Discretization of the double curl equation by discrete differential forms and collocation techniques

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Contents

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- Introduction to discrete forms
- Boundary Value Problems, Discretization of Cauchy data
- Adaptive Cross Approximation of BEM matrices
- Numerical Examples

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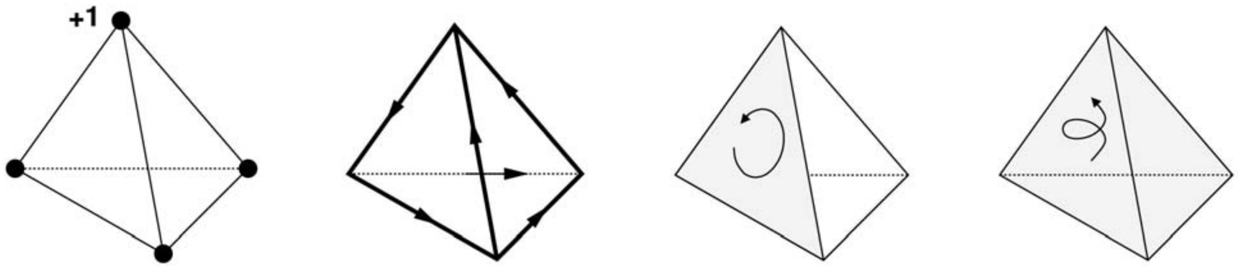
- A **discretization** Ω_h of the domain $\Omega \in \mathbb{R}^3$ is a finite collection of oriented **cells** $\mathcal{S}_3(\Omega_h)$, **faces** $\mathcal{S}_2(\Omega_h)$, **edges** $\mathcal{S}_1(\Omega_h)$ and **vertices** $\mathcal{S}_0(\Omega_h)$.
- The elements of $\mathcal{S}_p(\Omega_h)$ are called p**-facets**.



- A p**-chain** is a weighted sum of p-facets: $\Omega_h^p = \sum_{s_i \in \mathcal{S}_p(\Omega_h)} \gamma_i \, s_i \,, \;\; \gamma_i \in \mathbb{R}$, represented by a vector $\{\Omega_h^p\} = (\gamma_1 \,, \ldots , \gamma_{k_p})^T$.
- Integration over a p-chain: $\omega \mid \Omega_h^p = \sum_{i=1}^{k_p} \gamma_i \, (\omega \mid s_i) = \sum_{i=1}^{k_p} \gamma_i \, \omega_i$
- p**-cochain** $\{\omega\} = (\omega_1 \,, \ldots , \omega_{k_p})^T$ represents a discrete DF on mesh Ω_h.

- p-forms assign to any p-dimensional manifold Ω^p a real number

$$\omega \mid \Omega^p \quad := \quad \int_{\Omega}^{p} \omega$$

- Discrete p-forms assign to any p-chain Ω_h^p a real number

$$\omega_h \mid \Omega_h^p \quad = \quad \sum_{i=1}^{k_p} \gamma_i \, \omega_i$$

- Discretization:
  - 0-form $\{\omega\} = (\omega \mid n_i)_{i=1}^{k_0}$ nodal values
  - 1-form $\{\omega\} = (\omega \mid e_i)_{i=1}^{k_1}$ circulation across the edges
  - 2-form $\{\omega\} = (\omega \mid f_i)_{i=1}^{k_2}$ flux through the faces
  - 3-form $\{\omega\} = (\omega \mid t_i)_{i=1}^{k_3}$ cell values

- Denote the vector space of p-cochains on Ω_h by $\mathcal{C}^p(\Omega_h)$.

Incidence matrices and discrete outer derivative

- Incidence matrix $D^p \in \mathbb{R}^{k_{p+1} \times k_p}$ of p-facets and $p+1$-facets:

$$D^p_{i,j} = \begin{cases} 0\,, & \text{facet } s^p_j \text{ is not a subset of the boundary of facet } s^{p+1}_i \\ \pm 1\,, & \text{facet } s^p_j \text{ is a subset of the boundary of facet } s^{p+1}_i \\ & \text{with orientation } \pm 1\,. \end{cases}$$

