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Numerical Aspects of a Poroelastic Time Domain Boundary Element Formulation

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Adaptive Fast Boundary Element Methods in Industrial Applications

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Content

Biot's theory of poroelastic continua

constitutive equation $\sigma_{ij} = \sigma^S_{ij} + \sigma^F \delta_{ij}$

$$\sigma_{ij} = G\left(u_{i,j} + u_{j,i}\right) + \left(\left(K - \frac{2}{3}G\right) u_{k,k} - \alpha p\right)\delta_{ij}$$

$$\zeta = \alpha u_{k,k} + \frac{\phi^2}{R} p$$

equilibrium $\rho = \rho_s (1 - \phi) + \phi \rho_f$

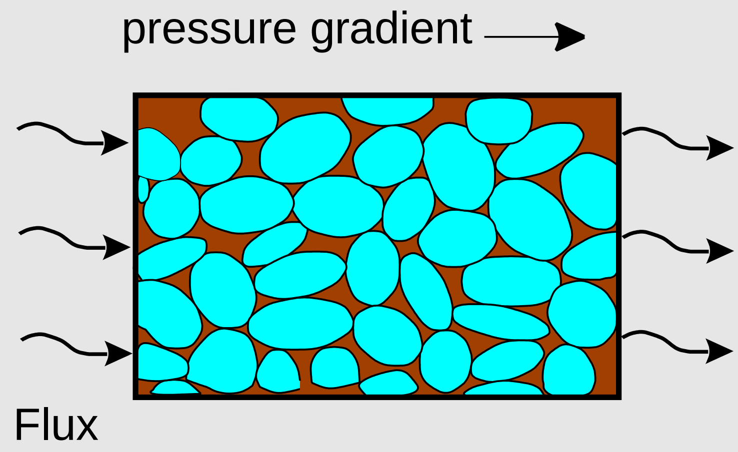
$$\sigma_{ij,j} + F_i = \rho \frac{\partial^2}{\partial t^2} u_i + \rho_f \frac{\partial}{\partial t} w_i$$

Darcy's law

$$q_i = -\kappa \left(p_{,i} + \rho_f \frac{\partial^2}{\partial t^2} u_i + \frac{\rho_a + \phi \rho_f}{\phi} \frac{\partial}{\partial t} w_i \right)$$

continuity equation

$$\frac{\partial}{\partial t}\zeta + q_{i,i} = a$$



Nomenclature

σ_{ij}	total stress	q_i	specific flux	u_i	solid displacement	ζ	'fluid strain'
α	Biot's stress coefficient	p	pore pressure	w_i	seepage velocity	F_i	bulk body force
G, K	shear, bulk modulus	ϕ	porosity	ρ_a	apparent mass density	ρ	bulk density

Governing equations

- representation in Laplace domain ($\mathscr{L}\{f(t)\} = \hat{f}$)

$$\left.\begin{aligned} G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - (\alpha - \beta)\hat{p}_{,i} - s^2(\rho - \beta\rho_f)\hat{u}_i &= -\hat{F}_i \\ \frac{\beta}{\rho_f s}\hat{p}_{,ii} - \frac{\phi^2 s}{R}\hat{p} - (\alpha - \beta)s\hat{u}_{i,i} &= -\hat{a} \end{aligned}\right\} \quad \mathbf{B}^* \begin{bmatrix} \hat{u}_i \\ \hat{p} \end{bmatrix} = - \begin{bmatrix} \hat{F}_i \\ \hat{a} \end{bmatrix}$$

Governing equations

- representation in Laplace domain ($\mathscr{L}\{f(t)\} = \hat{f}$)

$$\left.\begin{aligned} G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - (\alpha-\beta)\,\hat{p}_{,i} - s^2\left(\rho - \beta\rho_f\right)\hat{u}_i &= -\hat{F}_i \\ \frac{\beta}{\rho_f s}\hat{p}_{,ii} - \frac{\phi^2 s}{R}\hat{p} - (\alpha-\beta)\,s\hat{u}_{i,i} &= -\hat{a} \end{aligned}\right\} \qquad \mathbf{B}^*\begin{bmatrix}\hat{u}_i \\ \hat{p}\end{bmatrix} = -\begin{bmatrix}\hat{F}_i \\ \hat{a}\end{bmatrix}$$

- weak singular fundamental solutions

$$\hat{U}^S_{ij} = \frac{1+\nu}{8\pi E(1-\nu)}\left\{r_{,i}r_{,j} + \delta_{ij}(3-4\nu)\right\}\frac{1}{r} + \mathscr{O}\left(r^0\right) \qquad \hat{P}^F = \frac{\rho_f s}{4\pi\beta}\frac{1}{r} + \mathscr{O}\left(r^0\right)$$

$$\hat{T}^F_i = \frac{\rho_f s^2}{8\pi\beta}\frac{1-2\nu}{1-\nu}\left\{(\alpha-\beta)\,r_{,i}r_{,n} + n_i\left(\alpha + \beta\frac{1}{1-2\nu}\right)\right\}\frac{1}{r} + \mathscr{O}\left(r^0\right)$$

$$\hat{Q}^S_j = \frac{1+\nu}{8\pi E(1-\nu)}\left\{\alpha(1-2\nu)\left(r_{,n}r_{,j} - n_j\right) - 2\beta(1-\nu)\left(r_{,n}r_{,j} + n_j\right)\right\}\frac{1}{r} + \mathscr{O}\left(r^0\right)$$

Governing equations

- representation in Laplace domain ($\mathscr{L}\{f(t)\} = \hat{f}$)

$$\left.\begin{aligned} G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - (\alpha - \beta)\,\hat{p}_{,i} - s^2(\rho - \beta\rho_f)\,\hat{u}_i &= -\hat{F}_i \\ \frac{\beta}{\rho_f s}\hat{p}_{,ii} - \frac{\phi^2 s}{R}\hat{p} - (\alpha - \beta)\,s\hat{u}_{i,i} &= -\hat{a} \end{aligned}\right\} \qquad \mathbf{B}^* \begin{bmatrix} \hat{u}_i \\ \hat{p} \end{bmatrix} = - \begin{bmatrix} \hat{F}_i \\ \hat{a} \end{bmatrix}$$

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$$\hat{U}_{ij}^S = \frac{1+\nu}{8\pi E(1-\nu)}\left\{r_{,i}r_{,j} + \delta_{ij}(3-4\nu)\right\}\frac{1}{r} + \mathscr{O}\left(r^0\right) \qquad \hat{P}^F = \frac{\rho_f s}{4\pi\beta}\frac{1}{r} + \mathscr{O}\left(r^0\right)$$

$$\hat{T}_i^F = \frac{\rho_f s^2}{8\pi\beta}\frac{1-2\nu}{1-\nu}\left\{(\alpha-\beta)\,r_{,i}r_{,n} + n_i\left(\alpha + \beta\frac{1}{1-2\nu}\right)\right\}\frac{1}{r} + \mathscr{O}\left(r^0\right)$$

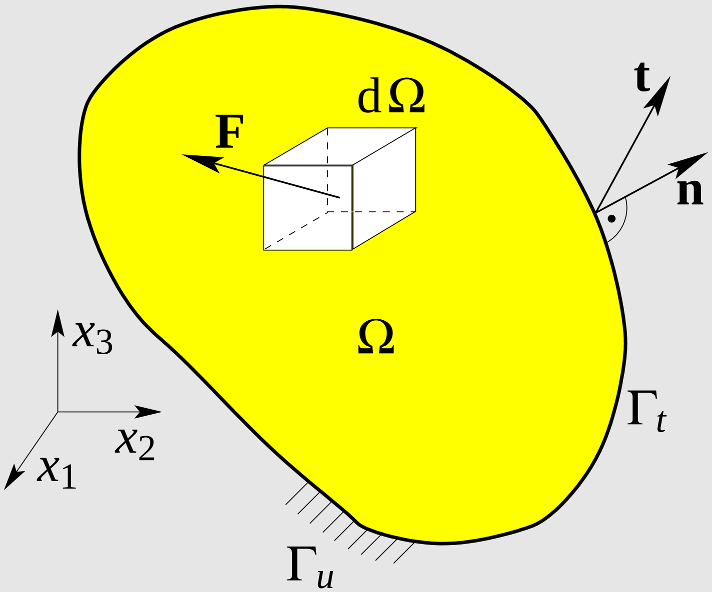
$$\hat{Q}_j^S = \frac{1+\nu}{8\pi E(1-\nu)}\left\{\alpha(1-2\nu)\left(r_{,n}r_{,j} - n_j\right) - 2\beta(1-\nu)\left(r_{,n}r_{,j} + n_j\right)\right\}\frac{1}{r} + \mathscr{O}\left(r^0\right)$$

- strong singular fundamental solutions

$$\hat{T}_{ij}^S = \frac{-\left[(1-2\nu)\,\delta_{ij} + 3r_{,i}r_{,j}\right]r_{,n} + (1-2\nu)\left(r_{,j}n_i - r_{,i}n_j\right)}{8\pi(1-\nu)\;\;r^2} + \mathscr{O}\left(r^0\right) \qquad \hat{Q}^F = -\frac{1}{4\pi}\frac{r_{,n}}{r^2} + \mathscr{O}\left(r^0\right)$$

