Integral Representation and Time Evolution of Superoscillations

Jussi Behrndt, Fabrizio Colombo, and Peter Schlosser

Contents

Abstract

Superoscillations are a phenomenon where a band-limited signal locally oscillates faster than its highest frequency component, achieved by combining low-frequency waves to mimic high-frequency oscillations. They are important because they enable super-resolution imaging and play a significant role in weak values in quantum mechanics. The aim of this chapter is to summarize some of the main facts of the mathematics of the theory of superoscillations. Superoscillations will be defined as sequences of holomorphic functions that admit integral representations with respect to complex Borel measures and

J. Behrndt (✉) · P. Schlosser
Institute of Applied Mathematics, Graz University of Technology, Graz, Austria
e-mail: behrndt@tugraz.at; pschlosser@math.tugraz.at

F. Colombo
Politecnico di Milano, Dipartimento di Matematica, Milan, Italy
e-mail: fabrizio.colombo@polimi.it

D. Alpay et al. (eds.), *Operator Theory*,
https://doi.org/10.1007/978-3-0348-0692-3_100-1

converge to a plane wave in the space $\mathcal{A}_1(\mathbb{C})$ of exponentially bounded entire functions. Additionally, it aims to explain the main results obtained in the study of the evolution of superoscillations via the Schrödinger equation.

Keywords

Superoscillations · Integral representation · Green's function · Schrödinger equation

AMS Classification

81Q05 · 35A20 · 35A08

Introduction

This chapter explores the integral representation of superoscillations, aiming to establish a comprehensive definition based on existing examples. Superoscillations can be represented through sequences of functions involving sums or integrals of exponentials with bounded frequencies, which converge to a higher frequency exponential in the space of entire functions with exponential growth. In the following, this space is denoted by $\mathcal{A}_1(\mathbb{C})$. The prototypical superoscillatory function, which appears in the theory of weak values in quantum mechanics developed by Aharonov and collaborators, is of the form

$$F_n(z) = \sum_{j=0}^{n} C_j(n) e^{ik_j(n)z}, \qquad z \in \mathbb{C},\ n \in \mathbb{N}, \tag{1}$$

where for some fixed $a > 1$ the coefficients are given by

$$C_j(n) = \binom{n}{j}\left(\frac{1+a}{2}\right)^{n-j}\left(\frac{1-a}{2}\right)^{j} \qquad \text{and} \qquad k_j(n) = 1 - \frac{2j}{n}. \tag{2}$$

Note that every F_n is a linear combination of plane waves with frequencies $k_j(n) \in [-1, 1]$. The superoscillatory behavior now comes from the convergence

$$\lim_{n\to\infty} F_n(z) = e^{iaz}, \tag{3}$$

to a plane wave with frequency $a > 1$, in the natural convergence space $\mathcal{A}_1(\mathbb{C})$. This function has inspired various new constructions of superoscillatory functions, such as Berry's integral formulas (6), which highlight how suitable sequences of functions built from bounded frequency components can exhibit superoscillatory behavior.

A key concept in finding a unified description of the various constructions of superoscillations in the existing literature is representing them as integrals over plane waves with respect to measures supported in $[-1, 1]$,

$$F_n(z) = \int_{-1}^{1} e^{ikz} d\mu_n(k), \qquad z \in \mathbb{C}. \tag{4}$$

Furthermore, in this note, we discuss some methods for constructing new superoscillatory functions from existing ones, as solutions of a generalized free Schrödinger equation, emphasizing the importance of integral representations and holomorphic properties.

We then discuss the evolution of superoscillations under the time-dependent Schrödinger equation with potentials whose corresponding Green's function satisfies certain regularity and growth assumptions. In particular, potentials such as the centrifugal, the Pöschl-Teller, and the Dirac δ potential are covered by this approach.

The plan of the chapter. In Section Integral Representation of Superoscillations, more precisely in (5), (6), (7), and (8), we collect some functions with superoscillatory behavior that often appear in the literature. In (9), we then find a unified representation for all of these functions of the form (4). In Section Construction of Superoscillating Functions, we introduce two methods, (12) and (13), for constructing new superoscillating functions from existing ones. Section Time Evolution of Superoscillations explores how a superoscillating initial condition evolves over time, under the Schrödinger equation. A general approach is presented in Theorem 2, which relies only on qualitative assumptions about the corresponding Green's function of a potential, without requiring its explicit form. Finally, in Section Examples of Green's Functions, we illustrate these results using explicit Green's functions for specific potentials.

