# Coupled FETI/BETI for Nonlinear Potential Problems

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# Outline

- Motivation

  Nonlinear Magnetostatic $\longleftarrow$ Maxwell

- Coupled FETI/BETI for Linear Potential Problems

  Weak Formulation – Tearing and Interconnecting – Schur Complement
  Regularization – Dual Problem – Preconditioning

- Nonlinear Potential Problems

  Approximation of Nonlinear Parameter – Newton + Coupled FETI-BETI

- First Numerical Results (Linear)

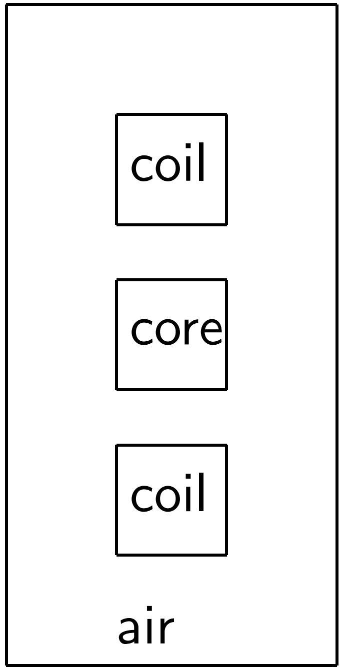
- Concluding Remarks and Outlook

# Motivation for Coupled FETI/BETI

- Solving Nonlinear Field Problems
  Magnetostatic 2D $\longleftarrow$ Maxwell

- Why FETI/BETI?
  $\longrightarrow$ Parallel Computing

- Why Coupling?
  $\longrightarrow$ Benefit from both FEM and BEM techniques




*References*

FETI: [Farhat, Roux, Klawonn, Widlund, Brenner]

BETI/Coupling: [Langer, Steinbach]

# Non-overlapping Domain Decomposition

$\Omega \subset \mathbb{R}^n \quad n = 2, 3 \quad$ bounded

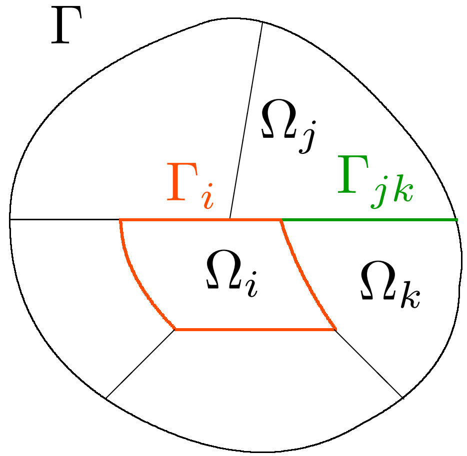
$\bar{\Omega} = \bigcup_{i \in I} \bar{\Omega}_i$

$\Gamma = \partial\Omega$

$\Gamma_i = \partial\Omega_i$

$\Gamma_{ij} = \Gamma_i \cap \Gamma_j$

$\vec{n}_i \ldots$ outward unit normal vector to $\Omega_i$



# Linear Potential Problem

Potential Problem $\alpha_i = \alpha_i(x)$ !

$$
\begin{aligned}
-\nabla \cdot \left[\alpha_i \nabla u\right] &= f \quad \text{in } \Omega_i \\
u &= 0 \quad \text{on } \Gamma \\
(\alpha_i \nabla u) \cdot \vec{n}_i + (\alpha_j \nabla u) \cdot \vec{n}_j &= 0 \quad \text{on } \Gamma_{ij}
\end{aligned}
$$

Weak formulation – Find $u \in H_0^1(\Omega)$:

$$
\sum_{i \in I} \int_{\Omega_i} \alpha_i \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\Omega} f \, v \, \mathrm{d}x \qquad \forall v \in H_0^1(\Omega)
$$

# BEM Subdomains

$I = I^{\text{BEM}} \dot{\cup} I^{\text{FEM}}$ such that $\alpha_i \equiv \text{const}$ for $i \in I^{\text{BEM}}$

Local representation:

$$\int_{\Omega_i} \alpha_i \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\Omega_i} \underbrace{-\alpha_i \Delta u}_{=f} \, v \, \mathrm{d}x + \int_{\Gamma_i} \underbrace{\alpha_i \frac{\partial u}{\partial \vec{n}_i}}_{=S_i u - N_i f} \, v \, \mathrm{ds}_x \qquad \text{for } i \in I^{\text{BEM}}$$

Global formulation:

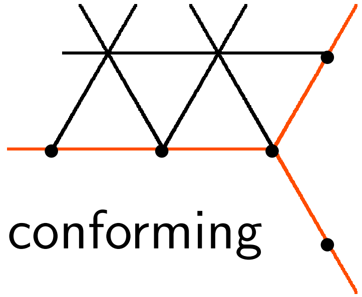
$$\sum_{i \in I^{\text{FEM}}} \int_{\Omega_i} \alpha_i \nabla u \cdot \nabla v \, \mathrm{d}x + \sum_{i \in I^{\text{BEM}}} \int_{\Gamma_i} (S_i u) \, v \, \mathrm{ds}_x =$$

$$= \sum_{i \in I^{\text{FEM}}} \int_{\Omega_i} f \, v \, \mathrm{d}x + \sum_{i \in I^{\text{BEM}}} \int_{\Gamma_i} (N_i f) \, v \, \mathrm{ds}_x \qquad \forall v \in H_0^1(\Omega)$$