- Discrete representation of the boundary

$$\{\partial\Omega^{p+1}_h\} = [D^p]^T \{\Omega^p_h\}$$

- Discretization of the outer derivative

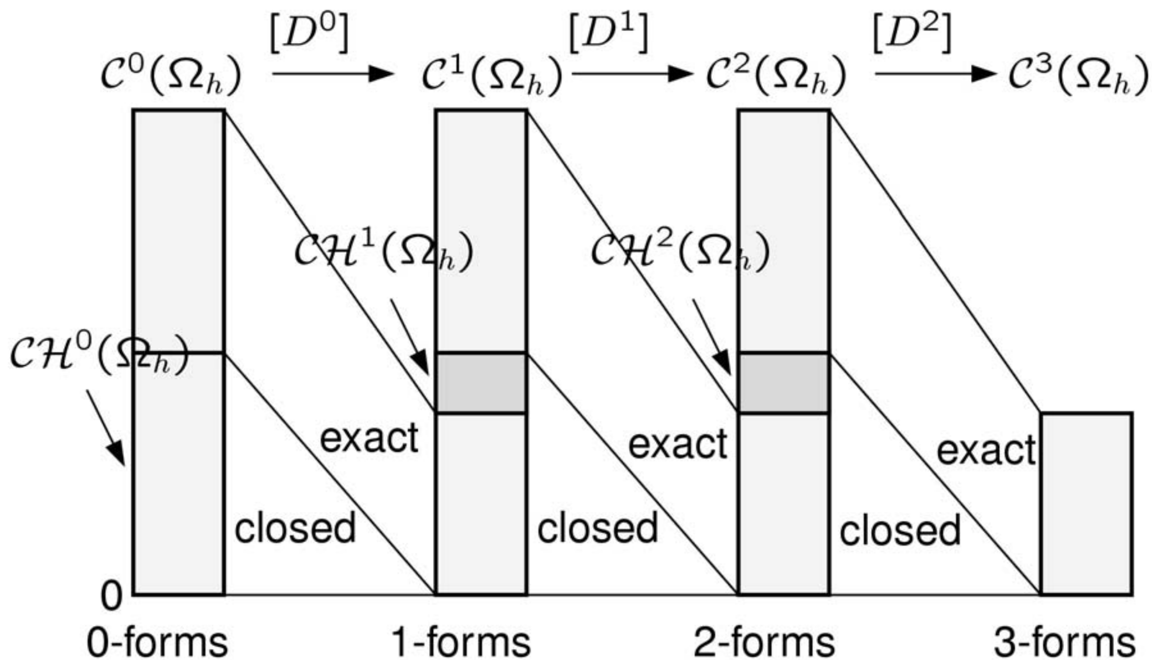
$$\begin{aligned} \{\mathrm{d}\omega\} \mid \{\Omega\} &= \{\omega\} \mid \{\partial\Omega\} \qquad \text{(discrete Stokes' theorem)} \\ &= \{\omega\} \mid [D^p]^T \{\Omega\} \\ &= [D^p]\{\omega\} \mid \{\Omega\} \end{aligned}$$

$$\Rightarrow \quad \{\mathrm{d}\omega\} = [D^p]\{\omega\}$$

DeRham theorem for cochains

- There are finite-dim. spaces $\mathcal{HC}^p(\Omega_h) \subset \mathcal{C}^p(\Omega_h)$: for $\{\omega\} \in \mathcal{C}^p(\Omega_h)$ holds:

$$[D^p]\{\omega\} = 0 \iff \exists \{\eta\} \in \mathcal{C}^{p-1}(\Omega_h)\,,\ \{\gamma\} \in \mathcal{HC}^p(\Omega_h)$$
$$\text{satisfying } \{\omega\} = [D^{p-1}]\{\eta\} + \{\gamma\}\,.$$



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- Cochains are **representatives** of discrete forms. Interpolation of p-cochains with Whitney p-forms yields discrete p-forms.
- Whitney p-form of a p-facet s_i of a tetrahedral mesh:

$$\beta^p_{s_i} = p \sum_{j=0}^{p} (-1)^j \lambda_{n_j} \mathrm{d}\lambda_{n_0} \wedge \cdots \wedge \mathrm{d}\lambda_{n_{j-1}} \wedge \mathrm{d}\lambda_{n_{j+1}} \wedge \cdots \wedge \mathrm{d}\lambda_{n_p} \,,$$

  where λ_{n_j} is the linear nodal ansatz function of the node n_j.
- Interpolation $\{\beta^p\} : \mathcal{C}^p(\Omega_h) \longrightarrow \mathcal{W}^p(\Omega_h)$

$$\omega_h = \sum_{i=1}^{k_p} \omega_i \, \beta^p_{s_i} =: \{\omega\}^T \{\beta^p\}$$

  $\mathcal{W}^p(\Omega_h)$ is the vector space of discrete p-forms.