Content

Boundary integral equation



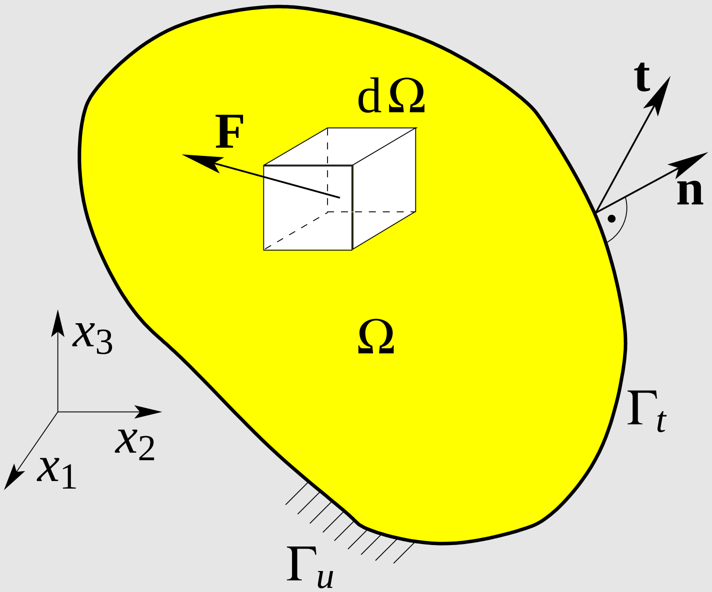
$$\Gamma = \Gamma_u + \Gamma_t$$

- weighted residuals

$$\int_{\Omega} \mathbf{G}^T \mathbf{B}^* \begin{bmatrix} \hat{u}_i(\mathbf{x}, s) \\ \hat{p}(\mathbf{x}, s) \end{bmatrix} \mathrm{d}\Omega = \mathbf{0}$$

with $\mathbf{G} = \begin{bmatrix} \hat{U}^S_{ij}(\mathbf{x}, \mathbf{y}, s) & \hat{U}^F_i(\mathbf{x}, \mathbf{y}, s) \\ \hat{P}^S_j(\mathbf{x}, \mathbf{y}, s) & \hat{P}^F(\mathbf{x}, \mathbf{y}, s) \end{bmatrix}$

Boundary integral equation



$$\Gamma = \Gamma_u + \Gamma_t$$

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- two partial integrations
- singular behavior
- transformation to time domain

$$\int_0^t \int_\Gamma \begin{bmatrix} U^S_{ij}(t-\tau,\mathbf{y},\mathbf{x}) & -P^S_j(t-\tau,\mathbf{y},\mathbf{x}) \\ U^F_i(t-\tau,\mathbf{y},\mathbf{x}) & -P^F(t-\tau,\mathbf{y},\mathbf{x}) \end{bmatrix} \begin{bmatrix} t_i(\tau,\mathbf{x}) \\ q(\tau,\mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma\,\mathrm{d}\tau =$$

$$\int_0^t \oint_\Gamma \begin{bmatrix} T^S_{ij}(t-\tau,\mathbf{y},\mathbf{x}) & Q^S_j(t-\tau,\mathbf{y},\mathbf{x}) \\ T^F_i(t-\tau,\mathbf{y},\mathbf{x}) & Q^F(t-\tau,\mathbf{y},\mathbf{x}) \end{bmatrix} \begin{bmatrix} u_i(\tau,\mathbf{x}) \\ p(\tau,\mathbf{x}) \end{bmatrix} \mathrm{d}\Gamma\,\mathrm{d}\tau + \begin{bmatrix} c_{ij}(\mathbf{y}) & 0 \\ 0 & c(\mathbf{y}) \end{bmatrix} \begin{bmatrix} u_i(t,\mathbf{y}) \\ p(t,\mathbf{y}) \end{bmatrix}$$

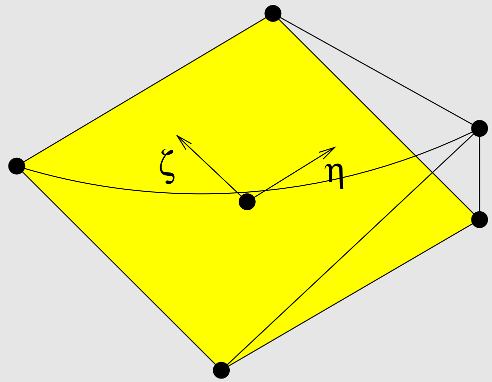
- spatial discretization

$$u_i\left(\mathbf{x},t\right) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f\left(\mathbf{x}\right) u_i^{ef}\left(t\right)$$

$$t_i\left(\mathbf{x},t\right) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f\left(\mathbf{x}\right) t_i^{ef}\left(t\right)$$

$$p\left(\mathbf{x},t\right) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f\left(\mathbf{x}\right) p^{ef}\left(t\right)$$

$$q\left(\mathbf{x},t\right) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f\left(\mathbf{x}\right) q^{ef}\left(t\right)$$



e.g. linear ansatz function

$$N_e^1\left(\eta,\zeta\right) = \frac{1}{4}\left(1-\eta\right)\left(1-\zeta\right)$$

$$N_e^2\left(\eta,\zeta\right) = \frac{1}{4}\left(1-\eta\right)\left(1+\zeta\right)$$

$$N_e^3\left(\eta,\zeta\right) = \frac{1}{4}\left(1+\eta\right)\left(1+\zeta\right)$$

$$N_e^4\left(\eta,\zeta\right) = \frac{1}{4}\left(1+\eta\right)\left(1-\zeta\right)$$

Spatial discretization

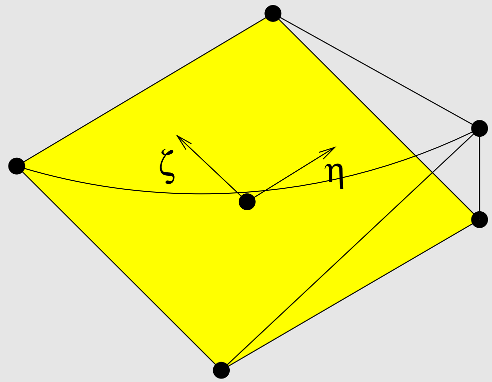
- spatial discretization

$$u_i(\mathbf{x},t) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})\, u_i^{ef}(t)$$

$$t_i(\mathbf{x},t) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})\, t_i^{ef}(t)$$

$$p(\mathbf{x},t) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})\, p^{ef}(t)$$

$$q(\mathbf{x},t) = \sum_{e=1}^{E}\sum_{f=1}^{F} N_e^f(\mathbf{x})\, q^{ef}(t)$$



e.g. linear ansatz function

$$N_e^1(\eta,\zeta) = \frac{1}{4}(1-\eta)(1-\zeta)$$

$$N_e^2(\eta,\zeta) = \frac{1}{4}(1-\eta)(1+\zeta)$$

$$N_e^3(\eta,\zeta) = \frac{1}{4}(1+\eta)(1+\zeta)$$

$$N_e^4(\eta,\zeta) = \frac{1}{4}(1+\eta)(1-\zeta)$$

- discretized integral equation

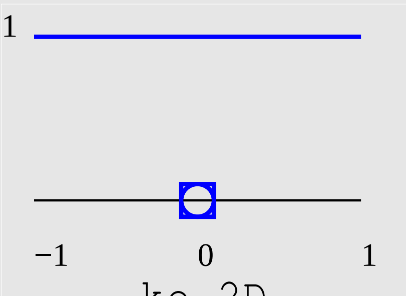
$$\begin{bmatrix} c_{ij}(\mathbf{y})\, u_i(\mathbf{y},t) \\ c(\mathbf{y})\, p(\mathbf{y},t) \end{bmatrix} = \sum_{e=1}^{E}\sum_{f=1}^{F}\Bigg\{ \int_0^t \int_\Gamma \begin{bmatrix} U_{ij}^S(\mathbf{x},\mathbf{y},t-\tau) & -P_j^S(\mathbf{x},\mathbf{y},t-\tau) \\ U_i^F(\mathbf{x},\mathbf{y},t-\tau) & -P^F(\mathbf{x},\mathbf{y},t-\tau) \end{bmatrix} N_e^f(\mathbf{x})\, \mathrm{d}\Gamma \begin{bmatrix} t_i^{ef}(\tau) \\ q^{ef}(\tau) \end{bmatrix} \mathrm{d}\tau$$

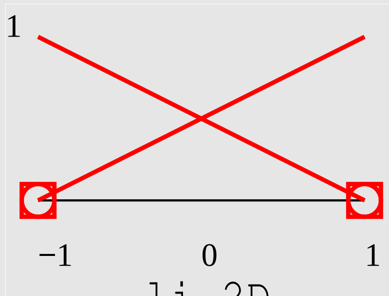
$$- \int_0^t \oint_\Gamma \begin{bmatrix} T_{ij}^S(\mathbf{x},\mathbf{y},t-\tau) & Q_j^S(\mathbf{x},\mathbf{y},t-\tau) \\ T_i^F(\mathbf{x},\mathbf{y},t-\tau) & Q^F(\mathbf{x},\mathbf{y},t-\tau) \end{bmatrix} N_e^f(\mathbf{x})\, \mathrm{d}\Gamma \begin{bmatrix} u_i^{ef}(\tau) \\ p^{ef}(\tau) \end{bmatrix} \mathrm{d}\tau \Bigg\}$$

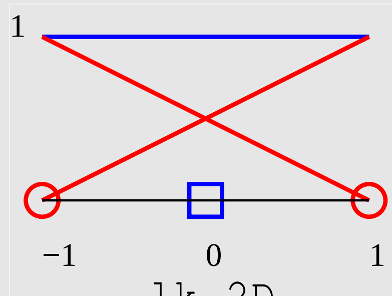
Mixed shape functions

- Isoparametric elements - shape functions identical for all quantities and geometry

- Mixed elements – using different shape functions for different quantities (common for finite elements), e.g. $N_e^f(\mathbf{x})$ linear for u, t and constant for p, q

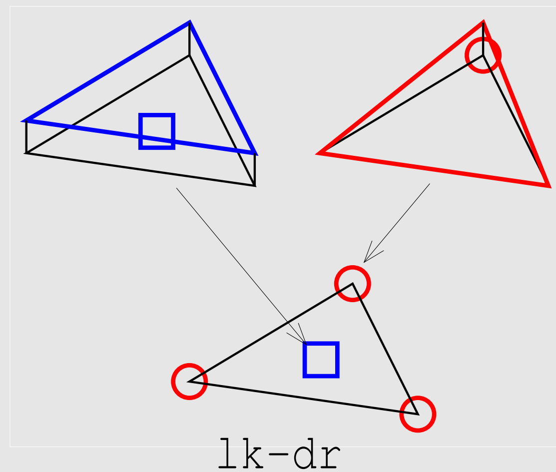






Shape functions in 2-d and 3-d

Element	${}^uN_e^f, {}^tN_e^f$	${}^pN_e^f, {}^qN_e^f$
ko-2D, ko-dr	constant	constant
li-2D, li-dr	linear	linear
lk-2D, lk-dr	linear	constant



- quadrature rule for $n = 0, 1, \ldots, N$ time steps:

$$y(t) = f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau \quad \Rightarrow \quad y(n\Delta t) = \sum_{k=0}^{n} \omega_{n-k}\left(\hat{f}, \Delta t\right) g(k\Delta t)$$

