Comparison of Geometric and Algebraic Multigrid for Data-Sparse Boundary Element Matrices

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Outline of the Talk

- Preliminaries
- Boundary Element Method and
  Adaptive Cross Approximation
- Multigrid Methods

  Geometric Multigrid
  Algebraic Multigrid
- Convergence of the V-Cycle for ACA Matrices
- Numerical Results
- Conclusions

Problem Formulation

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\Gamma = \partial\Omega$.

1. **Interior Dirichlet Problem**:

$$\begin{array}{rcll} -\Delta u & = & 0 & x \in \Omega \\ u(x) & = & g(x) & x \in \Gamma \end{array}$$

2. **Representation Formula**:

$$\sigma(y)u(y) = -\int_\Gamma u(x)\frac{\partial E}{\partial n_x}(x,y)ds_x + \int_\Gamma \frac{\partial u}{\partial n_x}(x)E(x,y)ds_x$$

$$\begin{array}{rcll} \sigma(y) & = & 0 & y \notin \bar{\Omega} \\ \sigma(y) & = & \frac{\Theta}{2\Pi} & y \in \Gamma \\ \sigma(y) & = & 1 & y \in \Omega \end{array}$$

Problem Formulation

we obtain the compact operator-form:

$$Vv = (\frac{1}{2}I + K)u$$

$V \ldots$ single layer potential operator

$K \ldots$ double layer potential operator

$u \ldots$ Dirichlet data

$v = \partial u / \partial n \ldots$ Neumann data

BEM - Sparse Representation (ACA)

Problem: Dense matrix, vector multiplication is of order $\mathcal{O}(N_h^2)$ Idea: Low rank approximation $\rightarrow$ Adaptive Cross Approximation
M. Bebendorf [2001]
M. Bebendorf, S. Rjasanow [2001]
$\rightarrow$ split the dense BE-matrix V_h into 'far-field' and 'near-field' contributions

$$V_h = V_h^{near} + V_h^{far}$$

BEM - Sparse Representation (ACA)

Let D_1, D_2 clusters corresponding to indices of V_h^{far} with

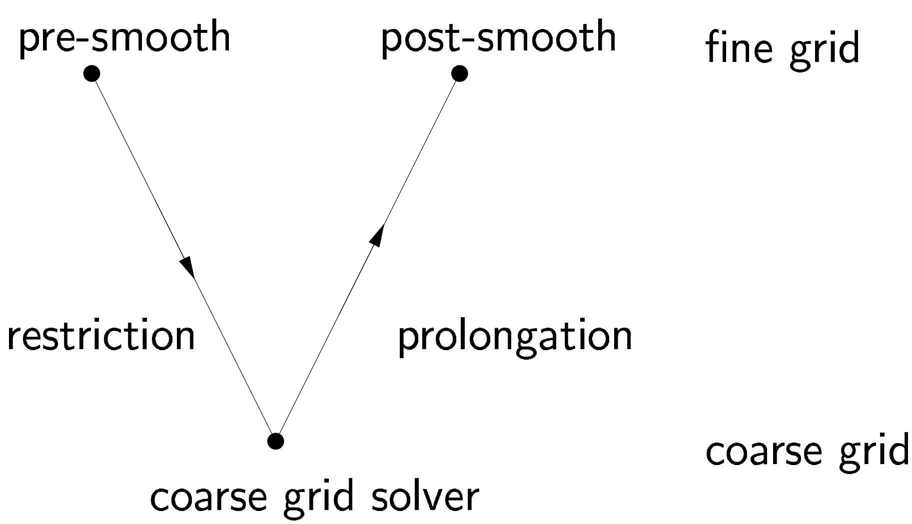
$$\text{diam } D_2 < \eta \, \text{dist}(D_1, D_2)$$

$$\rightarrow \eta\text{-admissible}$$

find a low rank approximation $\tilde{A} = \sum_{i=1}^{r} u_i v_i^\top$
for some η-admissible matrix $A \in \mathbb{R}^{n \times m}$

$$\rightarrow \widetilde{V}_h = V_h^{near} + \sum_{i=1}^{N_B} \sum_{j=1}^{r_i} u_j^i v_j^{i\top}$$

Multigrid Methods - Twogrid



Motivation: split the frequencies of the error

- smoother for highly oscillating contributions
- coarse grid correction for smooth components

Components of Multigrid Algorithms

1. **Matrix Hierarchy**

   separate grids (GMG)
   splitting into fine and coarse grid nodes (AMG)

2. **Transfer operators**

   prolongation $P_h : X_H \mapsto X_h$, restriction $R_h = (P_h)^T : X_h \mapsto X_H$

3. **Coarse grid operator**

   Galerkin's method: $K_H = P_h^T K_h P_h$

4. **Smoothing operator**

   Gauss-Seidel method (FEM, Hyp.sing.Op.)
   Bramble-Leyk-Pasciak smoother (Sing.Op.)