# Discretization

- Triangulation of $\Omega_i$, $i \in I^{\mathsf{FEM}}$
  $\rightsquigarrow V_{i,h} \subset H^1(\Omega_i) \cap H^1_0(\Omega)$

- Triangulation of $\Gamma_i$, $i \in I^{\mathsf{BEM}}$
  $\rightsquigarrow V_{i,h} \subset \{v \in H^{1/2}(\Gamma_i) : v|_{\Gamma_i \cap \Gamma} = 0\})$


conforming

$$(K_{i,h}\,\underline{u}_i, \underline{v}_i) = \int_{\Omega_i} \alpha_i \nabla u_{i,h} \cdot \nabla v_{i,h}\,\mathsf{d}x \qquad (\underline{f}_i, \underline{v}_i) = \int_{\Omega_i} f\, v_{i,h}\,\mathsf{d}x$$

$$(S^{\mathsf{BEM}}_{i,h}\underline{u}_i, \underline{v}_i) \approx \int_{\Gamma_i} (S_i\, u_{i,h}) v_{i,h}\,\mathsf{ds}_x \qquad (f^{\mathsf{BEM}}_{i,h}, \underline{v}_i) = \int_{\Gamma_i} (N_i f)\, v_{i,h}\,\mathsf{ds}x$$

$$S^{\mathsf{BEM}}_{i,h} := D_{i,h} + \Big[\frac{1}{2} M^\top_{i,h} + K^\top_{i,h}\Big] V^{-1}_{i,h} \Big[\frac{1}{2} M_{i,h} + K_{i,h}\Big]$$

# Tearing and Interconnecting

*Tearing:* Unknowns are now $(\underline{u}_i)_{i\in I}$

*Interconnecting:* Continuity enforced by $\sum_{i\in I} B_i \underline{u}_i = \underline{0}$

Linear System:

$$I^{\mathsf{FEM}} = \{1, \ldots, q\}$$
$$I^{\mathsf{BEM}} = \{q+1, \ldots, p\}$$

$$\begin{pmatrix} K_{1,h} & & & & & & B_1^\top \\ & \ddots & & & & & \vdots \\ & & K_{q,h} & & & & B_q^\top \\ & & & S_{q+1,h}^{\mathsf{BEM}} & & & B_{q+1}^\top \\ & & & & \ddots & & \vdots \\ & & & & & S_{p,h}^{\mathsf{BEM}} & B_p^\top \\ B_1 & \cdots & B_q & B_{q+1} & \cdots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_q \\ \underline{u}_{q+1} \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_q \\ \underline{f}_{q+1}^{\mathsf{BEM}} \\ \vdots \\ \underline{f}_p^{\mathsf{BEM}} \\ 0 \end{pmatrix}$$

# FEM – Schur Complement

Optionally: Elimination of inner FEM-unknowns via Schur Complement

$$S_{i,h}^{\mathsf{FEM}} = K_{i,h}^{CC} - K_{i,h}^{IC}\,[K_{i,h}^{II}]^{-1}\,K_{i,h}^{CI}$$

$$\underline{f}_{i,h}^{\mathsf{FEM}} = \underline{f}_{i,h}^{C} - K_{i,h}^{IC}\,[K_{i,h}^{II}]^{-1}\,\underline{f}_{i,h}^{I}$$

Linear System:

$$\begin{pmatrix} S_{1,h}^{\mathsf{FEM}} & & & & & & B_1^\top \\ & \ddots & & & & & \vdots \\ & & S_{q,h}^{\mathsf{FEM}} & & & & B_q^\top \\ & & & S_{q+1,h}^{\mathsf{BEM}} & & & B_{q+1}^\top \\ & & & & \ddots & & \vdots \\ & & & & & S_{p,h}^{\mathsf{BEM}} & B_p^\top \\ B_1 & \cdots & B_q & B_{q+1} & \cdots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_q \\ \underline{u}_{q+1} \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1^{\mathsf{FEM}} \\ \vdots \\ \underline{f}_q^{\mathsf{FEM}} \\ \underline{f}_{q+1}^{\mathsf{BEM}} \\ \vdots \\ \underline{f}_p^{\mathsf{BEM}} \\ 0 \end{pmatrix}$$

# FETI/BETI – Regularization, Dual Problem

In floating domains, i.e. $\Gamma_I \cap \Gamma = \emptyset$:

$$\tilde{S}_{i,h}^{\text{FEM/BEM}} = S_{i,h}^{\text{FEM/BEM}} + \beta_i \, \underline{e}_i \, \underline{e}_i^\top$$

Elimination of $\underline{u}_i$:

$$\underline{u}_i = \left[\tilde{S}_{i,h}^{\text{FEM/BEM}}\right]^{-1} \left[\underline{f}_i^{\text{FEM/BEM}} - B_i^\top\right] + \gamma_i \, \underline{e}_i$$

Dual problem: $F := \sum_{i \in I} B_i \left[\tilde{S}_{i,h}^{\text{FEM/BEM}}\right]^{-1} B_i^\top, \quad G := (B_i e_i)_{i \in I^{\text{floating}}}$

$$\begin{pmatrix} F & -G \\ G^\top & \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ \underline{e} \end{pmatrix}$$

Projected dual problem: $P := I - G(G^\top G)^{-1} G^\top$

$$P F \underline{\lambda} = P \underline{d}$$

# FETI/BETI – Preconditioners

Projected dual problem solved with CG-iteration.