Lubich, C. (1988): Convolution Quadrature and Discretized Operational Calculus. I., *Numerische Mathematik*, Vol. 52, 129–145

Convolution Quadrature Method

- quadrature rule for $n = 0, 1, \ldots, N$ time steps:

$$y(t) = f(t) * g(t) = \int_0^t f(t-\tau)\, g(\tau)\, \mathrm{d}\tau \quad \Rightarrow \quad y(n\Delta t) = \sum_{k=0}^{n} \omega_{n-k}\left(\hat{f}, \Delta t\right) g(k\Delta t)$$

- integration weight:

$$\omega_n\left(\hat{f}, \Delta t\right) = \frac{1}{2\pi i} \int_{|z|=\mathscr{R}} \hat{f}\left(\frac{\gamma(z)}{\Delta t}\right) z^{-n-1}\, \mathrm{d}z \approx \frac{\mathscr{R}^{-n}}{L} \sum_{\ell=0}^{L-1} \hat{f}\left(\frac{\gamma\left(\mathscr{R} e^{i\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) e^{-in\ell\frac{2\pi}{L}}$$

  - $\gamma(z)$ A-stable multi step method, e.g. BDF 2: $\gamma(z) = \frac{3}{2} - 2z + \frac{1}{2}z^2$
  - Δt time step size of equal duration
  - $L = N$ effective choice for determining ω_n (FFT)
  - $\mathscr{R}^N = \sqrt{\varepsilon}$ with $\varepsilon \approx 10^{-10}$

Lubich, C. (1988): Convolution Quadrature and Discretized Operational Calculus. I., *Numerische Mathematik*, Vol. 52, 129–145

Temporal discretization

- temporal discretization with Convolution Quadrature Method yields for $n = 0, 1, \ldots, N$

$$
\begin{bmatrix} c_{ij}(\mathbf{y})\, u_i(n\Delta t) \\ c(\mathbf{y})\, p(n\Delta t) \end{bmatrix} = \sum_{e=1}^{E} \sum_{f=1}^{F} \sum_{k=0}^{n} \left\{ \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{U}_{ij}^{S}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}_{j}^{S}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{U}_{i}^{F}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}^{F}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} t_i^{ef}(k\Delta t) \\ q^{ef}(k\Delta t) \end{bmatrix} \right.
$$

$$
\left. - \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{T}_{ij}^{S}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}_{j}^{S}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{T}_{i}^{F}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}^{F}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} u_i^{ef}(k\Delta t) \\ p^{ef}(k\Delta t) \end{bmatrix} \right\}
$$

with integration weights, e.g.

$$
\omega_{n-k}^{ef}\left(\hat{U}_{ij}, \mathbf{y}, \Delta t\right) = \frac{\mathscr{R}^{k-n}}{L} \sum_{\ell=0}^{L-1} \int_{\Gamma} \hat{U}_{ij}\left(\mathbf{x}, \mathbf{y}, \frac{\gamma\left(\mathscr{R} e^{-\mathrm{i}\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) N_e^f(\mathbf{x})\, \mathrm{d}\Gamma \;\; e^{-\mathrm{i}(n-k)\ell\frac{2\pi}{L}}
$$

Temporal discretization

- temporal discretization with Convolution Quadrature Method yields for $n = 0, 1, \ldots, N$

$$\begin{bmatrix} c_{ij}(\mathbf{y})\, u_i(n\Delta t) \\ c(\mathbf{y})\, p(n\Delta t) \end{bmatrix} = \sum_{e=1}^{E} \sum_{f=1}^{F} \sum_{k=0}^{n} \left\{ \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{U}_{ij}^{S}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}_{j}^{S}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{U}_{i}^{F}, \mathbf{y}, \Delta t\right) & -\omega_{n-k}^{ef}\left(\hat{P}^{F}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} t_i^{ef}(k\Delta t) \\ q^{ef}(k\Delta t) \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} \omega_{n-k}^{ef}\left(\hat{T}_{ij}^{S}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}_{j}^{S}, \mathbf{y}, \Delta t\right) \\ \omega_{n-k}^{ef}\left(\hat{T}_{i}^{F}, \mathbf{y}, \Delta t\right) & \omega_{n-k}^{ef}\left(\hat{Q}^{F}, \mathbf{y}, \Delta t\right) \end{bmatrix} \begin{bmatrix} u_i^{ef}(k\Delta t) \\ p^{ef}(k\Delta t) \end{bmatrix} \right\}$$

with integration weights, e.g.

$$\omega_{n-k}^{ef}\left(\hat{U}_{ij}, \mathbf{y}, \Delta t\right) = \frac{\mathcal{R}^{k-n}}{L} \sum_{\ell=0}^{L-1} \int_{\Gamma} \hat{U}_{ij}\left(\mathbf{x}, \mathbf{y}, \frac{\gamma\left(\mathcal{R} e^{-\mathrm{i}\ell\frac{2\pi}{L}}\right)}{\Delta t}\right) N_e^f(\mathbf{x})\, \mathrm{d}\Gamma \;\; e^{-\mathrm{i}(n-k)\ell\frac{2\pi}{L}}$$

- quadrature formula
  - regular integrals: Gauss formula
  - weak singular integrals: Regularization with polar coordinate transformation
  - strong singular integrals: Formula by GUIGGIANI and GIGANTE

Time stepping procedure

solution with point collocation, i.e. moving $\mathbf{y}$ in every node and solving the system in each time step ($\mathbf{U}, \mathbf{T}, \mathbf{u}, \mathbf{t}$ are generalized variables here)

$$
\begin{aligned}
\omega_0 (\mathbf{T}) \mathbf{u} (\Delta t) &= \omega_0 (\mathbf{U}) \mathbf{t} (\Delta t) \\
\omega_1 (\mathbf{T}) \mathbf{u} (\Delta t) + \omega_0 (\mathbf{T}) \mathbf{u} (2\Delta t) &= \omega_1 (\mathbf{U}) \mathbf{t} (\Delta t) + \omega_0 (\mathbf{U}) \mathbf{t} (2\Delta t) \\
\omega_2 (\mathbf{T}) \mathbf{u} (\Delta t) + \omega_1 (\mathbf{T}) \mathbf{u} (2\Delta t) + \omega_0 (\mathbf{T}) \mathbf{u} (3\Delta t) &= \omega_2 (\mathbf{U}) \mathbf{t} (\Delta t) + \omega_1 (\mathbf{U}) \mathbf{t} (2\Delta t) + \omega_0 (\mathbf{U}) \mathbf{t} (\Delta t) \\
&\vdots
\end{aligned}
$$

Time stepping procedure

solution with point collocation, i.e. moving $\mathbf{y}$ in every node and solving the system in each time step ($\mathbf{U}, \mathbf{T}, \mathbf{u}, \mathbf{t}$ are generalized variables here)

$$\begin{aligned}
\omega_0(\mathbf{T})\,\mathbf{u}(\Delta t) &= \omega_0(\mathbf{U})\,\mathbf{t}(\Delta t)\\
\omega_1(\mathbf{T})\,\mathbf{u}(\Delta t) + \omega_0(\mathbf{T})\,\mathbf{u}(2\Delta t) &= \omega_1(\mathbf{U})\,\mathbf{t}(\Delta t) + \omega_0(\mathbf{U})\,\mathbf{t}(2\Delta t)\\
\omega_2(\mathbf{T})\,\mathbf{u}(\Delta t) + \omega_1(\mathbf{T})\,\mathbf{u}(2\Delta t) + \omega_0(\mathbf{T})\,\mathbf{u}(3\Delta t) &= \omega_2(\mathbf{U})\,\mathbf{t}(\Delta t) + \omega_1(\mathbf{U})\,\mathbf{t}(2\Delta t) + \omega_0(\mathbf{U})\,\mathbf{t}(\Delta t)\\
&\vdots
\end{aligned}$$