Integral Representation of Superoscillations

The aim of this section is to gather existing ideas and examples of superoscillating functions and to synthesize them into a generally valid definition of superoscillations.

First, superoscillations of the form (1) do not only appear with the frequencies $k_j(n)$ in (2), but it was shown in [10] that for any choice (pairwise disjoint if n is fixed) of frequencies $k_j(n) \in [-1, 1]$ and any $a > 1$, one can construct coefficients $C_j(n) \in \mathbb{C}$, such that

$$\lim_{n\to\infty} \sum_{j=0}^{n} C_j(n) e^{ik_j(n)z} = e^{iaz}. \tag{5}$$

Moreover, the convergence in (3) and also in (5) is understood in the space $\mathcal{A}_1(\mathbb{C})$ defined below, see [21, Lemma 2.4] for a proof.

Definition 1 The space of entire functions with exponential growth is defined as

$$\mathcal{A}_1(\mathbb{C}) := \Big\{ F : \mathbb{C} \to \mathbb{C} \text{ entire} \,\Big|\, \exists A, B \geq 0 \text{ such that } |F(z)| \leq Ae^{B|z|} \text{ for all } z \in \mathbb{C} \Big\}.$$

For any $F, (F_n)_n \in \mathcal{A}_1(\mathbb{C})$, we say that $F_n \to F$ converges in $\mathcal{A}_1(\mathbb{C})$, if there exists some $B \geq 0$, such that

$$\lim_{n\to\infty} \sup_{z\in\mathbb{C}} |F_n(z) - F(z)|e^{-B|z|} = 0.$$

A certain type of superoscillations, not a discrete sum of exponentials as in (1) or (5), was introduced by Berry in [15], namely

$$F_\delta(z) = \frac{1}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(u-ia)^2}{2\delta^2}} e^{ik(u)z} du, \qquad z \in \mathbb{C},\ \delta > 0, \tag{6}$$

with some fixed coefficient $a > 0$ and a frequency function k with values $k(u) \in [-1, 1]$ for every $u \in \mathbb{R}$. Instead of a convergence result of the form (3), it is shown in Berry's work that the local wavenumber exceeds the range $[-1, 1]$, while the intrinsic frequencies k take only values in $[-1, 1]$. Later in [14, Theorem 2.1], it was proved that the family of functions (6) indeed converges as

$$\lim_{\delta\to0^+} F_\delta(z) = e^{ik(ia)z} \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

If the function k is now chosen in a way that $k(ia) > 1$, this sequence can indeed be called superoscillating, since every F_δ is built out of plane waves $e^{ik(u)z}$, weighted by the factor $e^{-\frac{(u-ia)^2}{2\delta^2}}$, with frequencies $k(u) \in [-1, 1]$, but the limit, that is the exponential function $e^{ik(ia)z}$, has a frequency $k(ia) > 1$. Examples of frequency functions k with such a property are given by

$$k_1(u) = \frac{1}{1+\frac{u^2}{2}}, \qquad k_2(u) = \frac{1}{\cosh(u)}, \qquad k_3(u) = e^{-\frac{u^2}{2}}, \qquad k_4(u) = \cos(u).$$

In the paper [16], Berry provided another function

$$F_\delta(z) = \frac{2}{\delta} e^{-\frac{1}{\delta}} \operatorname{sinc}\left(\sqrt{z^2 - \frac{2iaz}{\delta} - \frac{1}{\delta^2}}\right), \qquad z \in \mathbb{C},\ \delta > 0, \tag{7}$$

with a superoscillating behavior in the sense that the local wavenumber exceeds the support $[-1, 1]$ of its Fourier transform

$$\widehat{F}_\delta(k) = \begin{cases} \frac{\sqrt{2\pi}}{\delta} e^{\frac{ak-1}{\delta}} J_0\Big(\frac{\sqrt{(a^2-1)(1-k^2)}}{\delta}\Big), & |k| \leq 1, \\ 0, & |k| > 1. \end{cases}$$

