



On Compressed Resolvents of Schrödinger Operators with Complex Potentials

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Abstract

The compression of the resolvent of a non-self-adjoint Schrödinger operator $-\Delta + V$ onto a subdomain $\Omega \subset \mathbb{R}^n$ is expressed in a Kreĭn–Naĭmark type formula, where the Dirichlet realization on Ω , the Dirichlet-to-Neumann maps, and certain solution operators of closely related boundary value problems on Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are being used. In a more abstract operator theory framework this topic is closely connected and very much inspired by the so-called coupling method that has been developed for the self-adjoint case by Henk de Snoo and his coauthors.

Keywords Schrödinger operator · Complex potential · Compressed resolvent · Generalized resolvent · Kreĭn–Naĭmark formula · Dirichlet-to-Neumann map

1 Introduction

Let $V \in L^\infty(\mathbb{R}^n)$, $n \geq 2$, be a real or complex function and consider the Schrödinger operator

$$A = -\Delta + V, \quad \text{dom } A = H^2(\mathbb{R}^n), \quad (1.1)$$

Happy Birthday, Henk! With great pleasure I dedicate this small note to my good friend, beer buddy, colleague, and coauthor Henk de Snoo on the occasion of his 75th birthday.

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in $L^2(\mathbb{R}^n)$. Note that, in general, this operator is non-self-adjoint in $L^2(\mathbb{R}^n)$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 -smooth boundary, let P_Ω be the orthogonal projection in $L^2(\mathbb{R}^n)$ onto $L^2(\Omega)$, and denote by ι_Ω the natural embedding of $L^2(\Omega)$ into $L^2(\mathbb{R}^n)$. The aim of this note is to derive an expression for the compression of the resolvent of A onto $L^2(\Omega)$, that is,

$$P_\Omega(A - \lambda)^{-1}\iota_\Omega, \quad \lambda \in \rho(A). \quad (1.2)$$

In the special case that the potential V is real the Schrödinger operator (1.1) is self-adjoint in $L^2(\mathbb{R}^n)$ and hence, from a physical point of view, the operator A can be seen as the Hamiltonian of a closed quantum system. In this situation the compressed resolvent (1.2) can be interpreted as a family of resolvents of non-selfadjoint operators in $L^2(\Omega)$ modeling an open quantum system, and vice versa the operator A in (1.1) can be viewed as the Hamiltonian describing the natural closed extension of an open quantum system. In the self-adjoint context it also follows from abstract operator theory principles that the compressed resolvent (1.2) can be described via the Kreĭn–Naĭmark formula or can be seen as a Štraus family of extensions of a symmetric operator in $L^2(\Omega)$, see, e.g., [3, Chapter 2.7], the contributions [4, 11–14, 20, 21] by Henk de Snoo and his coauthors, and also the classical works [22–25, 31]. However, it is of particular interest to determine the various operators and mappings that appear in the classical abstract Kreĭn–Naĭmark formula for the present case of a Schrödinger operator; for real potentials V an explicit expression for the compressed resolvent (1.2) was given in [3, Theorem 8.6.3] and for Lipschitz subdomains of Riemannian manifolds in [1, Corollary 5.5].

The main purpose of this note is to show that also in the general case of a non-self-adjoint Schrödinger operator (1.1) (that is, the values of the potential $V \in L^\infty(\mathbb{R}^n)$ are not real a.e.) the compressed resolvent (1.2) is given by

$$(A_\Omega - \lambda)^{-1} - \gamma_\Omega(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\tilde{\gamma}_\Omega(\bar{\lambda})^*, \quad (1.3)$$

where $A_\Omega = -\Delta + V_\Omega$ is the Dirichlet realization in $L^2(\Omega)$ and $V_\Omega \in L^\infty(\Omega)$ is the restriction of V onto Ω . Furthermore, M and τ turn out to be the (minus) λ -dependent Dirichlet-to-Neumann maps corresponding to the differential expression $-\Delta + V$ on Ω and on $\mathbb{R}^n \setminus \bar{\Omega}$, respectively, and $\gamma_\Omega(\lambda)$ and $\tilde{\gamma}_\Omega(\bar{\lambda})$ are closely related solution operators, also called Poisson operators in the theory of elliptic PDEs. Our analysis is strongly inspired by the abstract coupling method and other boundary triple techniques, which were originally developed for the self-adjoint case in [11, 13] (see also [27–30] for dual pairs) and some more explicit preparatory results on extension theory of non-self-adjoint Schrödinger operators from [2]. We do not make an attempt here to develop a systematic study in the non-self-adjoint context, but instead we derive (1.3) in a goal-oriented way using mostly PDE-techniques such as trace maps, the second Green identity, and well posedness of boundary value problems.