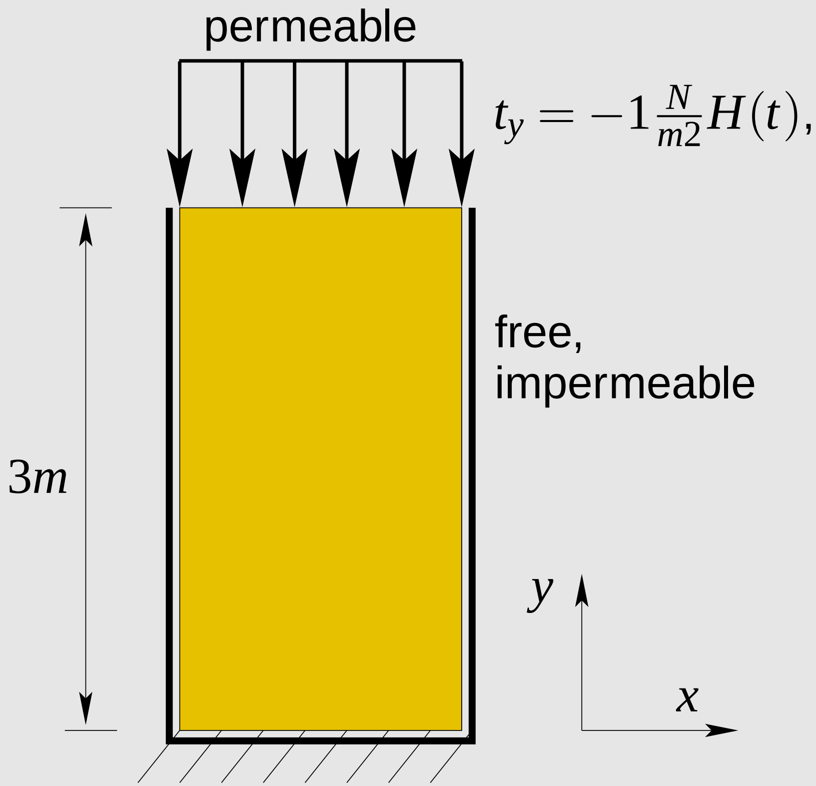
➠ final recursion formula

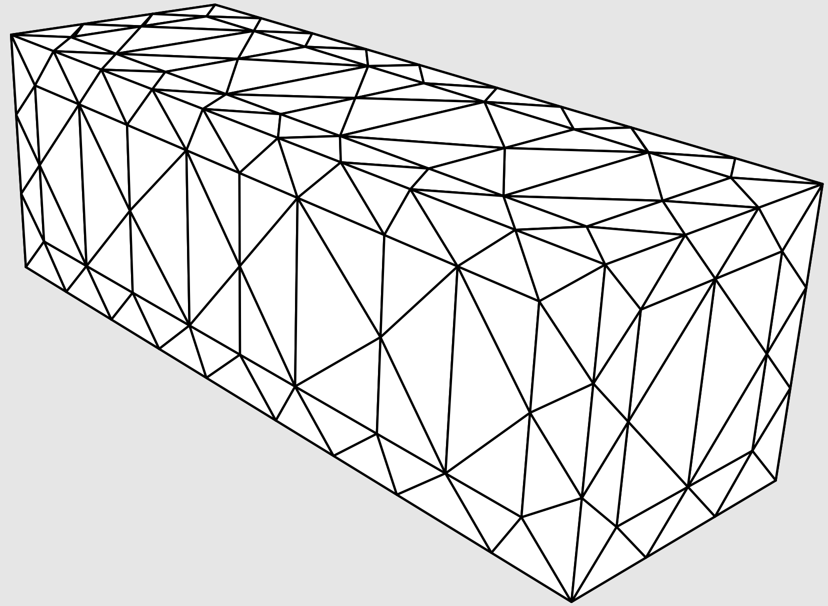
$$\omega_0(\mathbf{C})\,\mathbf{d}^n = \omega_0(\mathbf{D})\,\bar{\mathbf{d}}^n + \sum_{m=1}^{n}\left(\omega_m(\mathbf{U})\,\mathbf{t}^{n-m} - \omega_m(\mathbf{T})\,\mathbf{u}^{n-m}\right) \quad n = 1, 2, \ldots, N$$

with the vector of unknown boundary data $\mathbf{d}^n$ and the known boundary data $\bar{\mathbf{d}}^n$ in each time step

Content

Column: Problem description





324 elements on 188 nodes

	$K\ [\frac{N}{m^2}]$	$G\ [\frac{N}{m^2}]$	$\rho\ [\frac{kg}{m^3}]$	ϕ	$R\ [\frac{N}{m^2}]$	$\rho_f\ [\frac{kg}{m^3}]$	α	$\kappa\ [\frac{m^4}{Ns}]$
rock	$8 \cdot 10^9$	$6 \cdot 10^9$	2458	0.19	$4.7 \cdot 10^8$	1000	0.867	$1.9 \cdot 10^{-10}$
soil	$2.1 \cdot 10^8$	$9.8 \cdot 10^7$	1884	0.48	$1.2 \cdot 10^9$	1000	0.981	$3.55 \cdot 10^{-9}$

Dimensionless variables

Dimensionless variables

$$\tilde{x} = \frac{x}{A} \qquad \tilde{t} = \frac{t}{B} \qquad \tilde{K} = \frac{K}{C} \qquad \tilde{G} = \frac{G}{C} \qquad \tilde{\rho} = \frac{A^2}{B^2 C}\rho \qquad \tilde{\kappa} = \frac{BC}{A^2}\kappa$$

- **Fall 1, 2, 3** $\Rightarrow$ all material data $\mathscr{O}(\lambda)$

$$A = \kappa\lambda^2\sqrt{\rho C} \qquad B = \frac{\rho\kappa}{\lambda^2} \qquad C = \frac{1}{\lambda}\left(K + \frac{4}{3}G + \frac{\alpha^2}{\phi^2}R\right) \qquad \lambda = 1, 10^{-3}, 10^3$$

Dimensionless variables

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$$\tilde{x} = \frac{x}{A} \qquad \tilde{t} = \frac{t}{B} \qquad \tilde{K} = \frac{K}{C} \qquad \tilde{G} = \frac{G}{C} \qquad \tilde{\rho} = \frac{A^2}{B^2 C}\rho \qquad \tilde{\kappa} = \frac{BC}{A^2}\kappa$$

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- **Fall 4, 5** $\Rightarrow$ only normalization of modules

$$A = 1 \qquad B = 1 \qquad C = \lambda E = \lambda\frac{9KG}{6K+G} \qquad \lambda = 1, 10$$

Dimensionless variables

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$$\tilde{x} = \frac{x}{A} \qquad \tilde{t} = \frac{t}{B} \qquad \tilde{K} = \frac{K}{C} \qquad \tilde{G} = \frac{G}{C} \qquad \tilde{\rho} = \frac{A^2}{B^2 C}\rho \qquad \tilde{\kappa} = \frac{BC}{A^2}\kappa$$

- **Fall 1, 2, 3** $\Rightarrow$ all material data $\mathcal{O}(\lambda)$

$$A = \kappa\lambda^2\sqrt{\rho C} \qquad B = \frac{\rho\kappa}{\lambda^2} \qquad C = \frac{1}{\lambda}\left(K + \frac{4}{3}G + \frac{\alpha^2}{\phi^2}R\right) \qquad \lambda = 1, 10^{-3}, 10^3$$

- **Fall 4, 5** $\Rightarrow$ only normalization of modules

$$A = 1 \qquad B = 1 \qquad C = \lambda E = \lambda\frac{9KG}{6K+G} \qquad \lambda = 1, 10$$

- **Fall 6** $\Rightarrow$ scaling of Young's modules to the permeability

$$A = 1 \qquad B = 1 \qquad C = \sqrt{\frac{E}{\kappa}}$$

Dimensionless variables

Dimensionless variables

$$\tilde{x} = \frac{x}{A} \qquad \tilde{t} = \frac{t}{B} \qquad \tilde{K} = \frac{K}{C} \qquad \tilde{G} = \frac{G}{C} \qquad \tilde{\rho} = \frac{A^2}{B^2 C}\rho \qquad \tilde{\kappa} = \frac{BC}{A^2}\kappa$$

- **Fall 1, 2, 3** $\Rightarrow$ all material data $\mathcal{O}(\lambda)$

$$A = \kappa\lambda^2\sqrt{\rho C} \qquad B = \frac{\rho\kappa}{\lambda^2} \qquad C = \frac{1}{\lambda}\left(K + \frac{4}{3}G + \frac{\alpha^2}{\phi^2}R\right) \qquad \lambda = 1, 10^{-3}, 10^3$$

- **Fall 4, 5** $\Rightarrow$ only normalization of modules

$$A = 1 \qquad B = 1 \qquad C = \lambda E = \lambda\frac{9KG}{6K+G} \qquad \lambda = 1, 10$$

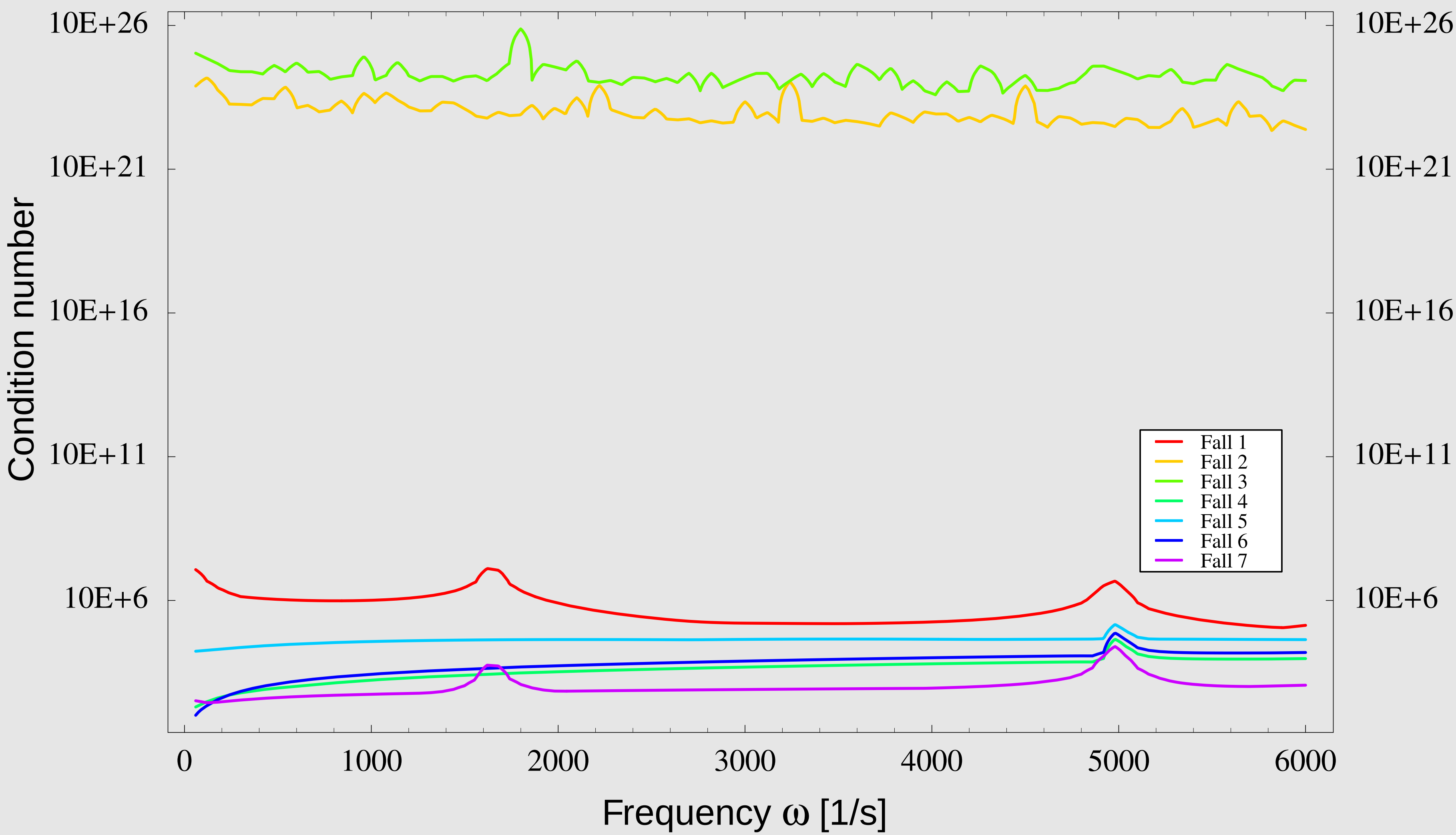
- **Fall 6** $\Rightarrow$ scaling of Young's modules to the permeability