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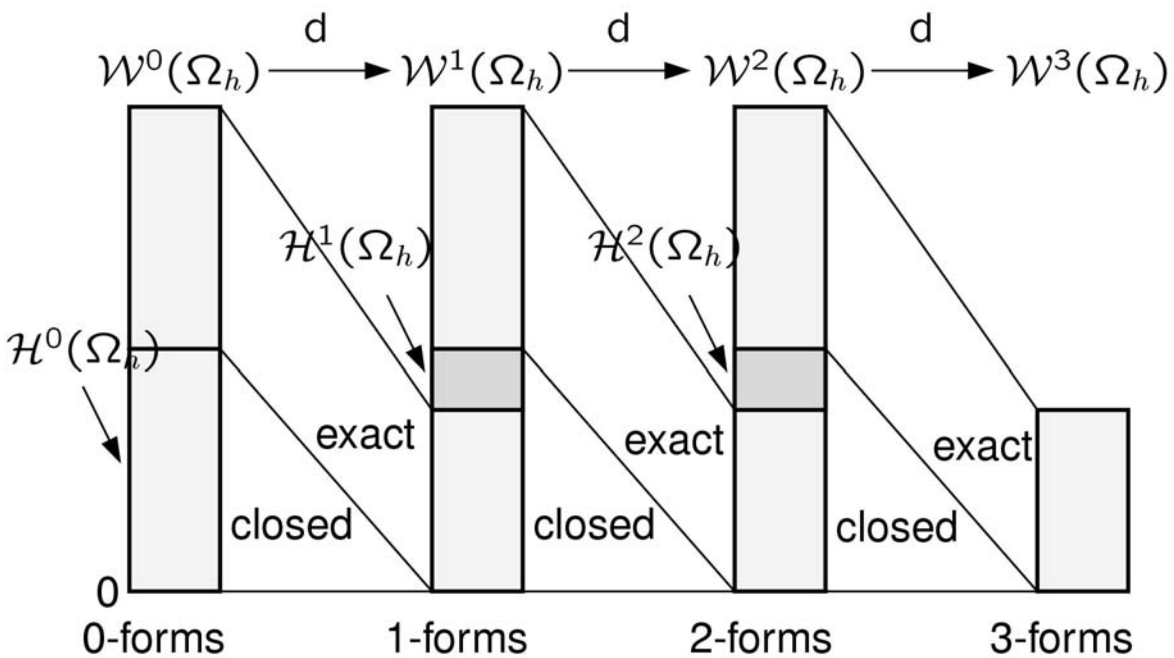
- DeRham complex is invariant w.r.t. interpolation



Discrete Dirichlet problem

- The discrete Dirichlet problem reads:
  For $\beta \in \mathcal{W}^1(\Gamma_h)$ *find* $\gamma \in \mathcal{W}^1(\mathrm{d}_0, \Gamma_h)$ *such that*

$$\mathcal{V}\gamma \mid \Gamma_h^1 = \left(\frac{\Theta}{4\pi}\mathcal{I} + \mathcal{K}\right)\beta \mid \Gamma_h^1, \; \forall \Gamma_h^1 \in \mathcal{C}_1(\partial_0, \Gamma_h)$$

- From DeRham theorem follows

$$\mathcal{W}^1(\mathrm{d}_0, \Gamma_h) = \mathrm{d}\mathcal{W}^0(\Gamma_h) \oplus \mathcal{H}^1(\Gamma_h)$$

- Discretization of $\mathcal{W}^0(\Gamma_h)$: nodal ansatz functions
  Discretization of $\mathcal{H}^1(\Gamma_h)$: cycles [R.Hiptmair and J.Ostrowski, 2001]
- Assume in the sequel trivial topology $\mathcal{W}^1(\mathrm{d}_0, \Gamma_h) = \mathrm{d}\mathcal{W}^0(\Gamma_h)$
- Collocation on boundaries of dual faces