This was later investigated in [14, Theorem 3.2], where it was shown that also the functions (7) converge as

$$\lim_{\delta \to 0^+} F_\delta(z) = e^{iaz} \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

Another approach of constructing superoscillating functions was taken by Ferreira, Kempf, and Lee in a series of papers [25, 26, 27, 28, 29, 30]. There, functions of the form

$$F_n(x) = \sum_{j=0}^{n} C_j(n) \operatorname{sinc}(x - x_j), \quad x \in \mathbb{R},$$

were considered, where the coefficients $C_j(n)$ are chosen such that $F_n(x_j) = a_j$, $j \in \{0, \dots, n\}$, admits the prescribed values $(a_j)_j$ at the prescribed points $(x_j)_j$. Choosing now, for example, $x_j = j\delta$ for some arbitrary small $\delta > 0$ and $a_j = (-1)^j$ with alternating sign, the function F admits an arbitrary large number of oscillations in an arbitrary small interval, while its Fourier transform

$$\widehat{F}_n(k) = \begin{cases} \sum_{j=0}^{n} C_j(n) e^{ikx_j}, & |k| \leq 1, \\ 0, & |k| > 1, \end{cases}$$

is always supported in the bounded interval $[-1, 1]$.

In [14, Section 4], this idea was revisited, and the $\mathcal{A}_1$ convergence was shown, demonstrating that there exist coefficients $C_j(n) \in \mathbb{C}$ such that the functions

$$F_n(z) := \sum_{j=0}^{n} C_j(n) f^{(j)}(z), \qquad z \in \mathbb{C}, \tag{8}$$

converge as

$$\lim_{n \to \infty} F_n(z) = e^{iaz}, \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

Here $f^{(j)}(z)$ denotes the j-th derivative of a function f whose Fourier transform $\widehat{f}$ is supported in the interval $[-1, 1]$ and is of the Szegö class, i.e., for some $\alpha < \beta \in [-1, 1]$, it satisfies

$$\int_{\alpha}^{\beta} \frac{\ln(\widehat{f}(k))}{\sqrt{(\beta-k)(k-\alpha)}} dk > -\infty.$$

Also note that the values $C_j(n)$ are given as the solution of the linear system of equations

$$\begin{pmatrix} \int_{-1}^{1} k^0 \widehat{f}(k)dk & \int_{-1}^{1} k^1 \widehat{f}(k)dk & \dots & \int_{-1}^{1} k^n \widehat{f}(k)dk \\ \int_{-1}^{1} k^1 \widehat{f}(k)dk & \int_{-1}^{1} k^2 \widehat{f}(k)dk & \dots & \int_{-1}^{1} k^{n+1} \widehat{f}(k)dk \\ \vdots & \vdots & \ddots & \vdots \\ \int_{-1}^{1} k^n \widehat{f}(k)dk & \int_{-1}^{1} k^{n+1} \widehat{f}(k)dk & \dots & \int_{-1}^{1} k^{2n} \widehat{f}(k)dk \end{pmatrix} \begin{pmatrix} C_0(n) \\ C_1(n) \\ \vdots \\ C_n(n) \end{pmatrix} = \begin{pmatrix} a^0 \\ a^1 \\ \vdots \\ a^n \end{pmatrix}.$$

Keeping in mind that all the above examples of superoscillating functions converge in the same function space $\mathcal{A}_1(\mathbb{C})$, we are in the position to give a general definition of superoscillations that contains the previous examples as particular cases. The two main features of this general definition have to be: First, the sequence $F_n(z)$ of entire functions has to converge to a plane wave e^{iaz} with frequency $a > 1$, in the space $\mathcal{A}_1(\mathbb{C})$. Secondly, to cover discrete sums like (1), integrals of the form (6) as well as functions with a compactly supported Fourier transform, we require the functions F_n to be represented as an integral over some complex Borel measure. To also make sure the F_n's only consist of small frequencies, these Borel measures have to be supported in the interval $[-1, 1]$.