$$A = 1 \qquad B = 1 \qquad C = \sqrt{\frac{E}{\kappa}}$$

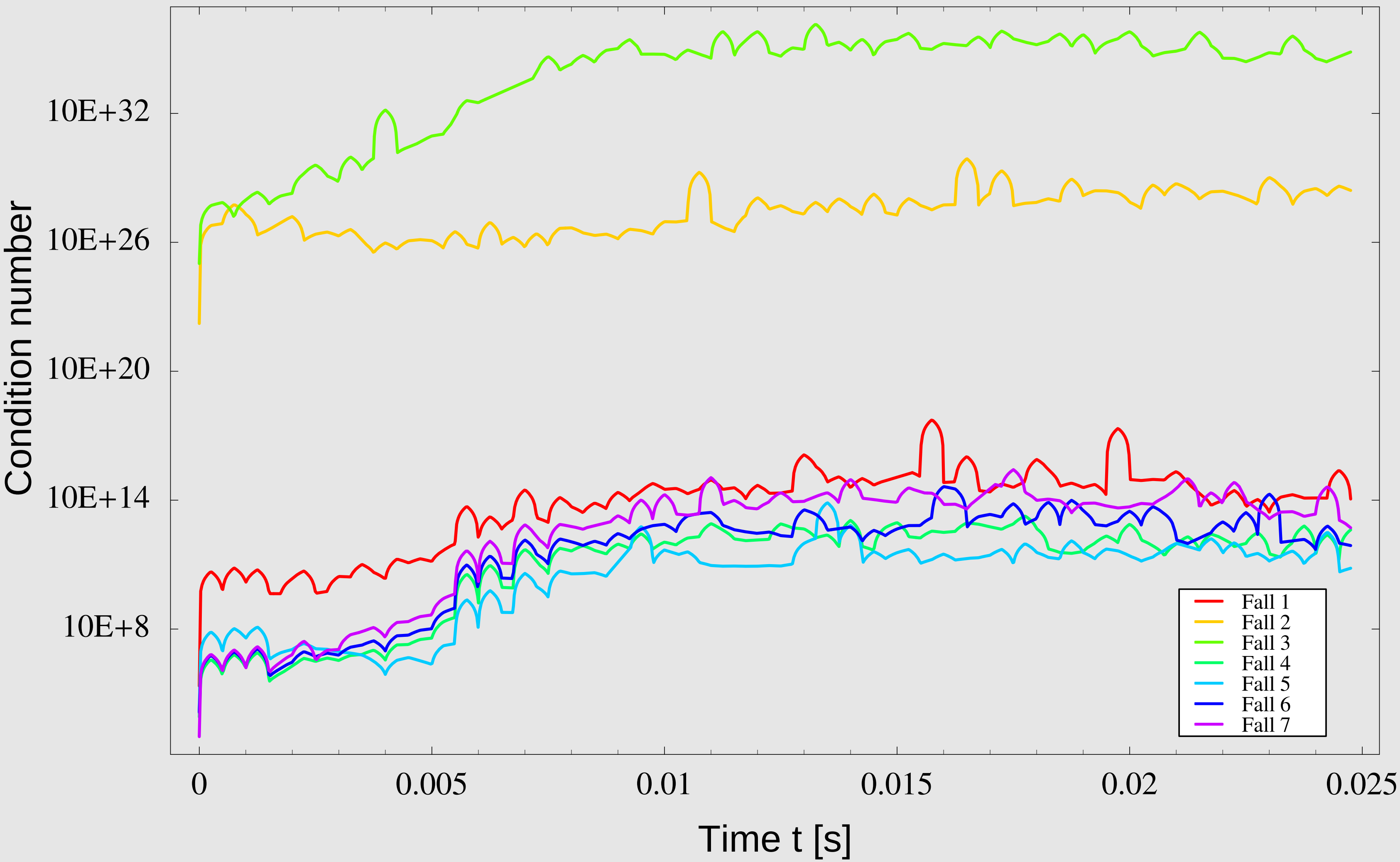
- **Fall 7** $\Rightarrow$ simple normalization

$A = r_{max}$ maximum radius $\qquad B = t_e$ maximum time $\qquad C = E$

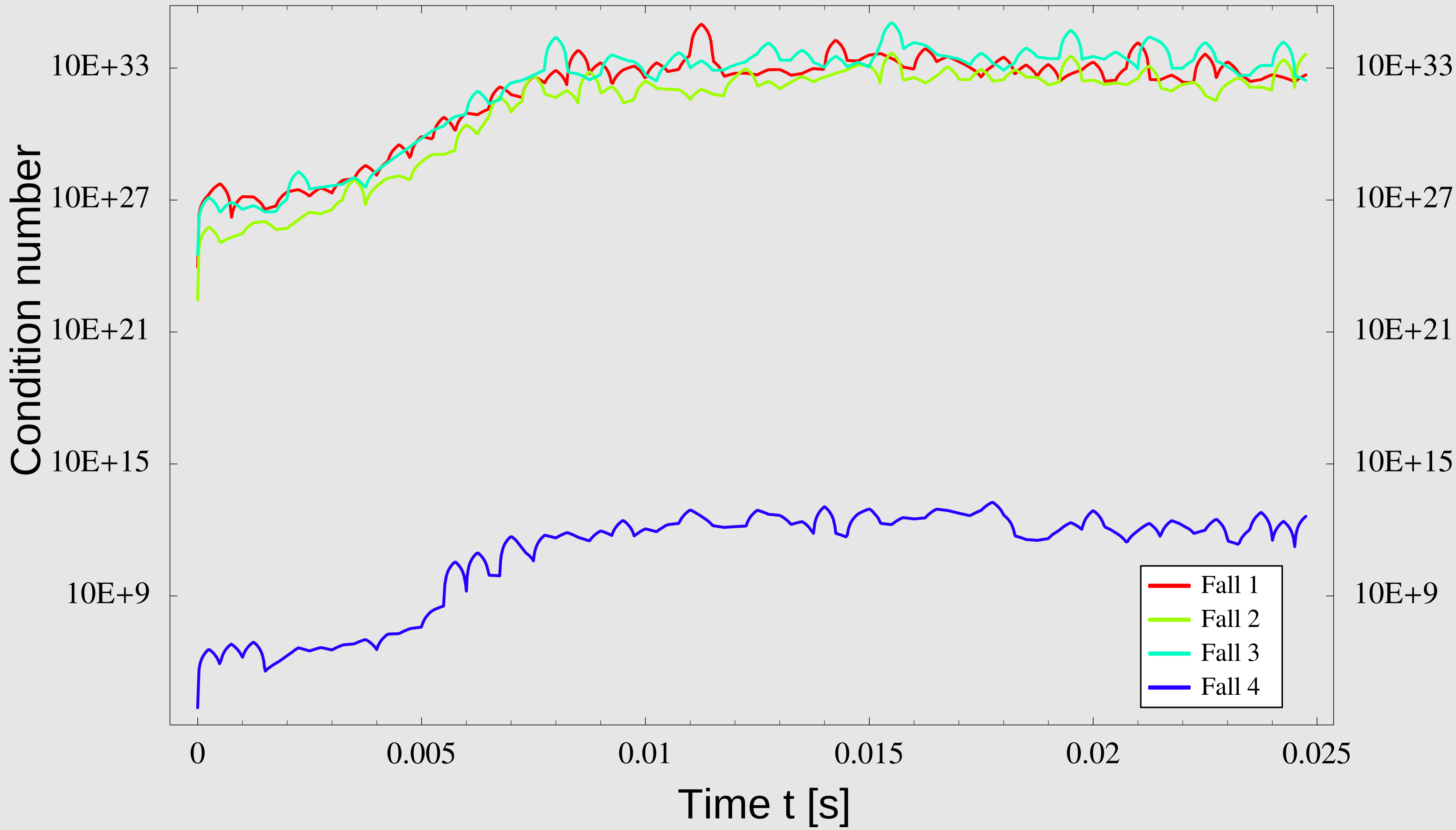
Condition number in frequency domain



Condition number in time domain



Condition number: Different parameters



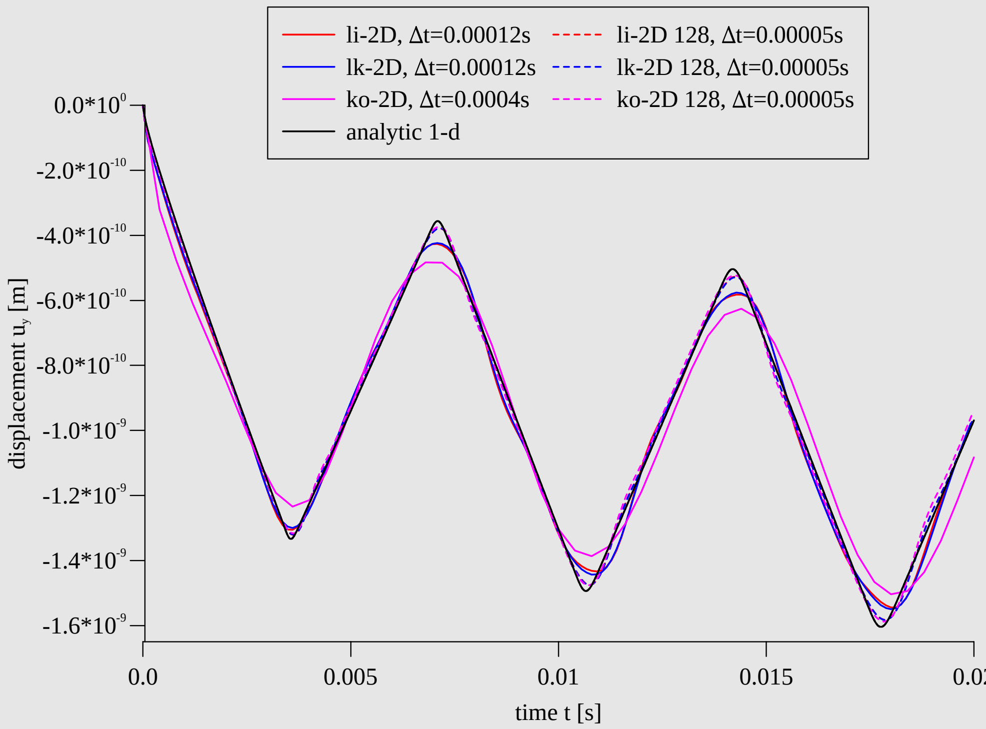
Fall 1 $\hat{=}$ $A = r_{max}$ $B = 1$ $C = 1$