- The discrete Neumann problem reads:
  For $\gamma \in \mathcal{W}^1(\mathrm{d}_0, \Gamma_h)$ *find* $\beta \in \mathcal{W}^1(\Gamma_h)$ *such that*

$$\mathcal{V}\gamma \mid \Gamma_h^1 = \left(\frac{\Theta}{4\pi}\mathcal{I} + \mathcal{K}\right)\beta \mid \Gamma_h^1, \ \forall \Gamma_h^1 \in \mathcal{C}_1(\Gamma_h)$$

- Discretization of $\mathcal{W}^1(\Gamma_h)$: edge ansatz functions
- Collocation on primal edges

Discretization of potentials, BEM matrices

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- Single layer potential $\mathcal{V}$, double layer potential $\mathcal{K}$:

$$\mathcal{V}\,\gamma = \int_{\Gamma} (\gamma_D \mathbf{G}) \wedge \gamma\,, \quad \mathcal{K}\,\boldsymbol{\beta} = \int_{\Gamma} (\gamma_N \mathbf{G}) \wedge \boldsymbol{\beta}$$

$$\gamma \approx \sum_{i=1}^{N} \gamma_i\, \mathbf{curl}_{\Gamma}\, \phi_i\,, \quad \boldsymbol{\beta} \approx \sum_{i=1}^{E} \beta_i\, \boldsymbol{\omega_i}$$

- Entries of the matrices

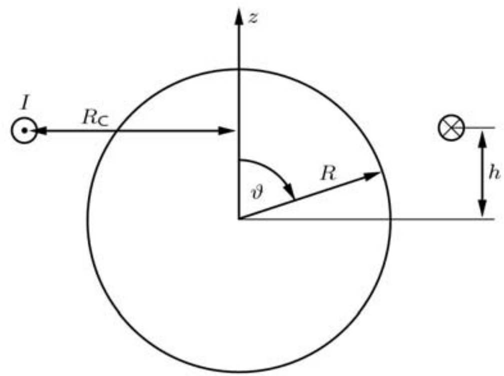
$$V_{ij} = \int_{C_i} \langle \int_{\Gamma_h} \mathbf{G}(\mathbf{x},\mathbf{y})\, \mathbf{curl}_{\Gamma} \phi_j(\mathbf{x})\, dF_x\,,\, dl_y \rangle\,,$$

$$K_{ij} = \int_{C_i} \langle \int_{\Gamma_h} (\mathbf{curl}_x \mathbf{G}(\mathbf{x},\mathbf{y}) \times \mathbf{n}(\mathbf{x}))^T\, \boldsymbol{\omega}_j(\mathbf{x})\, dF_x\,,\, dl_y \rangle\,,$$

where C_i are collocation contours.

- Dirichlet problem: $V \in \mathbb{R}^{N \times N}\,,\ K \in \mathbb{R}^{N \times E}$
  Neumann problem: $V \in \mathbb{R}^{E \times N}\,,\ K \in \mathbb{R}^{E \times E}$
- Analytic inner integration for triangular and rectangular meshes.

- Sphere with Radius $R = 0.05$m immersed in the field of a current $I = 20kA$ in a circular coil, $R_C = 0.07$m , $h = 0.03$m. [Z.Rhen]



- Vector potential for the exterior domain can be written analytically:

$$\boldsymbol{\alpha}(r,\theta) = \boldsymbol{\alpha}_S(r,\theta) - \frac{R}{r}\boldsymbol{\alpha}_S(\frac{R^2}{r},\theta)\,,\ r \geq R\,,$$

where $\boldsymbol{\alpha}_S = \int_{\Omega^C} \mathbf{G} \wedge \eta$ is the source potential due to the coil's excitation.

- The corresponding Neumann data γ can be calculated analytically.

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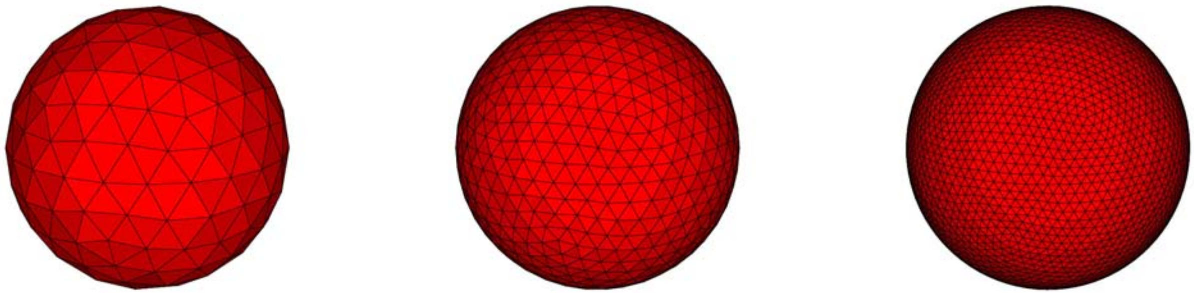
Mesh data for Dirichlet Problem