We also mention that the analysis and spectral theory of non-self-adjoint Schrödinger operators has attracted a lot of attention in the recent past. In particular, eigenvalue bounds, Lieb–Thirring inequalities, and other spectral properties of

Schrödinger operators with complex potentials were derived in, e.g., [5,6,9,10,15–19,26]. The resolvent formula in Theorem 3.2 below and the compressed resolvent (1.2)–(1.3) are intimately connected with the spectral analysis of Schrödinger operators as in (1.1). Roughly speaking, the isolated eigenvalues of A coincide with the isolated singularities of the function $\lambda \mapsto (M(\lambda) + \tau(\lambda))^{-1}$ and also other spectral data of A can be characterized with the limit behaviour of this function; cf. [7,8,29].

2 Preparations

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with C^2 -smooth boundary $\Sigma = \partial\Omega$ and denote the outward normal unit vector by ν . Furthermore, let $V_\Omega \in L^\infty(\Omega)$ be a real or complex function. We consider the differential expressions

$$\mathcal{L}_\Omega = -\Delta + V_\Omega \quad \text{and} \quad \tilde{\mathcal{L}}_\Omega = -\Delta + \bar{V}_\Omega, \tag{2.1}$$

which are formally adjoint to each other. Recall that the trace mapping $C^\infty(\bar{\Omega}) \ni f \mapsto \{f|_\Sigma, \partial_\nu f|_\Sigma\}$ can be extended to a continuous surjective mapping

$$H^2(\Omega) \ni f \mapsto \{\Gamma_\Omega^D f, \Gamma_\Omega^N f\} \in H^{3/2}(\Sigma) \times H^{1/2}(\Sigma), \tag{2.2}$$

and that for $f, g \in H^2(\Omega)$ the second Green identity in this context reads as

$$(\mathcal{L}_\Omega f, g)_{L^2(\Omega)} - (f, \tilde{\mathcal{L}}_\Omega g)_{L^2(\Omega)} = (\Gamma_\Omega^D f, \Gamma_\Omega^N g)_{L^2(\Sigma)} - (\Gamma_\Omega^N f, \Gamma_\Omega^D g)_{L^2(\Sigma)}; \tag{2.3}$$

here $H^2(\Omega)$ and $H^t(\Sigma)$, $t = \frac{1}{2}, \frac{3}{2}$, denote the usual L^2 -based Sobolev spaces on Ω and Σ , respectively.

In the following the Dirichlet operators

$$\begin{aligned} A_\Omega &= -\Delta + V_\Omega, & \text{dom } A_\Omega &= \{f \in H^2(\Omega) : \Gamma_\Omega^D f = 0\}, \\ \tilde{A}_\Omega &= -\Delta + \bar{V}_\Omega, & \text{dom } \tilde{A}_\Omega &= \{f \in H^2(\Omega) : \Gamma_\Omega^D f = 0\}, \end{aligned}$$

will be useful. Both operators A_Ω and \tilde{A}_Ω are closed, densely defined in $L^2(\Omega)$, and adjoint to each other,

$$A_\Omega^* = \tilde{A}_\Omega. \tag{2.4}$$

Moreover, as Ω is a bounded domain it follows from the compactness of the embedding $H^2(\Omega) \hookrightarrow L^2(\Omega)$ that the resolvents of A_Ω and \tilde{A}_Ω are compact operators in $L^2(\Omega)$. Note also that (2.4) implies $\lambda \in \rho(A_\Omega)$ if and only if $\bar{\lambda} \in \rho(\tilde{A}_\Omega)$.