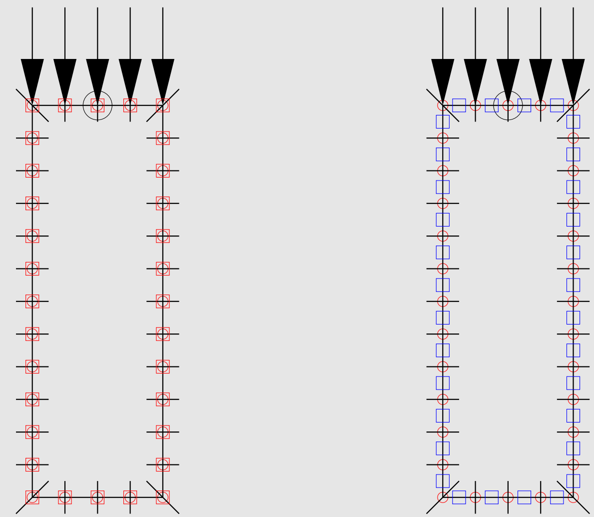
Fall 2 $\hat{=}$ $A = 1$ $B = t_{max}$ $C = 1$

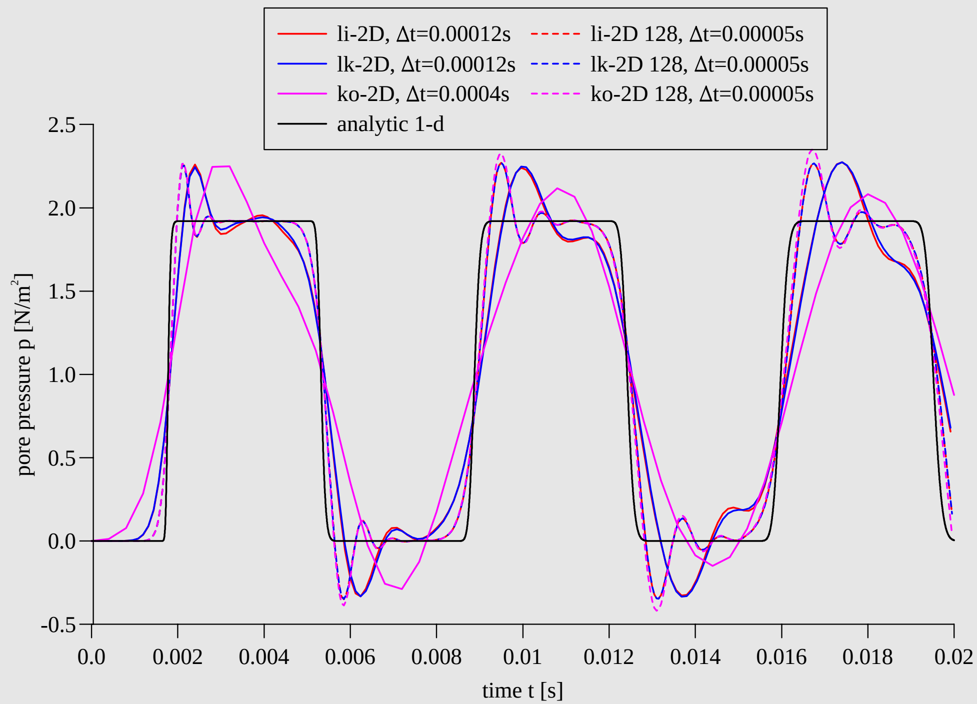
Fall 3 $\hat{=}$ $A = r_{max}$ $B = t_{max}$ $C = 1$