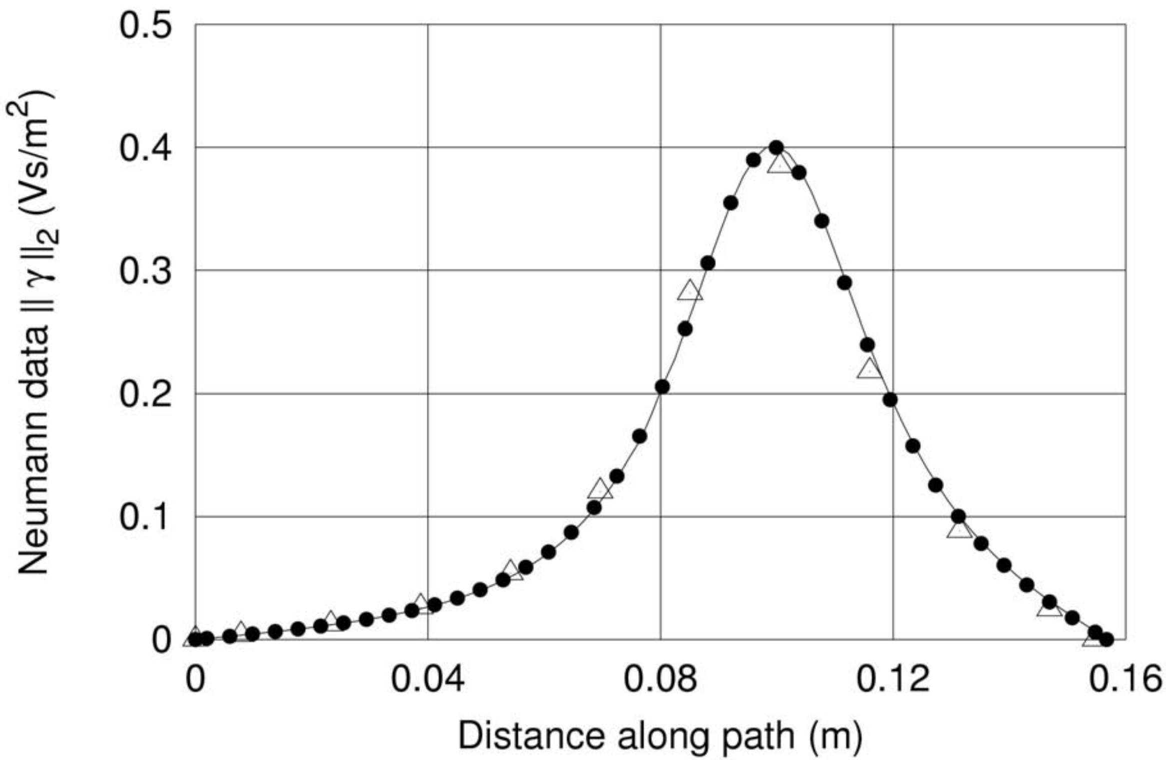
- Three meshes of the sphere are considered



Mesh	Coarse	Medium	Fine
# Nodes	162	642	2562
# Edges	480	1920	7680
# Trias	320	1280	5120
cond(V)	4.40	8.93	18.16
ε	8.36%	3.55%	1.62%

$$\varepsilon = \max_{\mathbf{x}\in C} \frac{\|\gamma_{\mathrm{ana}}(\mathbf{x}) - \gamma_{\mathrm{num}}(\mathbf{x})\|_2}{\|\gamma_{\mathrm{ana}}(\mathbf{x})\|_2}$$

- Comparison of the analytic and the numerical solutions along a meridian C.