For $\lambda \in \rho(A_\Omega)$ and $\varphi \in H^{3/2}(\Sigma)$ we consider the boundary value problem

$$(\mathcal{L}_\Omega - \lambda)f = 0, \quad \Gamma_\Omega^D f = \varphi, \tag{2.5}$$

and analogously for $\mu \in \rho(\tilde{A}_\Omega)$ and $\psi \in H^{3/2}(\Sigma)$ we consider the boundary value problem

$$(\tilde{\mathcal{L}}_\Omega - \mu)g = 0, \quad \Gamma_\Omega^D g = \psi. \tag{2.6}$$

From the the assumptions $\lambda \in \rho(A_\Omega)$ and $\mu \in \rho(\tilde{A}_\Omega)$ and the fact that $\Gamma_\Omega^D : H^2(\Omega) \rightarrow H^{3/2}(\Sigma)$ is onto it follows that both boundary value problems (2.5) and (2.6) admit a unique solution $f(\varphi, \lambda) \in H^2(\Omega)$ and $g(\psi, \mu) \in H^2(\Omega)$. We shall use the notation

$$\gamma_\Omega(\lambda)\varphi := f(\varphi, \lambda) \quad \text{and} \quad \tilde{\gamma}_\Omega(\mu)\psi := g(\psi, \mu) \tag{2.7}$$

for the solution operators of (2.5) and (2.6). The next lemma is essentially a consequence of the second Green identity (2.3); we leave the proof to the reader.

Lemma 2.1 *For $\lambda \in \rho(A_\Omega)$ and $\mu \in \rho(\tilde{A}_\Omega)$ the solution operators $\gamma(\lambda)$ and $\tilde{\gamma}(\mu)$ are bounded from $L^2(\Sigma)$ to $L^2(\Omega)$ with dense domain $H^{3/2}(\Sigma)$ and range contained in $H^2(\Omega)$. Their adjoints are everywhere defined bounded operators from $L^2(\Omega)$ to $L^2(\Sigma)$ given by*

$$\gamma_\Omega(\lambda)^* = -\Gamma_\Omega^N(\tilde{A}_\Omega - \bar{\lambda})^{-1} \quad \text{and} \quad \tilde{\gamma}_\Omega(\mu)^* = -\Gamma_\Omega^N(A_\Omega - \bar{\mu})^{-1}.$$

In particular, one has $\text{ran } \gamma_\Omega(\lambda)^ = H^{1/2}(\Sigma) = \text{ran } \tilde{\gamma}_\Omega(\mu)^*$.*

A further important object in our study will be the (minus) Dirichlet-to-Neumann map $M(\cdot)$ corresponding to \mathcal{L}_Ω . Let again $\lambda \in \rho(A_\Omega)$ and $\varphi \in H^{3/2}(\Sigma)$. Then the operator $M(\lambda)$ is defined by

$$M(\lambda)\varphi = -\Gamma_\Omega^N \gamma_\Omega(\lambda)\varphi = -\Gamma_\Omega^N f(\varphi, \lambda),$$

where $\gamma_\Omega(\lambda)\varphi = f(\varphi, \lambda) \in H^2(\Omega)$ is the unique solution of the boundary value problem (2.5). The Dirichlet-to-Neumann map is an operator mapping $H^{3/2}(\Sigma)$ into $H^{1/2}(\Sigma)$, but can also be viewed as a densely defined unbounded (nonclosed) operator in $L^2(\Sigma)$.

Besides the domain Ω and the operators introduced above we shall also make use of their counterparts acting on the unbounded (exterior) domain $\Omega' := \mathbb{R}^n \setminus \bar{\Omega}$ with C^2 -smooth boundary $\Sigma = \partial\Omega'$. The operators will be denoted in the same way, except that we shall use the subindex Ω' instead of Ω , e.g., $A_{\Omega'}$ stands for the Dirichlet realization of $\mathcal{L}_{\Omega'} = -\Delta + V_{\Omega'}$ in $L^2(\Omega')$. For the (minus) Dirichlet-to-Neumann map we will use the symbol $\tau(\cdot)$; note that the values $\tau(\lambda)$ are well defined for all $\lambda \in \rho(A_{\Omega'})$. The above statements all remain valid on the unbounded domain, with the only exception that the embedding $H^2(\Omega') \hookrightarrow L^2(\Omega')$ is not compact and the resolvents of $A_{\Omega'}$ and $\tilde{A}_{\Omega'}$ are not compact in $L^2(\Omega')$.