Preconditioners: [Langer, Steinbach, Klawonn, Widlund, Brenner]

$$C_{\mathsf{FETI}}^{-1} = (BC_\alpha^{-1}B^\top)^{-1}BC_\alpha^{-1}\Big[\sum_{i\in I} B_i S_{i,h}^{\mathsf{FEM/BEM}} B_i^\top\Big] C_\alpha^{-1}B^\top)(BC_\alpha^{-1}B^\top)^{-1}$$

$$C_{\mathsf{BETI}}^{-1} = (BC_\alpha^{-1}B^\top)^{-1}BC_\alpha^{-1}\Big[\sum_{i\in I} B_i D_{i,h} B_i^\top\Big] C_\alpha^{-1}B^\top)(BC_\alpha^{-1}B^\top)^{-1}$$

*Spectral Equivalence:*

$$S_{i,h}^{\mathsf{BEM}} \simeq S_{i,h}^{\mathsf{FEM}} \simeq S_{i,h} \simeq D_{i,h}$$

*Condition Estimate:*

$$\kappa(P\,C_{\mathsf{FETI/BETI}}^{-1}P^\top P^\top F P) \preceq (1+\log(H/h))^2$$

# Nonlinear Potential Problems

Nonlinear Coefficient $\nu_i : \mathbb{R}_0^+ \to \mathbb{R}^+$, $\quad \nu_i(\cdot) \equiv$ const for $i \in I^{\mathsf{BEM}}$
$s \mapsto \nu(s)s$ strongly monotone and Lipschitz-continuous

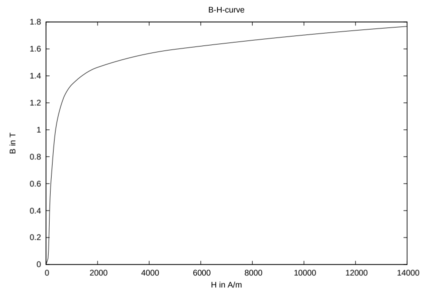
$$\begin{aligned} -\nabla \cdot \left[\nu_i(|\nabla u|)\nabla u\right] &= f \quad \text{in } \Omega_i \\ u &= 0 \quad \text{on } \Gamma \\ \nu_i(|\nabla u|)\nabla u \cdot \vec{n}_i + \nu_j(|\nabla u|)\nabla u \cdot \vec{n}_j &= 0 \quad \text{on } \Gamma_{ij} \end{aligned}$$

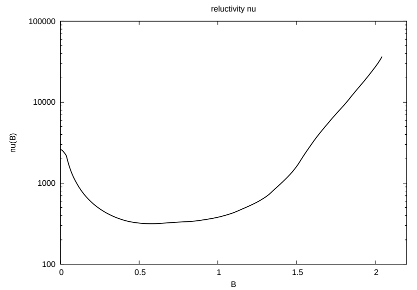
Problem: In magnetostatics, $\nu_i$ not available in analytical form
$\rightsquigarrow$ Approximation

# Approximation of $B$-$H$-Curves

Magnetostatics / Maxwell:

$$-\nabla \cdot [\nu_i(|\nabla u|)\nabla u] \quad \overset{2D\leftrightarrow 3D}{\longleftrightarrow} \quad \nabla \times [\underbrace{\nu_i(|\nabla \times A|)\,\underbrace{\nabla \times A}_{=B}}_{=H}]$$




$$|H| = \nu(|B|)\,|B| \quad \overset{\nu(s) := \frac{f^{-1}(s)}{s}}{\Longleftrightarrow} \quad |B| = f(|H|)$$

'Physical' Properties:

$B$-$H$-Curve $f$

$f \in \mathcal{C}^1(\mathbb{R}_0^+ \to \mathbb{R}_0^+)$

$f(0) = 0$

$f'(s) \geq \mu_0 > 0$

$\lim\limits_{s\to\infty} f'(s) = \mu_0$

$\Longrightarrow \; L := \max\limits_{s\geq 0} |f'(s)| < \infty$

$\Longrightarrow$

reluctivity $\nu$

$\nu \in \mathcal{C}^1(\mathbb{R}^+ \to \mathbb{R}^+)$

$\nu(s)s$ strongly monotne:

$$\big(\nu(s)s - \nu(t)t\big)(s-t) \geq 1/L|s-t|^2$$

$\nu(s)s$ Lipschitz-continuous:

$$|\nu(s)s - \nu(t)t| \leq 1/\mu_0|s-t|$$

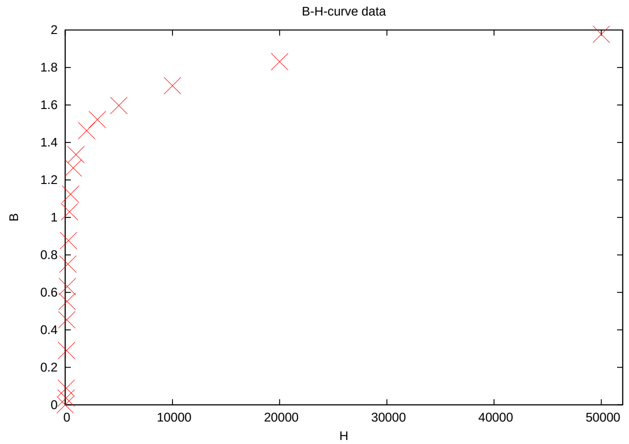
# Approximation of $B$-$H$-Curves

[Pechstein, Jüttler, Gfrerer]