Multigrid - Smoothing

Single Layer Potential:

Problem: Converse behavior of eigenvalues and eigenfunctions w.r.t. Hypersingular operator

Idea: suggested by Bramble, Leyk and Pasciak:
The Analysis of Multigrid Algorithms for Pseudo-Differential Operators of Order Minus One [1994]

$\rightarrow$ choose $H^{-1}(\Omega)$ inner product for MG development

$\rightarrow$ only multiplications with a sparse matrix are needed

Multigrid - Smoothing

Consider an appropriate BLP-smoothing sweep

$$\underline{u}_h = \underline{u}_h + \tau_h A_h(\underline{f} - K_h \underline{u}_h)$$

with $0 < \tau_h < 1/\lambda_u$ and λ_u the largest eigenvalue of

$$K_h \underline{w} = \lambda A_h^{-1} \underline{w}.$$

Note, A_h is discretization of the boundary corresponding to the Laplace-Beltrami operator

Multigrid - Transfer Operators

Geometric Multigrid:

basis of coarse and fine level $X_H = \operatorname{span} \Phi_i,\ X_h = \operatorname{span} \phi_i$

$$\Phi_i = \sum_{j=1}^{n} p_{ij} \phi_j$$

→ Prolongation/Restriction Operators

$$v_H = P_h v_h, \qquad V_H = P_h^\top V_h P_h$$

Algebraic Multigrid:

entries of P_h depend on the splitting into coarse node and fine nodes

$$\omega = \omega_C \cup \omega_F$$

Convergence Analysis

convergence for approximated single layer potential $\widetilde{V}(.,.)$

$\rightarrow$ uniform reduction per iteration
$V(\mathcal{E}v, v) \leq (1 - 1/C)V(v, v)$

$\rightarrow$ less than full regularity & approximation
$\|(I - P_{k-1}v\|^2 \leq c\lambda_k^{-1}V(v, v)$

$\rightarrow$ continuous form as well as matrix form

Convergence Analysis

proof for single layer potential $\rightarrow$

[1] J. Bramble, Z. Leyk and J. Pasciak.
The Analysis of Multigrid Algorithms for Pseudo-Differential Operators of Order Minus One. 1994

basically two assumptions $\rightarrow$

(A1) *lower estimate*

$$V(v,v) \leq \tilde{C}(V(P_1 v, v) + \sum_{k=2}^{J} \lambda_k^{-1} \|V_k P_k v\|_{-1}^2) \quad \forall v \in M$$

(A2) *upper estimate*

$$\|V_i v\|_{-1}^2 \leq \tilde{\tilde{C}}(\epsilon^{i-k})^2 \lambda_i V(v,v) \quad \forall v \in M_k \quad 1 \leq i \leq k \leq J \quad \epsilon < 1$$

Convergence Analysis

consider $\widetilde{V}(.,.)$ as perturbed form of $V(.,.)$

$\rightarrow$ conditions (A1), (A2) will also hold for $\widetilde{V}(.,.)$

$\rightarrow$ additional assumptions

(B1) *spectral equivalence*

$$c_1 V(v,v) \leq \widetilde{V}(v,v) \leq c_2 V(v,v) \quad v \in M$$

(B2) *approximation*

$$|V(v,w) - \widetilde{V}(v,w)| \leq c\lambda_J^{-\beta/2} |||v|||\,|||w||| \quad v,w \in M$$

with $|||v|||^2 = V(v,v)$

Convergence Analysis

(B1) induces

lower estimate (A1) for $\widetilde{V}(.,.)$

(B1), (B2) and inequality $c(v,v) \leq V(v,v)$ induce

upper estimate (A2) for $\widetilde{V}(.,.)$

$\rightarrow$ uniform reduction per iteration for $\widetilde{V}(.,.)$

Convergence Analysis

Verification for ACA Matrices

- low - rank approximation provides

$$\|V_h - \widetilde{V}_h\|_F \leq \epsilon \|V_h\|_F.$$

- norm equivalence between spectral norm / Frobenius norm

$$\|V_h\|_2 \leq \|V_h\|_F \leq \sqrt{n}\|V_h\|_2 \quad V_h \in \mathbb{R}^{n \times n}$$

Convergence Analysis

one can show, that $\forall \underline{v} \in \mathbb{R}^n$

$$\begin{aligned} \left(1 - \epsilon\sqrt{N_h}\kappa(V_h)\right) \underline{v}^\top V_h \underline{v} &\leq \underline{v}^\top \widetilde{V}_h \underline{v} \\ \underline{v}^\top \widetilde{V}_h \underline{v} &\leq \left(1 + \epsilon\sqrt{N_h}\kappa(V_h)\right) \underline{v}^\top V_h \underline{v} \end{aligned}$$