In our considerations we shall sometimes make use of a vector notation $(f, f')^\top : \mathbb{R}^n \rightarrow \mathbb{C}$ for functions $f : \Omega \rightarrow \mathbb{C}$ and $f' : \Omega \rightarrow \mathbb{C}$. The following simple observation will be useful in the proof of Theorem 3.2 in the next section.

Lemma 2.2 *Let $f \in H^2(\Omega)$ and $f' \in H^2(\Omega')$. Then*

$$\begin{pmatrix} f \\ f' \end{pmatrix} \in H^2(\mathbb{R}^n) \quad \text{if and only if} \quad \Gamma_\Omega^D f = \Gamma_{\Omega'}^D f' \quad \text{and} \quad \Gamma_\Omega^N f = -\Gamma_{\Omega'}^N f'. \tag{2.8}$$

Proof The implication (\Rightarrow) is clear from the definition of the trace maps and the implication (\Leftarrow) can be viewed as a consequence of the self-adjointness of the Laplacian $H = -\Delta$ defined on $\text{dom } H = H^2(\mathbb{R}^n)$ and the second Green identity. In fact, let $\tilde{g} \in \text{dom } H$ and assume that $f \in H^2(\Omega)$ and $f' \in H^2(\Omega')$ satisfy $\Gamma_{\Omega}^D f = \Gamma_{\Omega'}^D f'$ and $\Gamma_{\Omega}^N f = -\Gamma_{\Omega'}^N f'$. For $\tilde{g} = (g, g')^{\top} \in H^2(\mathbb{R}^n)$ we also have $\Gamma_{\Omega}^D g = \Gamma_{\Omega'}^D g'$ and $\Gamma_{\Omega}^N g = -\Gamma_{\Omega'}^N g'$, and hence for $\tilde{f} = (f, f')^{\top} \in H^2(\Omega) \times H^2(\Omega')$ it follows from (2.3) that

$$\begin{aligned} & (H\tilde{g}, \tilde{f})_{L^2(\mathbb{R}^n)} - (\tilde{g}, -\Delta\tilde{f})_{L^2(\mathbb{R}^n)} \\ &= (-\Delta g, f)_{L^2(\Omega)} - (g, -\Delta f)_{L^2(\Omega)} + (-\Delta g', f')_{L^2(\Omega')} - (g', -\Delta f')_{L^2(\Omega')} \\ &= (\Gamma_{\Omega}^D g, \Gamma_{\Omega}^N f + \Gamma_{\Omega'}^N f')_{L^2(\Sigma)} - (\Gamma_{\Omega}^N g, \Gamma_{\Omega}^D f - \Gamma_{\Omega'}^D f')_{L^2(\Sigma)} \\ &= 0. \end{aligned}$$

Therefore, $\tilde{f} = (f, f')^{\top} \in \text{dom } H^* = \text{dom } H = H^2(\mathbb{R}^n)$. □

3 A Formula for the Resolvent of the Operator A

In this section we obtain a Kreĭn type formula for the resolvent of the non-self-adjoint Schrödinger operator in (1.1), and as an immediate consequence we conclude the form (1.2) of the compressed resolvent. The construction is based on the abstract coupling method developed in [11,13], but is made more explicit here in the context of differential operators.

In the following let A be the non-self-adjoint Schrödinger operator in (1.1), let A_{Ω} and $A_{\Omega'}$ be the Dirichlet realizations of $\mathcal{L} = -\Delta + V$ in $L^2(\Omega)$ and $L^2(\Omega')$, respectively, and denote by $M(\cdot)$ and $\tau(\cdot)$ the (minus) Dirichlet-to-Neumann maps on Ω and Ω' . The Dirichlet realizations of $\tilde{\mathcal{L}} = -\Delta + \bar{V}$ in $L^2(\Omega)$ and $L^2(\Omega')$ are denoted by \tilde{A}_{Ω} and $\tilde{A}_{\Omega'}$, respectively. The next lemma is needed in the proof of our resolvent formula in Theorem 3.2 below. In the self-adjoint context this lemma was shown in [3, Lemma 8.6.1]. As the proof remains the same in the general non-self-adjoint situation we do not repeat it here.

