= \sum_{i=1}^{N_B} \sum_{j=1}^{r_i} P_h^\top u_j^i \, (P_h^\top v_j^i)^\top \end{aligned}$$

AMG designed for BEM - Coarse Grid Correction

Objective: Solving the equation system on the 'coarsest' grid

$$K_H \underline{u}_H = \underline{f}_H$$

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1. Full BEM: direct methods
2. ACA BEM: iterative methods (Richardson, Tschebyscheff, PCG)
   direct methods (evaluation of the ACA matrix)

Complexity Analysis

Matrix-by-Vector Multipl.: ops$(\widetilde{K}_h \underline{v}_h) = M_h := O(\varepsilon^{-\alpha} N_h^{1+\alpha})$

Preconditioning Operation: ops$(\widetilde{C}_h^{-1} \underline{d}_h) = O(M_h)$?

1. **Smoothing**

   Multiplication with K_h, $A_h \to O(M_h)$

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1. **Smoothing**

   Multiplication with K_h, $A_h \to O(M_h)$

2. **Transfer operators**

   Prolongation Operators are defined locally $\to O(N_h)$

Complexity Analysis

3. **Galerkin Projection**

   ops$(P^\top * (\tilde{K}_h P_h)) <$ ops$(\tilde{K}_h * P_h)$ and
   ops$(\tilde{K}_h * P_h) = O(M_h)$

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4. **Coarse Grid Operator**

   $O(N_H)$, which is less than $O(N_h)$

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5. **V - Cycle**

   1.- 4. hold for each level
   $N_H = \frac{1}{2} N_h$
   Counting the operations yields

$$\text{ops}(\widetilde{C}_h^{-1} \underline{d}_h) = O(M_h)$$

Numerical Results

1. Implementation in the AMG program package PEBBLES
   Parallel **E**lement **B**ased grey **B**ox **L**inear **E**quation **S**olver
   (www.numa.uni-linz.ac.at/Research/Projects/pebbles.html)

2. BEM-Matrix entries by OSTBEM (O. Steinbach)

3. AMG preconditioner for the conjugate gradient method

4. Sparse representation of K_h (ACA)

5. PC 2000MHz AMD Athlon(tm)

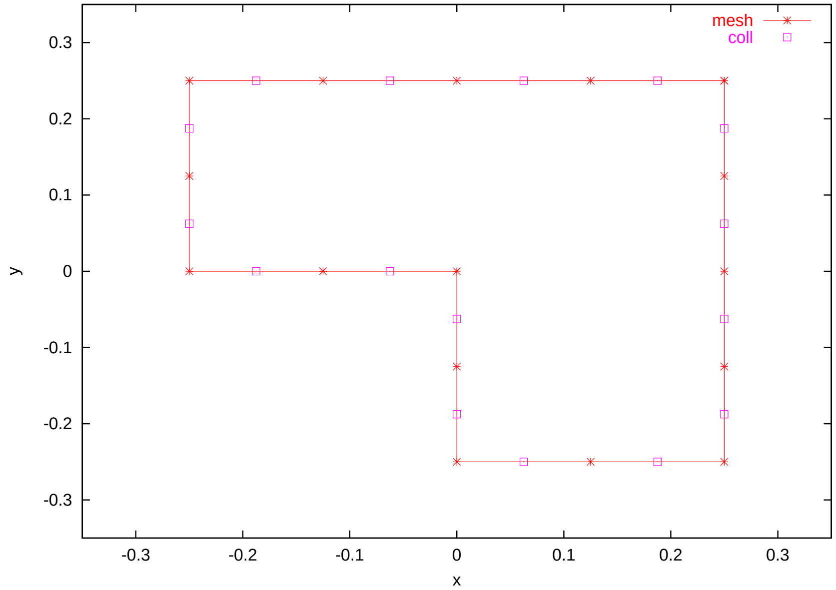
Interior Dirichlet Problem - L-Shape 2D

$$g(x) = \ln|x-y| \quad y \notin \bar{\Omega}$$

random initial guess u_0

rhs $f = (\frac{1}{2}I + K)g$

$\epsilon = 10^{-8}$

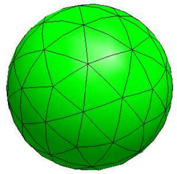


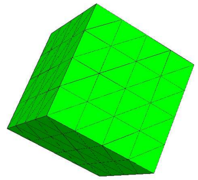
Interior Dirichlet Problem - L-Shape 2D

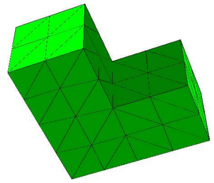
	N_h	Iterations	Setup	Cycle
AMG (ACA)	22000	5	7.3	1.3
	44000	5	15.4	2.9
	88000	5	32.0	6.2
	176000	5	69.1	14.7

Interior Dirichlet Problem 3D

	N_h	Setup [sec]	Cycle [sec]	It
Sphere	1920	1.9	0.2	9
	7680	8.2	1.4	10
	30720	33	8.3	11
	3072	6.6	0.3	6
Cube	12288	10	2.9	9
	49152	58	16.6	13
	1792	1.9	0.2	7
L-Shape	7168	7.5	1.3	7
	28672	30	8.0	10







Interior Dirichlet Problem 3D

	N_h	AMG w/ Setup	w/o Setup	CG	K_h^{-1}
Sphere	1920	3.4	1.5	3.1	11
	7680	22	13.8	24	611
	30720	125	92	151	$\sim$ 9h
L-Shape	1792	3.1	1.2	3.3	
	7168	17	9.5	24	
	28672	111	81	181	