Given Data Pairs

$$(H_k, B_k)_{k=1,\dots,N}$$

with $\quad |f(H_k) - B_k| \leq \delta_k$



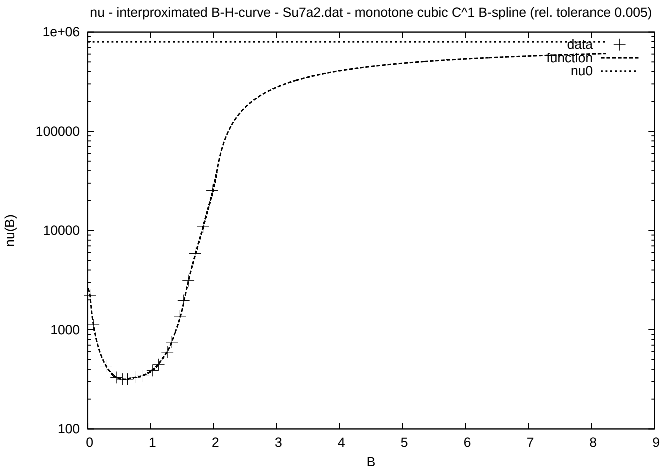

Approximation of $f$ in the class of strongly monotone cubic $\mathcal{C}^1$ splines, such that $|\tilde{f}(H_k) - B_k| \leq c\,\delta_k$

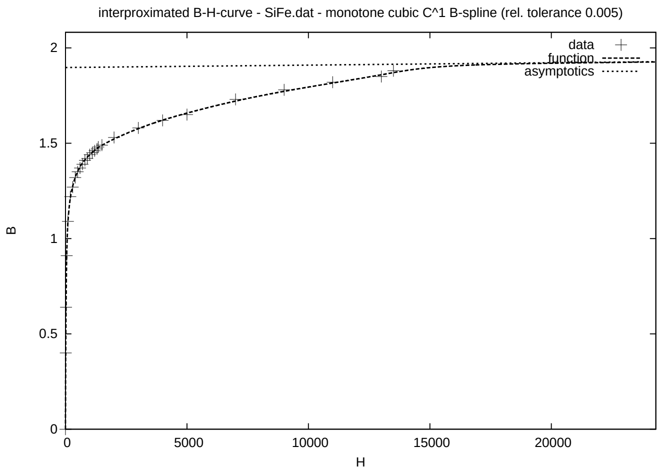
Minimization of $\int_0^{H_N} \left[f''(s)\right]^2 \frac{\mathrm{d}s}{\omega(s)}$ (Data Dependent Functional)

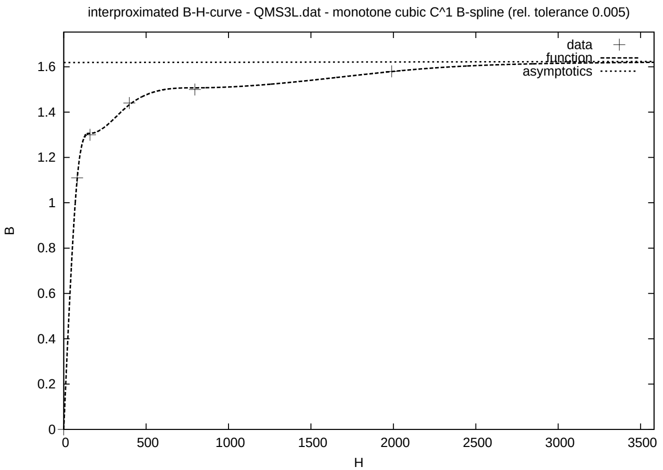
$\rightsquigarrow$ Quadratic Optimization Problem

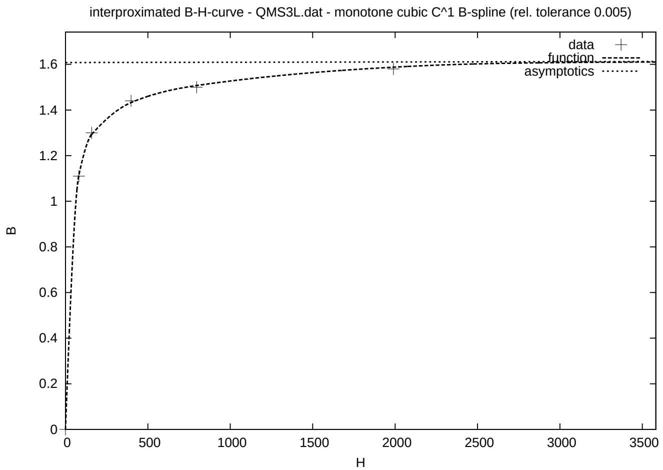
$\rightsquigarrow$ Spline Representation of $\tilde{f} \xrightarrow{\text{Newton+Trick}}$ Fast Evaluation of $\nu$, $\nu'$, etc.