Lemma 3.1 For $\lambda \in \rho(A_{\Omega}) \cap \rho(A_{\Omega'}) \cap \rho(A)$ the operator

$$M(\lambda) + \tau(\lambda) : H^{3/2}(\Sigma) \rightarrow H^{1/2}(\Sigma) \tag{3.1}$$

is bijective.

For our purposes it is convenient to use the notation $A_{\Omega, \Omega'} := A_{\Omega} \times A_{\Omega'}$ and we regard $A_{\Omega, \Omega'}$ as a closed operator in $L^2(\mathbb{R}^n) = L^2(\Omega) \times L^2(\Omega')$. Note that $\rho(A_{\Omega, \Omega'}) = \rho(A_{\Omega}) \cap \rho(A_{\Omega'})$ and

$$(A_{\Omega, \Omega'} - \lambda)^{-1} = \begin{pmatrix} (A_{\Omega} - \lambda)^{-1} & 0 \\ 0 & (A_{\Omega'} - \lambda)^{-1} \end{pmatrix}, \quad \lambda \in \rho(A_{\Omega}) \cap \rho(A_{\Omega'}).$$

Furthermore, we set

$$\begin{aligned} \gamma_{\Omega, \Omega'}(\lambda) &= \begin{pmatrix} \gamma_{\Omega}(\lambda) & 0 \\ 0 & \gamma_{\Omega'}(\lambda) \end{pmatrix}, \quad \lambda \in \rho(A_{\Omega}) \cap \rho(A_{\Omega'}), \\ \tilde{\gamma}_{\Omega, \Omega'}(\lambda) &= \begin{pmatrix} \tilde{\gamma}_{\Omega}(\lambda) & 0 \\ 0 & \tilde{\gamma}_{\Omega'}(\lambda) \end{pmatrix}, \quad \lambda \in \rho(\tilde{A}_{\Omega}) \cap \rho(\tilde{A}_{\Omega'}), \end{aligned} \tag{3.2}$$

and for $\lambda \in \rho(A_{\Omega}) \cap \rho(A_{\Omega'}) \cap \rho(A)$ we define

$$\Theta(\lambda) := \begin{pmatrix} (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \\ (M(\lambda) + \tau(\lambda))^{-1} & (M(\lambda) + \tau(\lambda))^{-1} \end{pmatrix}. \tag{3.3}$$

The next theorem is the main result of this note. We express the resolvent of the Schrödinger operator A in terms of the resolvent of the orthogonal sum $A_{\Omega, \Omega'}$ of the Dirichlet realizations and a perturbation term, which contains the Dirichlet-to-Neumann maps $M(\cdot)$ and $\tau(\cdot)$, the solution operators $\gamma_{\Omega}(\cdot)$ and $\gamma_{\Omega'}(\cdot)$, and their adjoints. In particular, since the solutions operators are analytic on the resolvent sets $\rho(A_{\Omega})$ and $\rho(A_{\Omega'})$, respectively, it follows that the poles of the resolvent of A (and hence also the isolated eigenvalues) in $\rho(A_{\Omega}) \cap \rho(A_{\Omega'})$ coincide with the isolated singularities of the function $\Theta(\cdot)$ in (3.3).

Theorem 3.2 *For $\lambda \in \rho(A_{\Omega}) \cap \rho(A_{\Omega'}) \cap \rho(A)$ the resolvent formula*

$$(A - \lambda)^{-1} = (A_{\Omega, \Omega'} - \lambda)^{-1} - \gamma_{\Omega, \Omega'}(\lambda)\Theta(\lambda)\tilde{\gamma}_{\Omega, \Omega'}(\bar{\lambda})^* \tag{3.4}$$

is valid. In particular, the compression of the resolvent of A onto $L^2(\Omega)$ is given by

$$P_{\Omega}(A - \lambda)^{-1}\iota_{\Omega} = (A_{\Omega} - \lambda)^{-1} - \gamma_{\Omega}(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\tilde{\gamma}_{\Omega}(\bar{\lambda})^*. \tag{3.5}$$