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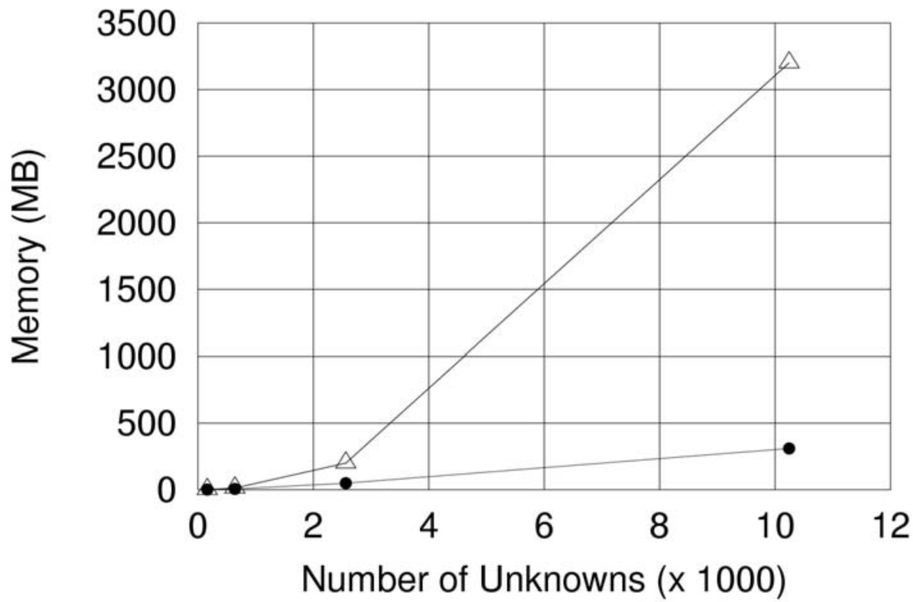
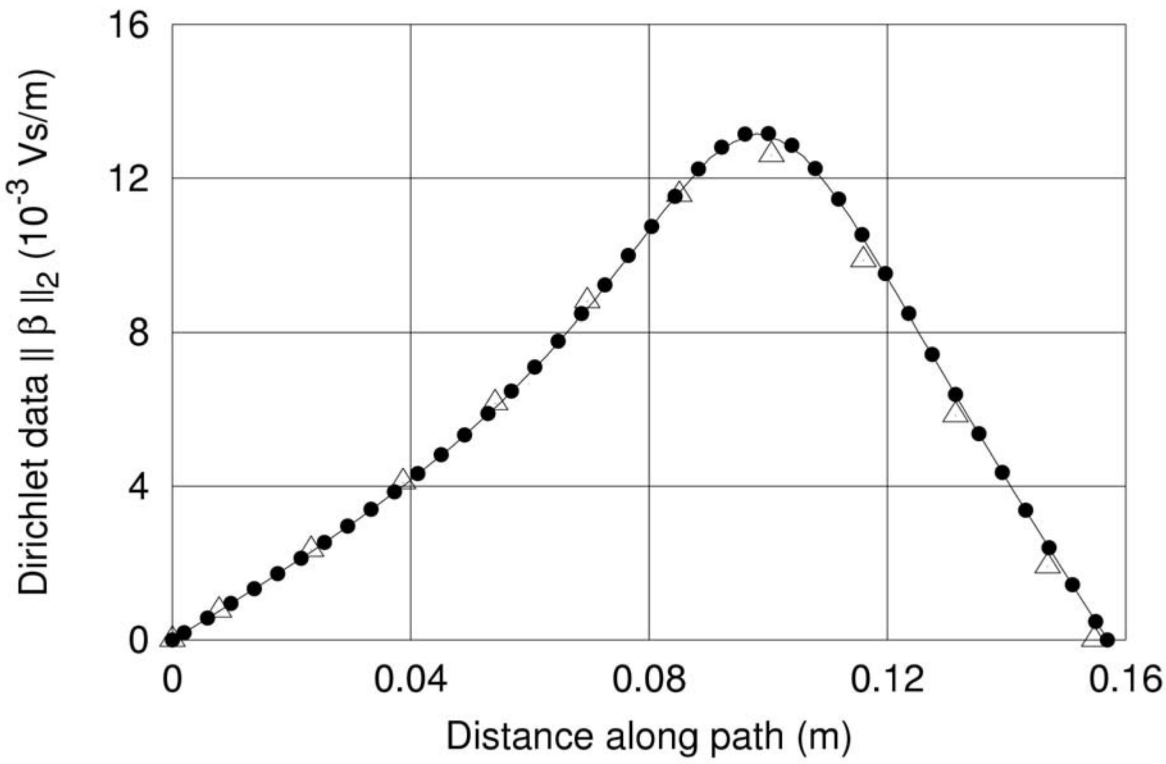
- ACA compression of the BEM matrices (Accuracy $\varepsilon = 10^{-3}$)