$\rightarrow$ verification of (B1)

Convergence Analysis

Further we have inequalities

$$\|u_h\|_{-1/2} \geq ch^{1/2}\|u_h\|_0,$$

$$\|\underline{u}_h\|_{\mathbb{R}^n} \leq ch^{-d/2}\|u_h\|_0$$

and

$$c_1\|u_h\|^2_{-1/2} \leq V(u_h, u_h) \leq c_2\|u_h\|^2_{-1/2}.$$

$$\|V_h\|_2 \simeq h^d$$

Convergence Analysis

Remark, $\lambda_J \simeq h^{-1}$, thus we can estimate

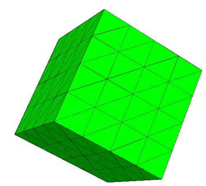
$$\begin{aligned}
|V(v_h, w_h) - \widetilde{V}(v_h, w_h)| &\leq \|V_h - \widetilde{V}_h\|_2 \|\underline{v}_h\|_{\mathbb{R}^{N_h}} \|\underline{u}_h\|_{\mathbb{R}^{N_h}} \\
&\leq \epsilon \sqrt{N_h} \|V_h\|_2 h^{-d-1} \|v_h\|_{-1/2} \|u_h\|_{-1/2} \\
&\leq \epsilon \|V_h\|_2 h^{-1-3d/2} |||v_h|||\ |||u_h||| \\
&\leq c\,\epsilon h^{-1-d/2-\beta/2} \lambda_J^{-\beta/2} |||v_h|||\ |||u_h|||
\end{aligned}$$

uniform reduction of perturbed bilinearform

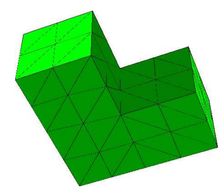
$$\rightarrow \widetilde{V}(\mathcal{E}v, v) \leq (1 - 1/C)\widetilde{V}(v, v)$$

Numerical Results

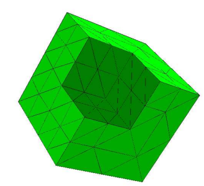
- AMG program package PEBBLES
(www.numa.uni-linz.ac.at/Research/Projects/pebbles.html)



- BEM-Matrix entries by OSTBEM (O. Steinbach)



- Sparse representation of V_h (ACA)
- AMG/GMG preconditioner for CG - Full MG



- PC 2000MHz Athlon

Assembling Time AMG - GMG

L-Shape N_h	V_h [sec] Assembling	AMG [sec] Galerkin Projection	GMG [sec] Matrix Hierarchy
1792	6.0	2.0	0.9
7168	32.5	7.5	6.9
28672	158	30	40

Iteration Numbers - CPU Time

Number of Unknowns	AMG		GMG	
	PCG-Cycle [sec]	Iterations	PCG-Cycle [sec]	Iterations
L-Shape				
1792	0.1	6	0.1	7
7168	0.8	6	0.6	7
28672	4.2	9	2.9	7
114688	-	-	33*)	7
Cube		*) memory exceeded - r/w effects		
3072	0.2	5	0.2	6
12288	1.7	8	1.1	7
49152	9.0	11	5.7	7
Fichera-Corner				
1920	0.1	14	0.1	15
7680	0.8	15	0.6	15
30720	5.0	17	3.2	15

Multigrid Preconditioner - Full Multigrid

Number of Unknowns	GMG		Full MG	
	PCG-Cycle [sec]	Iterations	Cycle [sec]	Iterations
L-Shape				
1792	0.1	7	0.1	10
7168	0.6	7	0.5	10
28672	2.9	7	2.8	9
Cube				
3072	0.2	6	0.2	8
12288	1.1	7	1.0	8
49152	5.7	7	5.2	8
Fichera-Corner				
1920	0.1	15	0.1	53
7680	0.6	15	0.6	52
30720	3.2	15	3.1	50

Grid Data - Level Data

Level	AMG	GMG
Cube		
fine 1	49152	49152
2	23828	12288
3	7311	3072
4	2025	768
coarse 5	625	-

Level	AMG	GMG
Fichera-Corner		
fine 1	30720	30720
2	14995	7680
3	4677	1920
coarse 4	1282	480

Conclusions and Outlook

+ GMG/AMG components for BE-matrices
+ 2D/3D examples for dense/sparse BE-matrices
+ First analysis results for GMG

+ Appropriate MG design for more complex geometries
+ Analysis for AMG