Proof Let $f \in L^2(\Omega)$ and $f' \in L^2(\Omega')$, and consider

$$\begin{pmatrix} g \\ g' \end{pmatrix} = (A_{\Omega, \Omega'} - \lambda)^{-1} \begin{pmatrix} f \\ f' \end{pmatrix} - \gamma_{\Omega, \Omega'}(\lambda)\Theta(\lambda)\tilde{\gamma}_{\Omega, \Omega'}(\bar{\lambda})^* \begin{pmatrix} f \\ f' \end{pmatrix}, \tag{3.6}$$

that is,

$$\begin{aligned} g &= (A_{\Omega} - \lambda)^{-1}f - \gamma_{\Omega}(\lambda)(M(\lambda) + \tau(\lambda))^{-1}(\tilde{\gamma}_{\Omega}(\bar{\lambda})^*f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^*f'), \\ g' &= (A_{\Omega'} - \lambda)^{-1}f' - \gamma_{\Omega'}(\lambda)(M(\lambda) + \tau(\lambda))^{-1}(\tilde{\gamma}_{\Omega}(\bar{\lambda})^*f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^*f'). \end{aligned} \tag{3.7}$$

It follows from $\text{ran } \tilde{\gamma}_{\Omega}(\bar{\lambda})^* = H^{1/2}(\Sigma) = \text{ran } \tilde{\gamma}_{\Omega'}(\bar{\lambda})^*$ and Lemma 3.1 that the products on the right-hand side of (3.6)–(3.7) are well-defined. Since $\text{dom } A_{\Omega} \subset H^2(\Omega)$ and $\text{ran } \gamma_{\Omega}(\lambda) \subset H^2(\Omega)$ by Lemma 2.1 one has $g \in H^2(\Omega)$. In the same way it follows that $g' \in H^2(\Omega')$. Moreover, as A_{Ω} and $A_{\Omega'}$ are Dirichlet realizations we have

$$\Gamma_{\Omega}^D(A_{\Omega} - \lambda)^{-1}f = 0 \quad \text{and} \quad \Gamma_{\Omega'}^D(A_{\Omega'} - \lambda)^{-1}f' = 0.$$

Together with the definition of the solution operators $\gamma_\Omega(\lambda)$ and $\gamma_{\Omega'}(\lambda)$ this leads to

$$\Gamma_\Omega^D g = -(M(\lambda) + \tau(\lambda))^{-1} (\tilde{\gamma}_\Omega(\bar{\lambda})^* f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^* f') \tag{3.8}$$

and

$$\Gamma_{\Omega'}^D g' = -(M(\lambda) + \tau(\lambda))^{-1} (\tilde{\gamma}_\Omega(\bar{\lambda})^* f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^* f'). \tag{3.9}$$

Using Lemma 2.1 and the definition of the Dirichlet-to-Neumann maps $M(\cdot)$ and $\tau(\cdot)$ we find

$$\begin{aligned} \Gamma_\Omega^N g &= \Gamma_\Omega^N (A_\Omega - \lambda)^{-1} f \\ &\quad - \Gamma_\Omega^N \gamma_\Omega(\lambda) (M(\lambda) + \tau(\lambda))^{-1} (\tilde{\gamma}_\Omega(\bar{\lambda})^* f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^* f') \\ &= -\tilde{\gamma}_\Omega(\bar{\lambda})^* f + M(\lambda) (M(\lambda) + \tau(\lambda))^{-1} (\tilde{\gamma}_\Omega(\bar{\lambda})^* f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^* f') \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \Gamma_{\Omega'}^N g' &= \Gamma_{\Omega'}^N (A_{\Omega'} - \lambda)^{-1} f' \\ &\quad - \Gamma_{\Omega'}^N \gamma_{\Omega'}(\lambda) (M(\lambda) + \tau(\lambda))^{-1} (\tilde{\gamma}_\Omega(\bar{\lambda})^* f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^* f') \\ &= -\tilde{\gamma}_{\Omega'}(\bar{\lambda})^* f' + \tau(\lambda) (M(\lambda) + \tau(\lambda))^{-1} (\tilde{\gamma}_\Omega(\bar{\lambda})^* f + \tilde{\gamma}_{\Omega'}(\bar{\lambda})^* f'). \end{aligned} \tag{3.11}$$