Fall 4 $\hat{=}$ $A = 1$ $B = 1$ $C = E$

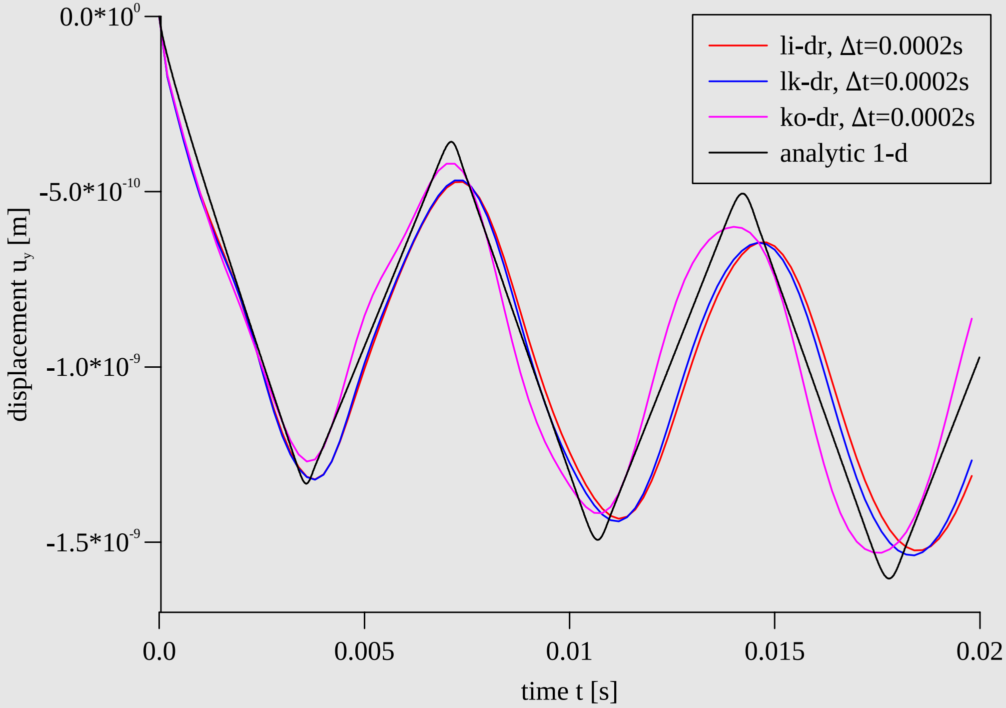
Mixed Elements in 2-d

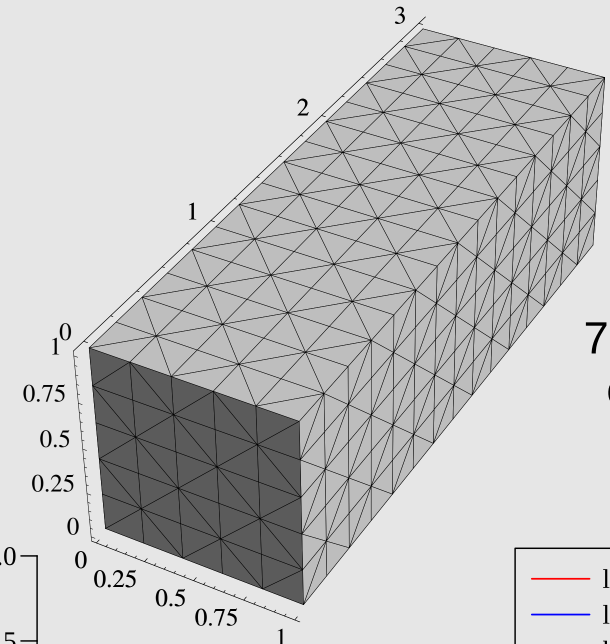




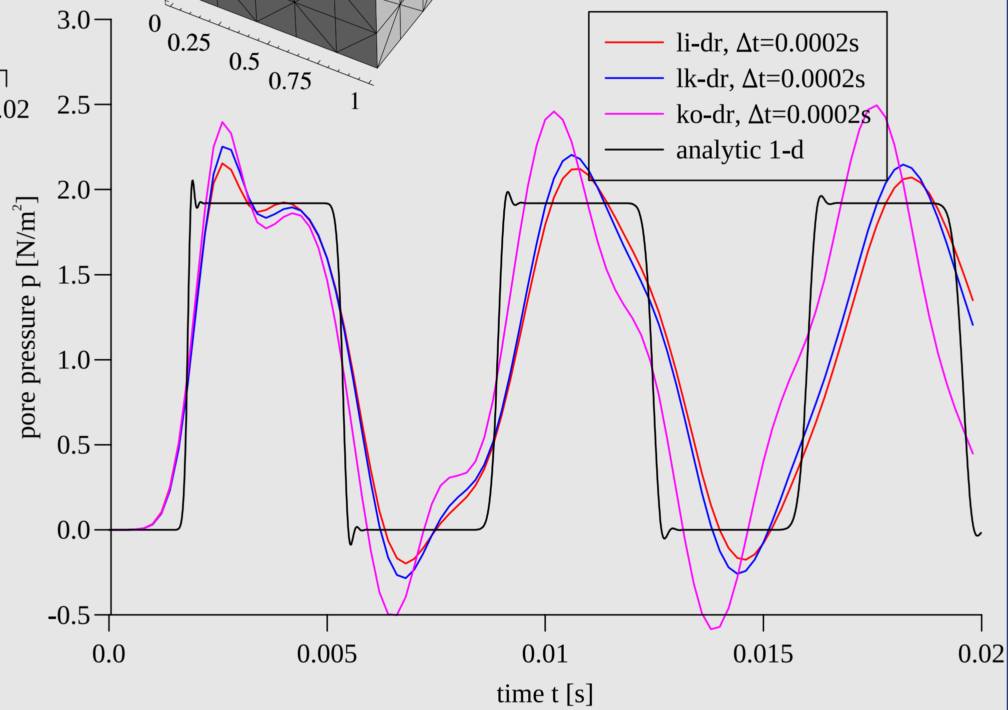


Mixed Elements in 3-d: Fixed surfaces

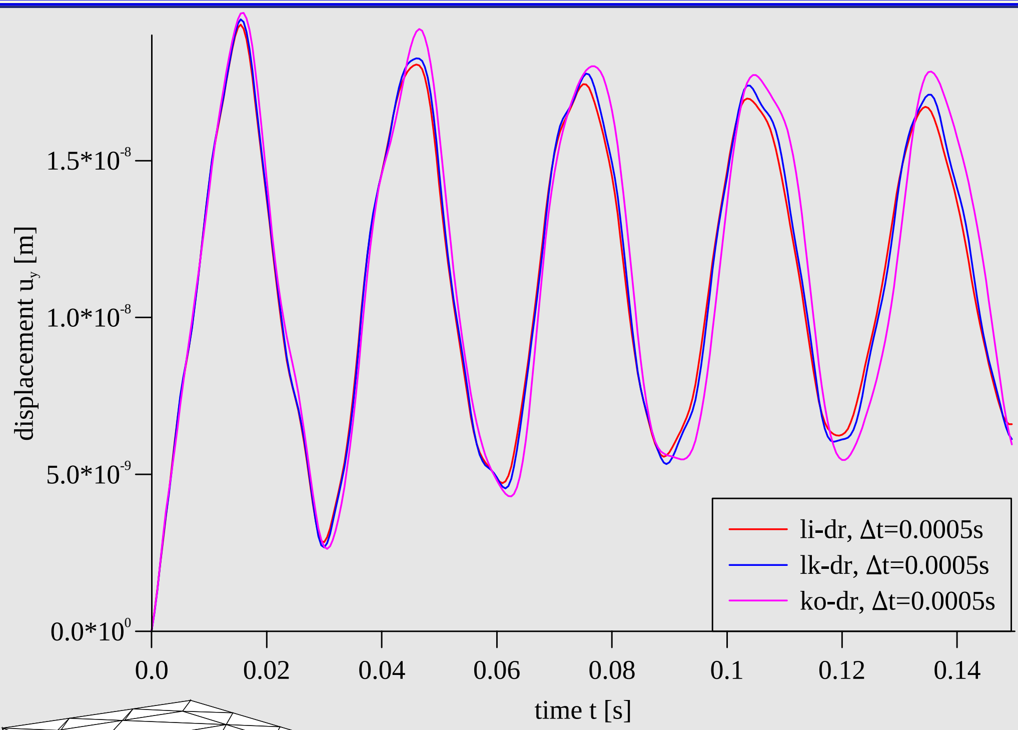


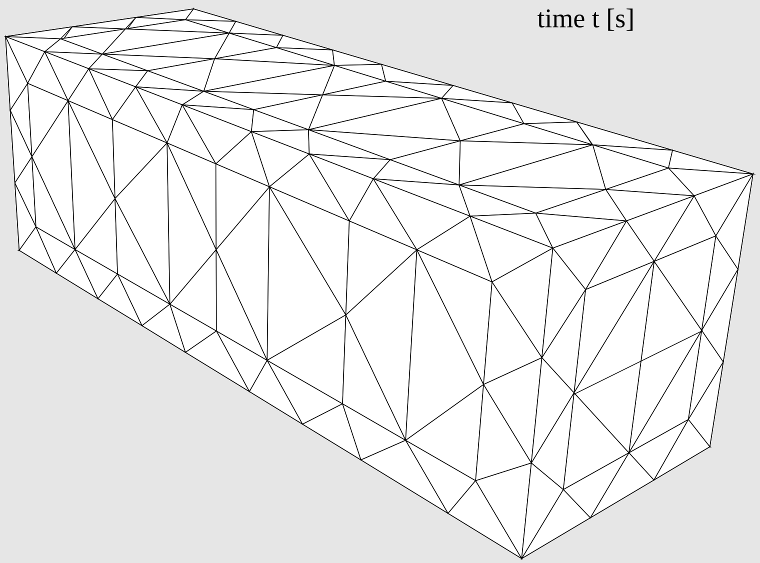


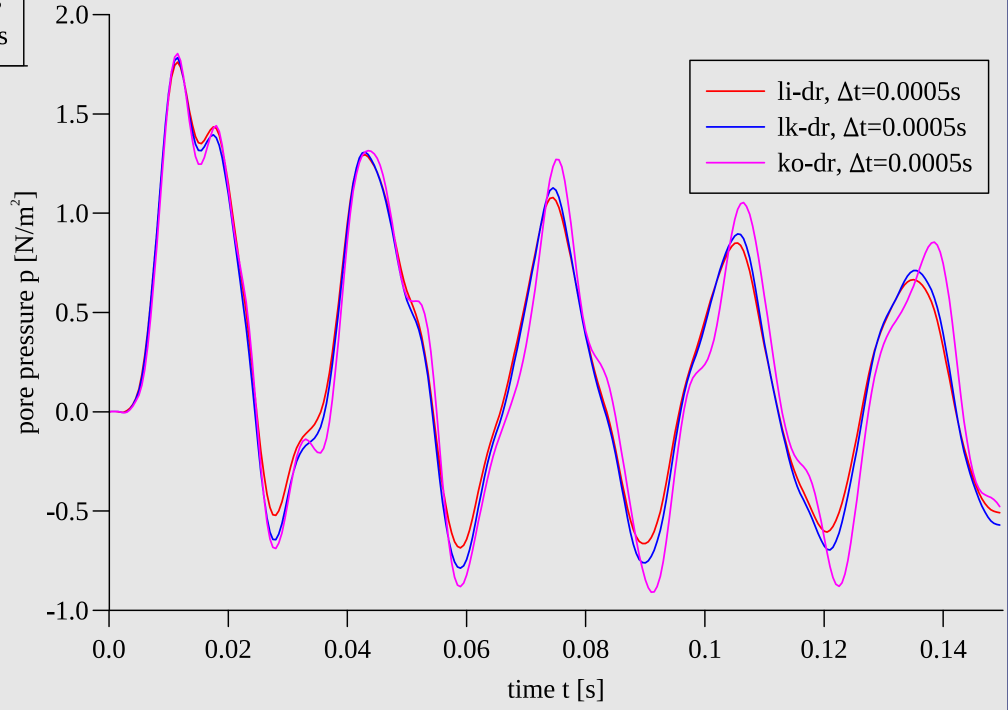
700 elements
on 392 nodes



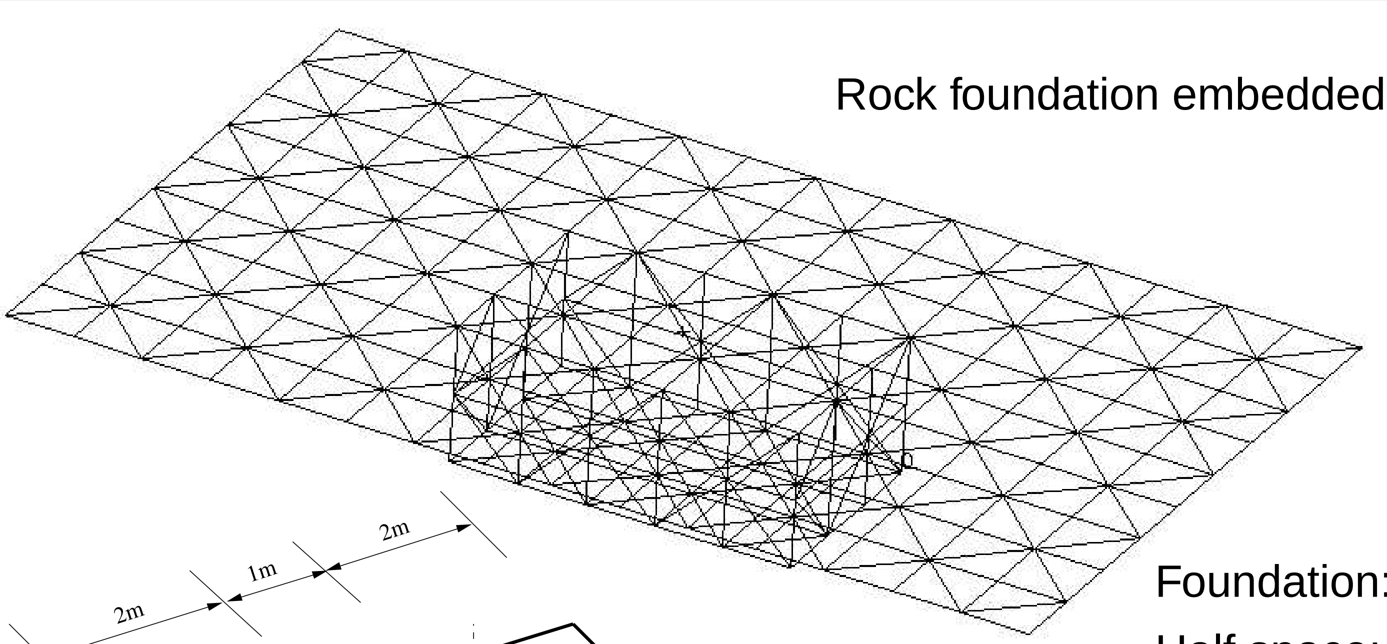
Mixed Elements in 3-d: Free surfaces



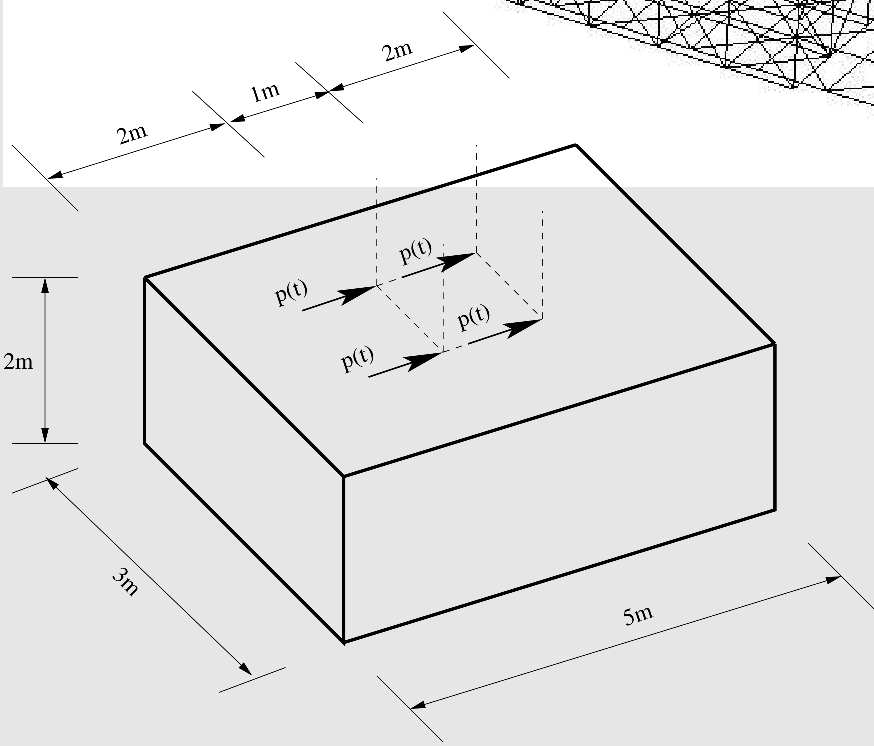




Half space: Problem description

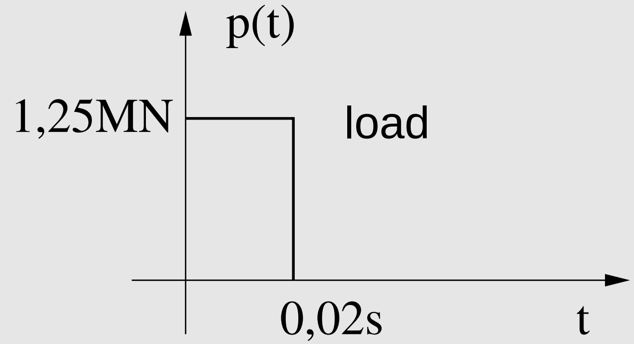


Rock foundation embedded in a soil half-space

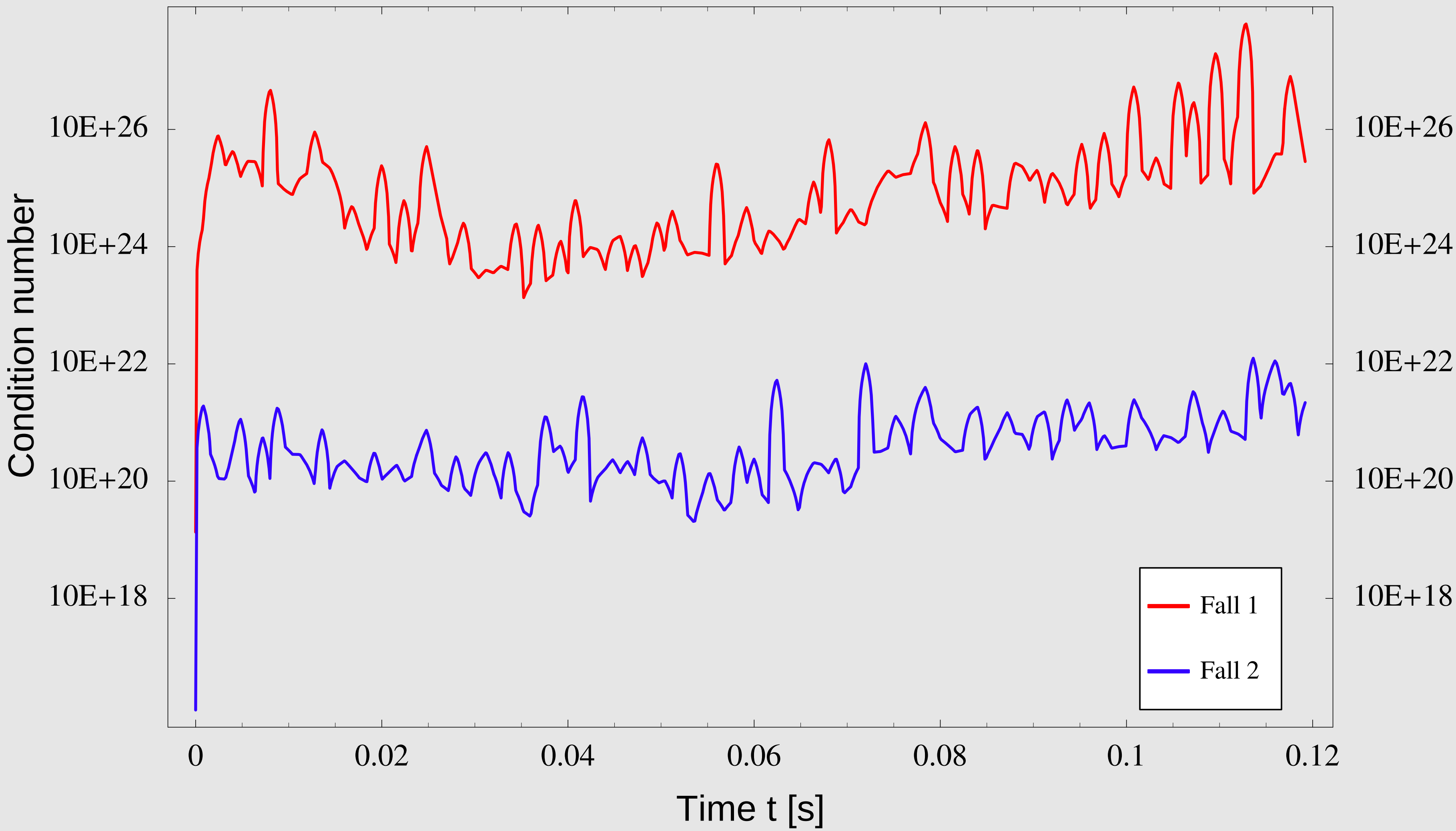


Foundation: 124 elements on 68 nodes

Half space: 334 elements on 192 nodes

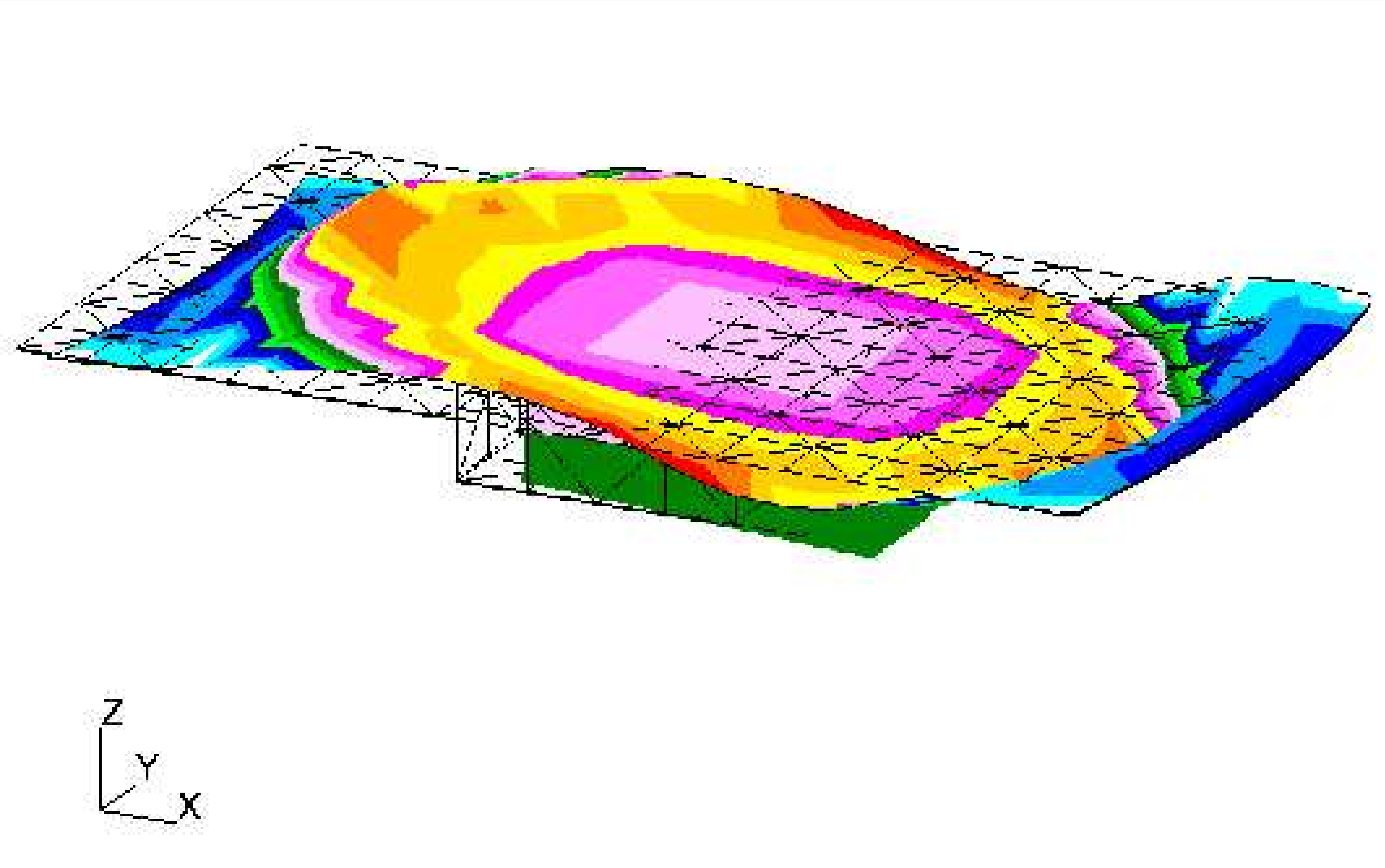


Half space: Condition number



Fall 1 $\hat{=}$ old dimensionless variables   Fall 2 $\hat{=}$ new more simpler suggestion

Half space: Numerical results



Summary

- Poroelastic BEM
  - Biot's theory
  - Based on Convolution Quadrature Method
  - Only Laplace transformed fundamental solutions are required
- Dimensionless variables
  - Normalization w.r.t. time, space, and Young's modulus
  - Largest influence due to the normalization to Young's modulus
- Mixed shape functions
  - Only sometimes improvement of stability and accuracy
  - Very CPU-time consuming
  - No justification for numerical effort

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Numerical Aspects of a Poroelastic Time Domain Boundary Element Formulation

Martin Schanz, Dobromil Pryl, Lars Kielhorn

Adaptive Fast Boundary Element Methods in Industrial Applications

Söllerhaus, 29.9.-2.10.2004

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