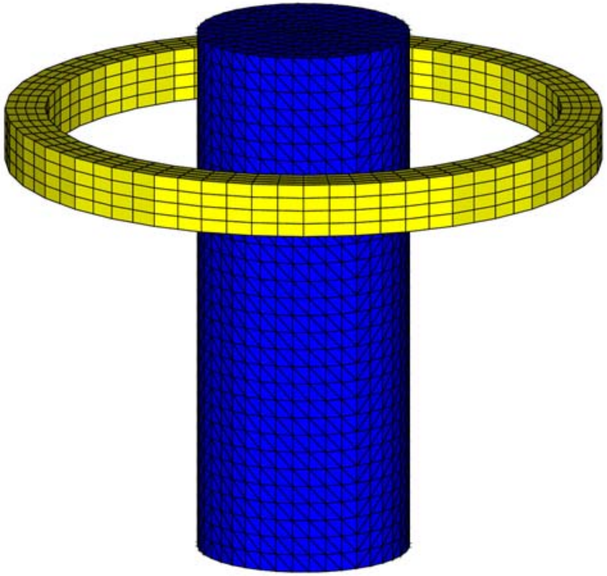
Mesh	Coarse	Medium	Fine	Very fine
Rel. size(V)	93.7%	63.3%	35.4%	14.0%
Rel. size(K)	85.3%	43.9%	20.7%	8.2%

- Comparison of the reference (2D-axisymmetric) and numerical solutions.



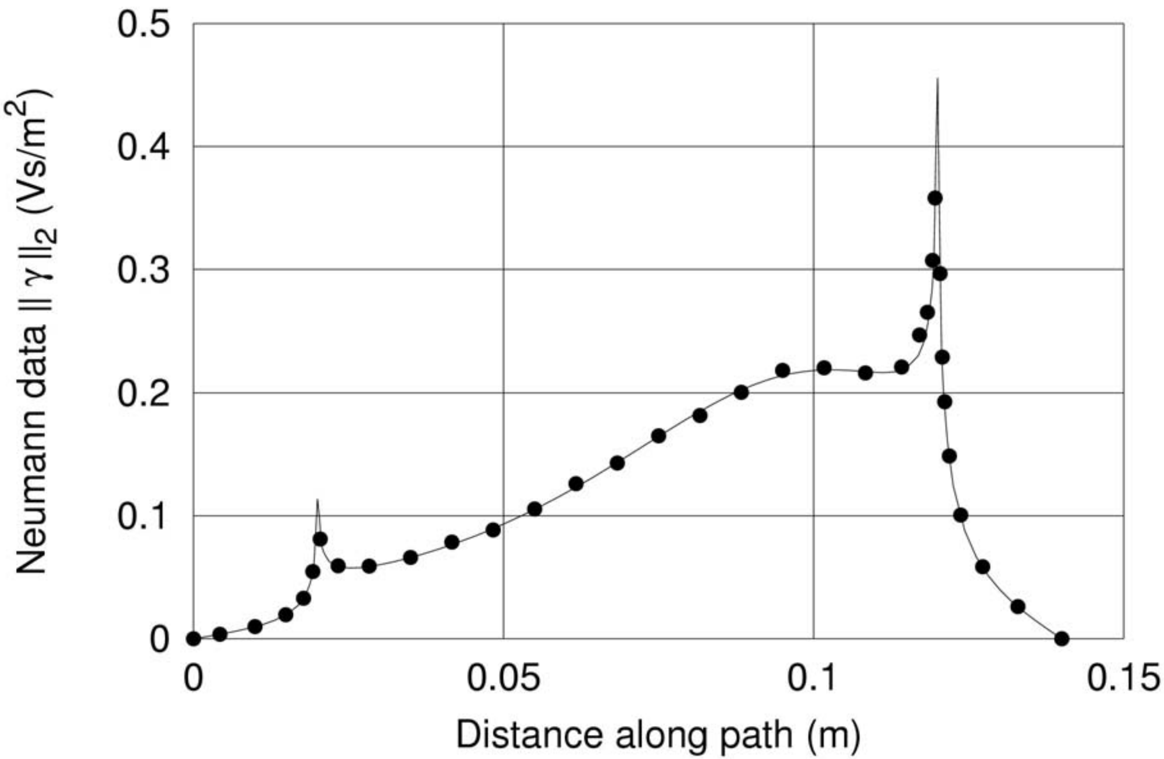
Numerical example. Cylinder.