Therefore, we have

$$\Gamma_\Omega^D g = \Gamma_{\Omega'}^D g' \quad \text{and} \quad \Gamma_\Omega^N g + \Gamma_{\Omega'}^N g' = 0,$$

and now Lemma 2.2 implies that the function in (3.6) is in $H^2(\mathbb{R}^n) = \text{dom } A$. As $(\mathcal{L}_\Omega - \lambda)\gamma_\Omega(\lambda)\varphi = 0$ and $(\mathcal{L}_{\Omega'} - \lambda)\gamma_{\Omega'}(\lambda)\psi = 0$ for all $\varphi, \psi \in H^{3/2}(\Sigma)$ it is also clear that

$$\begin{aligned} (A - \lambda) \begin{pmatrix} g \\ g' \end{pmatrix} &= (-\Delta + V - \lambda) \begin{pmatrix} g \\ g' \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_\Omega - \lambda \\ \mathcal{L}_{\Omega'} - \lambda \end{pmatrix} \begin{pmatrix} (A_\Omega - \lambda)^{-1} f \\ (A_{\Omega'} - \lambda)^{-1} f' \end{pmatrix} \\ &\quad - \begin{pmatrix} \mathcal{L}_\Omega - \lambda \\ \mathcal{L}_{\Omega'} - \lambda \end{pmatrix} \begin{pmatrix} \gamma_\Omega(\lambda) & 0 \\ 0 & \gamma_{\Omega'}(\lambda) \end{pmatrix} \Theta(\lambda) \tilde{\gamma}_{\Omega, \Omega'}(\bar{\lambda})^* \begin{pmatrix} f \\ f' \end{pmatrix} \\ &= \begin{pmatrix} f \\ f' \end{pmatrix}, \end{aligned}$$

which leads to (3.4). The formula (3.5) for the compressed resolvent is an immediate consequence of (3.4). □

From Lemma 2.1 it is clear that $\text{ran } \tilde{\gamma}_{\Omega, \Omega'}(\bar{\lambda})^* = H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$, $\lambda \in \rho(A_\Omega) \cap \rho(A_{\Omega'})$. Furthermore, since the embedding $H^{1/2}(\Sigma) \hookrightarrow L^2(\Sigma)$ is compact one concludes that $\tilde{\gamma}_{\Omega, \Omega'}(\bar{\lambda})^*$ is a compact operator from $L^2(\mathbb{R}^n)$ to $L^2(\Sigma) \times L^2(\Sigma)$. Since $(M(\lambda) + \tau(\lambda))^{-1}$, $\lambda \in \rho(A_\Omega) \cap \rho(A_{\Omega'}) \cap \rho(A)$, can be extended to a bounded

operator on $L^2(\Sigma)$ it follows that $\Theta(\lambda)$ in (3.3) admits a bounded extension to $L^2(\Sigma) \times L^2(\Sigma)$. Thus, the perturbation term in the resolvent formula in Theorem 3.2 is compact, and hence the resolvent difference

$$(A - \lambda)^{-1} - (A_{\Omega, \Omega'} - \lambda)^{-1}, \quad \lambda \in \rho(A_{\Omega}) \cap \rho(A_{\Omega'}) \cap \rho(A),$$

is compact in $L^2(\mathbb{R}^n)$ (and, in fact, it can be shown that the resolvent difference belongs to some Schatten–von Neumann ideal). Therefore, well known perturbation results imply that the essential spectra of A and $A_{\Omega, \Omega'}$ coincide, and as the resolvent of A_{Ω} is compact we conclude

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_{\Omega, \Omega'}) = \sigma_{\text{ess}}(A_{\Omega}) \cup \sigma_{\text{ess}}(A_{\Omega'}) = \sigma_{\text{ess}}(A_{\Omega'});$$

here the essential spectrum of a non-self-adjoint operator is defined as the complement of the isolated eigenvalues with finite algebraic multiplicities in the spectrum.

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