# Approximation of $B$-$H$-Curves – Results



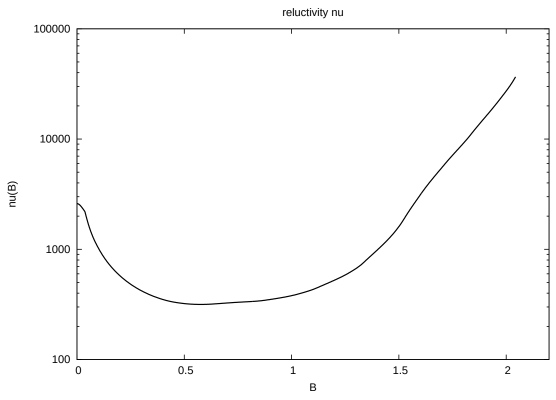



# Nonlinear Potential Problem

Nonlinear Coefficient $\nu_i \in \mathcal{C}^1(\mathbb{R}_0^+ \to \mathbb{R}^+)$, $\quad \nu_i(\cdot) \equiv \text{const for } i \in I^{\text{BEM}}$

$\nu_i(t)t$ strongly monotone, Lipschitz-continuous




$$-\nabla \cdot \left[\nu_i(|\nabla u|)\nabla u\right] = f \quad \text{in } \Omega_i$$

$$u = 0 \quad \text{on } \Gamma$$

$$\nu_i(|\nabla u|)\nabla u \cdot \vec{n}_i + \nu_j(|\nabla u|)\nabla u \cdot \vec{n}_j = 0 \quad \text{on } \Gamma_{ij}$$

Weak formulation: Find $u \in H_0^1(\Omega)$:

$$\sum_{i \in I^{\text{FEM}}} \int_{\Omega_i} \nu_i(|\nabla u|)\nabla u \cdot \nabla v \, \mathrm{d}x + \sum_{i \in I^{\text{BEM}}} \int_{\Gamma_i} (S_i^{(\nu_i)} u) v \, \mathrm{d}s_x =$$

$$= \sum_{i \in I^{\text{FEM}}} \int_{\Omega_i} f \, v \, \mathrm{d}x + \sum_{i \in I^{\text{BEM}}} \int_{\Gamma_i} (N_i f) v \, \mathrm{d}s_x \qquad \forall v \in H_0^1(\Omega)$$

# Newton Iteration

Initial $u^{(0)}$, e.g. $u^{(0)} = 0$

$u^{(k+1)} = u^{(k)} + \rho_k \, w^{(k)}$

$$\sum_{i \in I^{\mathsf{FEM}}} \int_{\Omega_i} \left[ \zeta_i(|\nabla u^{(k)}|) \nabla w^{(k)} \right] \cdot \nabla v \, \mathrm{d}x + \sum_{i \in I^{\mathsf{BEM}}} \int_{\Gamma_i} \left( S_i^{(\nu_i)} w^{(k)} \right) v \, \mathrm{ds}_x =$$

$$= \sum_{i \in I^{\mathsf{FEM}}} \int_{\Omega_i} f \, v - \nu_i(|\nabla u^{(k)}|) \nabla u^{(k)} \cdot \nabla v \, \mathrm{d}x + \sum_{i \in I^{\mathsf{BEM}}} \int_{\Gamma_i} \left[ (N_i f) - \left( S_i u^{(k)} \right) \right] v \, \mathrm{ds}_x$$

where

$$\zeta_i(p)q := \nu_i(|p|)q + \frac{\nu_i'(|p|)}{|p|}(p \cdot q)p \qquad \forall p \in \mathbb{R}^n \setminus \{0\} \quad \forall q \in \mathbb{R}^n$$

$$\zeta_i(0)q := \nu_i(|0|)q \qquad \forall q \in \mathbb{R}^n$$

# FETI/BETI for the $k$-th Newton Equation

Define

$$(K'_{i,h}(\underline{u})\underline{w},\underline{v}) = \int_{\Omega_i} [\zeta_i(u_h)\cdot\nabla w_h]\,\nabla v_h\,\mathrm{d}x$$

$$(\underline{r}_i^{(k)},\underline{v}) = (f_i - K_i(\underline{u}^{(k)},\underline{v}) \qquad \text{for } i\in I^{\mathsf{FEM}}$$

$$(\underline{r}_i^{(k),\mathsf{BEM}},\underline{v}) = (f_i^{\mathsf{BEM}} - S_{i,h}^{BEM}\underline{u}^{(k)}) \qquad \text{for } i\in I^{\mathsf{BEM}}$$

Linear System

$$\begin{pmatrix} K'_{1,h}(\underline{u}_i) & & & & & & B_1^\top \\ & \ddots & & & & & \vdots \\ & & K'_{q,h}(\underline{u}_i) & & & & B_q^\top \\ & & & S_{q+1,h}^{\mathsf{BEM}} & & & B_{q+1}^\top \\ & & & & \ddots & & \vdots \\ & & & & & S_{p,h}^{\mathsf{BEM}} & B_p^\top \\ B_1 & \cdots & B_q & B_{q+1} & \cdots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{w}_1^{(k)} \\ \vdots \\ \underline{w}_q^{(k)} \\ \underline{w}_{q+1}^{(k)} \\ \vdots \\ \underline{w}_p^{(k)} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{r}_1^{(k)} \\ \vdots \\ \underline{r}_q \\ \underline{r}_{q+1}^{(k),\mathsf{BEM}} \\ \vdots \\ \underline{r}_p^{(k),\mathsf{BEM}} \\ 0 \end{pmatrix}$$

# Inner vs. Outer Iteration

Outer Iteration: Newton
Inner Iteration: Projected Preconditioned CG

Stopping Criterion (PPCG):