- Cylinder immerced in the field of a circular coil

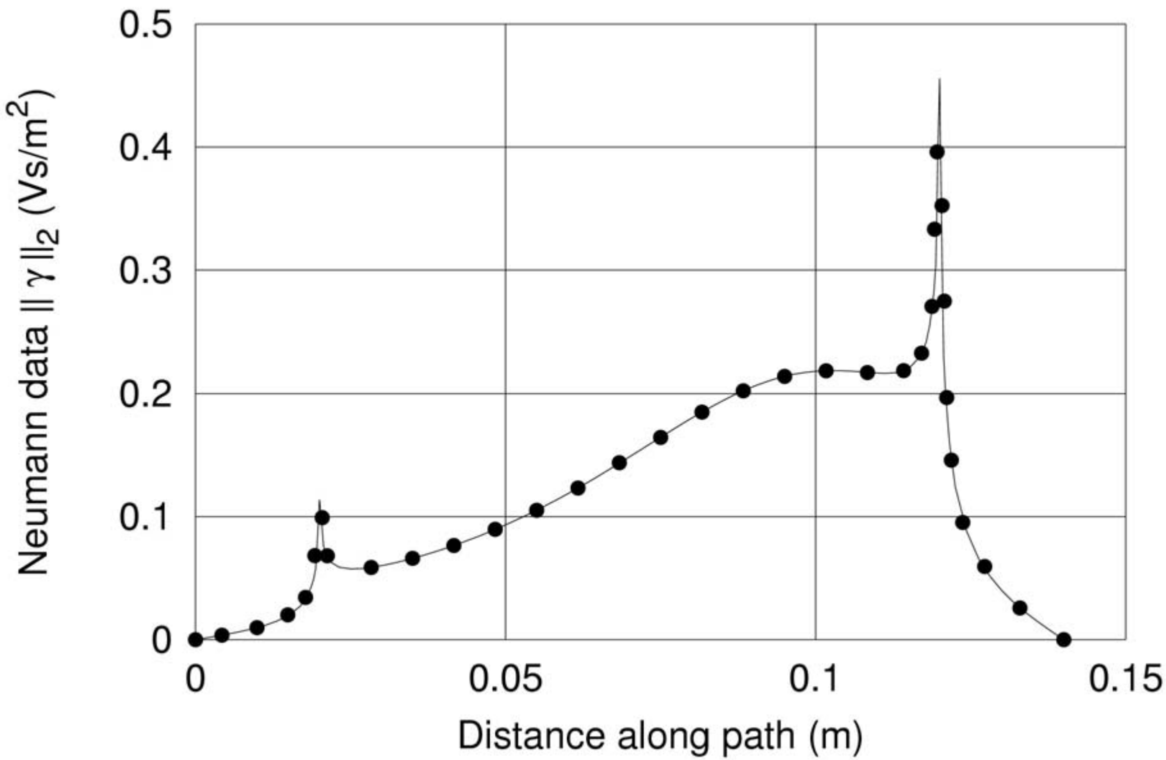


Nodal elements vs. reference solution

- Homogeneous Dirichlet problem on cylinder. Nodal elements. Fine mesh.



- Homogeneous Dirichlet problem on cylinder. Edge elements. Fine mesh.



Conclusions

- Discrete DFs provide a natural discretization of differential forms.
- DeRham theorem is identical for DFs, cohains and discrete DFs.
- Generalization of the point collocation technique by DeRham maps.
- BEM matrices for triangular and rectangular meshes can be computed semianalytically
- Due to the asymptotically smooth kernels BEM matrices can be compressed by the ACA.
- Numerical tests show a good approximation of the analytic solution (Dirichlet problem) and the reference 2D-axisymmetric solution (Neumann problem) on the sphere.
- On a cylinder edge elements achieve a better approximation of the reference solution, especially for the singular solution.