$\|PF\underline{\lambda}^{(i)}\| < \varepsilon_{\mathsf{CG}}^{(k)}\, \lambda_0$

Stopping Criterion (Newton):

$\|\left(r_i^{(k),\mathsf{inner}}\right)_{i\in I^{\mathsf{FEM}}}\| < \varepsilon_{\mathsf{Newton}}\, r_0$

additionally measure flux jump on the interfaces $\Gamma_{ij}$

# Numerical Results (Linear)

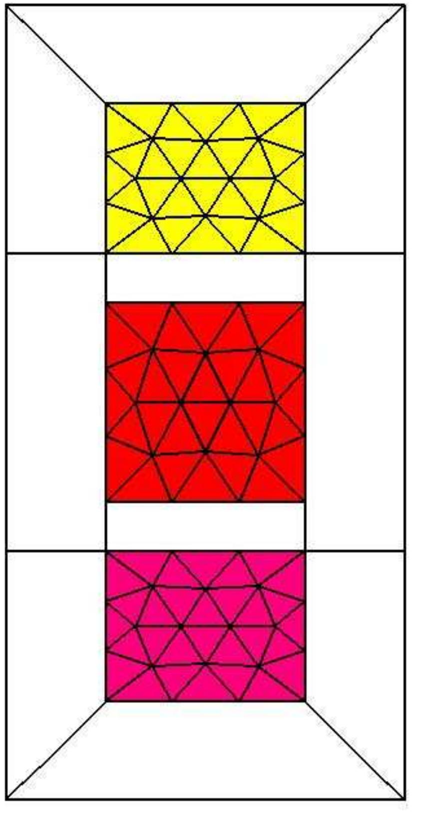
model problem:
coil-core configuration

| | |
|---|---|
| core | $\mu_r = 10^3$ |
| coil / coil | $f = \pm 10^3$ |
| elsewhere | $\mu_r = 1, f = 0$ |

Dirichlet: $u|_{\partial\Omega} = 0$

$\nu = \frac{1}{\mu_r \mu_0}$ $\quad \mu_0 = 4.\pi.10^{-7}$

OSTBEM



# Numerical Results (Linear)

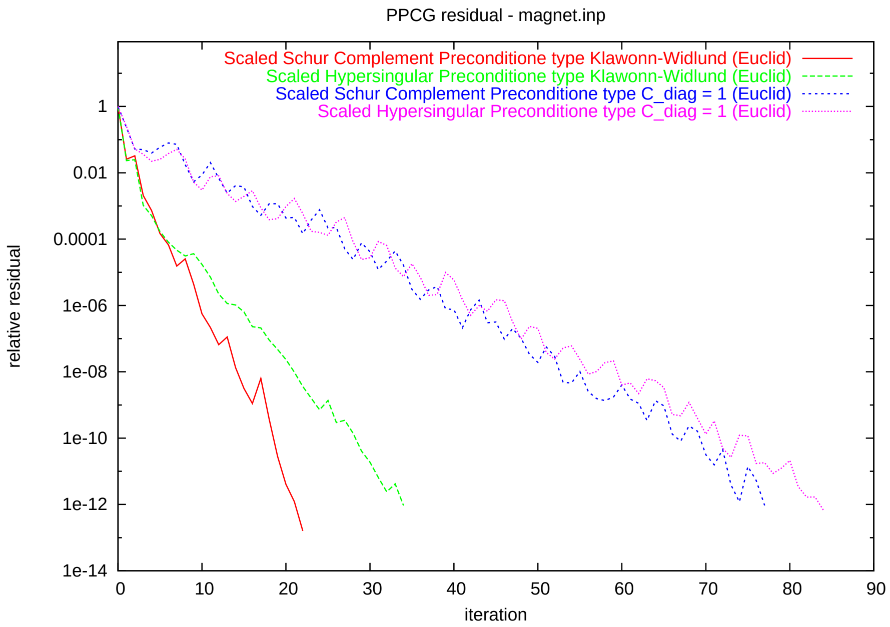
260 coupling nodes, 911 inner nodes, 276 Lagrange parameters

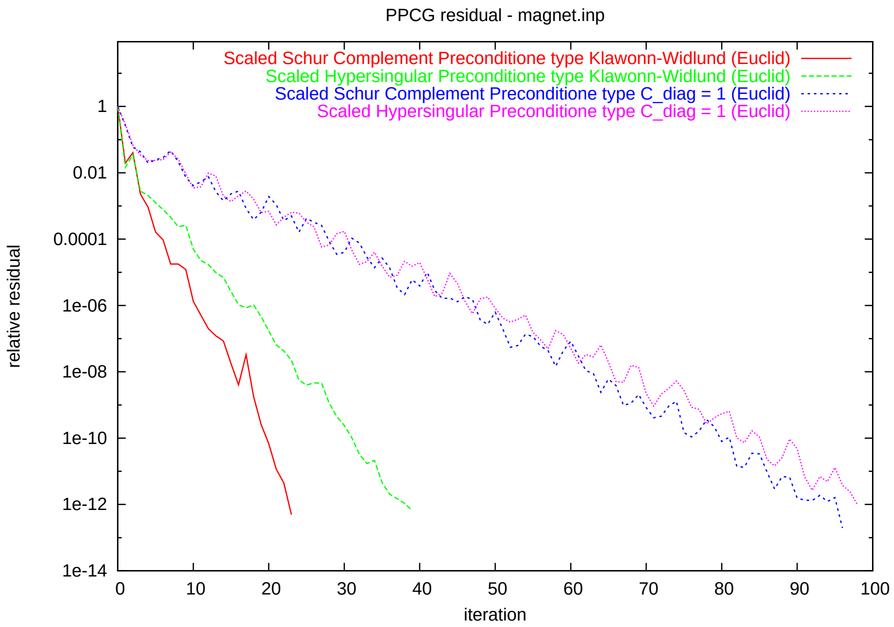
| preconditioner | iterations | residual | total (sec) | one step (sec) |
|---|---|---|---|---|
| FETI $(C_\alpha = 1)$ | 77 | 9.38e-13 | 5.97 | 0.0775 |
| BETI | 84 | 6.79e-13 | 5.73 | 0.0682 |
| FETI (Kla-Wid) | 22 | 1.60e-13 | 1.89 | 0.0859 |
| BETI | 34 | 9.26e-13 | 2.61 | 0.0768 |

532 coupling nodes, 3271 inner nodes, 548 Lagrange parameters

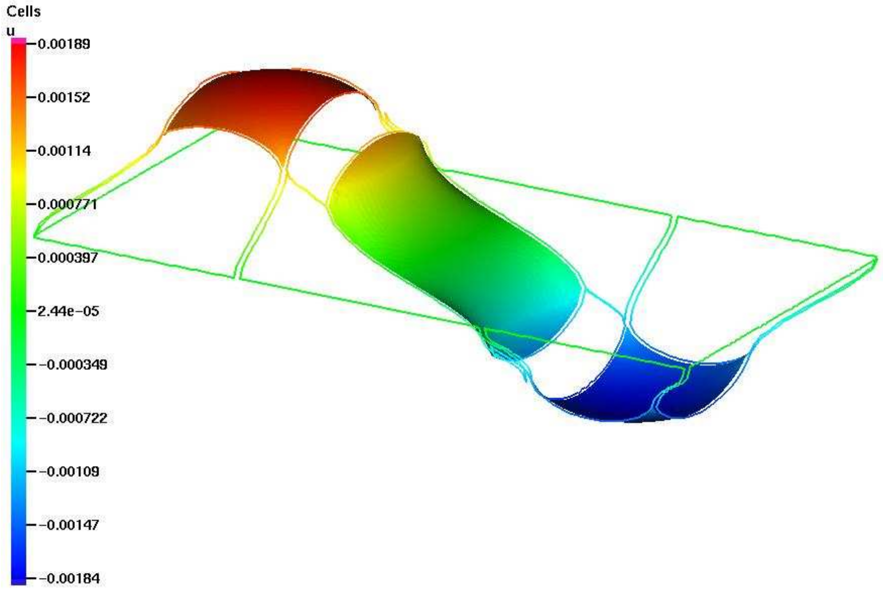
| preconditioner | iterations | residual | total (sec) | one step (sec) |
|---|---|---|---|---|
| FETI $(C_\alpha = 1)$ | 96 | 1.97e-13 | 38.8 | 0.404 |
| BETI | 98 | 9.60e-13 | 37.0 | 0.377 |
| FETI (Kla-Wid) | 23 | 4.94e-13 | 10.4 | 0.453 |
| BETI | 39 | 6.52e-13 | 15.8 | 0.404 |

# Numerical Results (Linear)

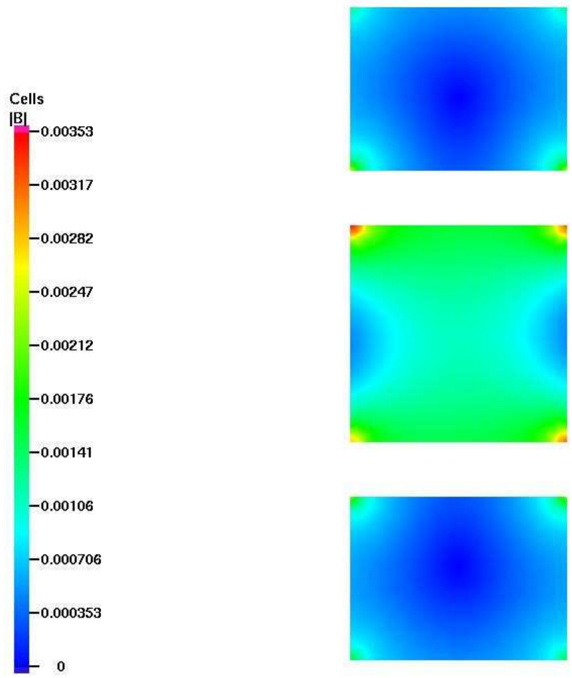


# Numerical Results (Linear)




potential $u$




$|B|$-field

# Concluding Remarks and Outlook

- Family of FEM/BEM Domain Decomposition Techniques
  Coupled FETI/BETI with two efficient and robust Preconditioners
- Efficient and fast handling of $B$-$H$-Curves
- Balancing inner and outer iteration ($\varepsilon_{\mathsf{CG}}^{(k)}$, $\varepsilon_{\mathsf{Newton}}$)
- Meshrefinement (adaptive)
  Use multilevel structure for good initials $u^{(0)}$ and Multigrid Preconditioning
- Exploit various levels of elimination w.r.t. parallel computing
  ($K_i$, $S_i$, BETI saddle-point-problem, etc.
- Nonlinear Tearing and Interconnecting

# Nonlinear Tearing and Interconnecting

Solution $u$ of the local nonlinear boundary value problem

$$\begin{aligned} -\nabla \cdot \left[\nu_i(|\nabla u|)\nabla u\right] &= f && \text{in } \Omega_i \\ u &= g && \text{on } \Gamma_i \cap \Gamma \\ u &= v_i && \text{on } \Gamma_i \setminus \Gamma \end{aligned}$$

defines the **Nonlinear Dirichlet-to-Neumann-map**

$$T_i[f,g] : H^{1/2}(\Gamma_i \setminus \Gamma) \to H^{-1/2}(\Gamma_i \setminus \Gamma) : v_i \mapsto \nu_i(|\nabla u|)\frac{\partial u}{\partial \vec{n}_i}$$

$\nu_i(\cdot) \equiv$ const $\implies T_i[f,g](v_i) = S_i g + S_i v_i - N_i f$

$\Omega_i$ floating: $T_i[f,g](v_i) = T_i[f](v_i)$, kernel $=$ constant functions

*Possible Realization* of $T_{i,h}[f,g](v_i)$:

Solve nonlinear Dirichlet problem $(g, v_i, f)$ with (damped) Newton
Last iterate $u_h^{(k)}$ determines Neumann data via Schur Complement.

# Nonlinear Tearing and Interconnecting

**Nonlinear Neumann-to-Dirichlet-Map (nonfloating)**
Solve Dirichlet problem

$$\begin{aligned} -\nabla \cdot \left[\nu_i(|\nabla u|)\nabla u\right] &= f, && \text{in } \Omega_i, \\ u &= g, && \text{on } \Gamma_i \cap \Gamma, \\ \nu_i(|\nabla u|)\partial u / \partial \vec{n}_i &= t_i, && \text{on } \Gamma_i \setminus \Gamma, \end{aligned}$$

with a (damped) Newton, take Dirichlet data $\rightsquigarrow T_i[f, g]^{-1}(t_i)$

**Nonlinear Neumann-to-Dirichlet-Map (floating)**
Solve a regularized version of the local Neumann problem

$$\begin{aligned} -\nabla \cdot \left[\nu_i(|\nabla u|)\nabla u\right] &= f && \text{in } \Omega_i \\ \nu_i(|\nabla u|)\partial u / \partial \vec{n}_i &= t_i && \text{on } \Gamma_i \end{aligned}$$

with (damped) Newton, take Dirichlet data $\rightsquigarrow \tilde{T}_i[f, g]^{-1}(t_i)$

# Nonlinear Tearing and Interconnecting

## Global Problem

$$t_i = T_i[f,g](u_i) \qquad \text{on } \Gamma_i$$

$$u_i = u_j \qquad \text{on } \Gamma_{ij}$$

$$t_i + t_j = 0 \qquad \text{on } \Gamma_{ij}$$

Eliminating $(t_i) \rightsquigarrow$ **Nonlinear Variational Skeleton Problem**

$$\sum_{i \in I} \int_{\Gamma_i \setminus \Gamma} T_i[f,g](u)\, v \,\mathsf{ds}_x = 0, \quad \forall v \in H^{1/2}(\cup_{i\in I}\Gamma_i \setminus \Gamma).$$

Discrete approximation of the nonlinear operators:

$$T_{i,h}^{\mathsf{FEM/BEM}}[f,g], \quad \tilde{T}_{i,h}^{\mathsf{FEM/BEM}}[f,g]$$

# Nonlinear Tearing and Interconnecting

## Nonlinear FETI/BETI System

$$T_{i,h}^{\mathsf{FEM/BEM}}[f,g](\underline{u}_i) + B_i^T \underline{\lambda} = 0, \qquad \forall i \in I,$$
$$\sum_{i \in I} B_i \underline{u}_i = 0.$$

Eliminate $(u_i)$ via solution of local (nonlinear) problems

$$u_i = T_{i,h}^{\mathsf{FEM/BEM}}[f,g]^{\dagger}\big(-B_i^T \lambda\big) \underbrace{+\gamma_i e_i}_{\text{“+ constant”}}$$

Define

$$F(\underline{\lambda}) := \sum_{i \in I} B_i\, T_{i,h}^{\mathsf{FEM/BEM}}[f,g]^{\dagger}\big(-B_i^T \underline{\lambda}\big),$$
$$G := (B_i e_i)_{i \in I,\ \Omega_i \text{ floating}},$$

# Nonlinear Tearing and Interconnecting

Nonlinear Dual Equation

$$\begin{array}{lclcl} F(\underline{\lambda}) & + & G\underline{\gamma} & = & 0 \\ G^T\underline{\lambda} & & & = & 0 \end{array}$$

Linear Projection $P := I - G(G^TG)^{-1}G^T$

$\rightsquigarrow$ **Projected Nonlinear Dual Equation**

$$PF(\underline{\lambda}) = 0$$

Solution $\underline{\lambda} \longrightarrow \underline{\gamma} \longrightarrow (\underline{u}_i) \longrightarrow$ global field $u_h$

# Nonlinear Tearing and Interconnecting - To Do

- Existence of a unique solution $\lambda$ (Regularity of $F(\cdot)$)
- Fixpoint or Newton Iteration $+$ Analysis
- Preconditioning