Definition 2 A sequence of functions of the form

$$F_n(z) = \int_{-1}^{1} e^{ikz} d\mu_n(k), \qquad z \in \mathbb{C}, \tag{9}$$

with complex Borel measures μ_n on $[-1, 1]$ is called superoscillating, if there exists some $a > 1$, such that

$$\lim_{n\to\infty} F_n(z) = e^{iaz} \quad \text{in } \mathcal{A}_1(\mathbb{C}). \tag{10}$$

Remark 1 Note that the Borel measures μ_n in (9) of the functions (1), (6), (7), and (8) are for every Borel set $B \subseteq [-1, 1]$ given by

$$\mu_n(B) = \sum_{j=0, k_j(n)\in B}^{n} C_j(n),$$

$$\mu_\delta(B) = \frac{1}{\delta\sqrt{2\pi}} \int_{k^{-1}(B)} e^{-\frac{(u-ia)^2}{2\delta^2}} du,$$

$$\mu_\delta(B) = \frac{1}{\delta}\int_B e^{\frac{ak-1}{\delta}} J_0\Big(\frac{\sqrt{(a^2-1)(1-k^2)}}{\delta}\Big)dk, \quad \text{and}$$

$$\mu_n(B) = \frac{1}{\sqrt{2\pi}}\sum_{j=0}^{n} C_j(n)\int_B k^j \widehat{f}(k)dk.$$

The derivation of these measures can be found in the respective sections of [14].

Construction of Superoscillating Functions

While Section Integral Representation of Superoscillations gathers various examples of superoscillations from the existing literature and puts them into a general framework, this section presents strategies to construct new families of superoscillating functions from given ones, using the theory of infinite-order differential operators acting on entire functions. Historically, this was the first approach used to generate new examples of superoscillatory functions from the prototypical one (1), demonstrating that this class was sufficiently large to become an independent mathematical theory. The precise mathematical concept is taken from [14, Section 5]. See also [2,3,7] for earlier considerations in this direction. The principle idea of this method is based on the generalized force free Schrödinger equation

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,z) &= -H\Big(-i\frac{\partial}{\partial z}\Big)\Psi(t,z), \qquad && t,z\in\mathbb{C}, \\ \Psi(0,z) &= F(z), && z\in\mathbb{C}, \end{aligned} \tag{11}$$

where for some entire function $H(z)=\sum_{l=0}^\infty h_l z^l$ the term $H(-i\frac{\partial}{\partial z})$ is understood as the corresponding infinite-order differential expression

$$H\Big(-i\frac{\partial}{\partial z}\Big) := \sum_{l=0}^\infty h_l(-i)^l\frac{d^l}{dz^l}.$$

If we now consider superoscillating initial conditions $\Psi_n(0,z)=F_n(z)$ of the form (9), then the Eq. (11) admits the explicit solution

$$\Psi_n(t,z) = \int_{-1}^1 e^{iH(k)t}e^{ikz}d\mu_n(k), \qquad t,z\in\mathbb{C};$$

cf. [14, Lemma 5.1]. One may now expect that the solutions $\Psi_n(t,\,\cdot\,)$ are superoscillatory also for times $t>0$. This method is specified in Theorem 1 i). Moreover, it turns out that in the point $z=0$, the functions $\Psi_n(\,\cdot\,,0)$ are superoscillatory in the time variable t as well, see Theorem 1 ii).

Theorem 1 *Consider a superoscillating sequence of functions* $(F_n)_n$ *according to Definition 2, with limit* $\lim_{n\to\infty} F_n(z) = e^{iaz}$*. Then one can construct the following two new families of superoscillating functions:*

i) *For every entire function* $H : \mathbb{C} \to \mathbb{C}$*, the functions*

$$F_n^{(1)}(z) := e^{-H(a)} \int_{-1}^{1} e^{H(k)} e^{ikz} d\mu_n(k), \qquad z \in \mathbb{C}, \tag{12}$$

are superoscillating with $\lim_{n\to\infty} F_n^{(1)}(z) = e^{iaz}$ *convergent in* $\mathcal{A}_1(\mathbb{C})$.

ii) *For every entire function* $H : \mathbb{C} \to \mathbb{C}$ *which satisfies* $H(x) \in [-1, 1]$ *for every* $x \in [-1, 1]$*, and* $H(a) \in \mathbb{R} \setminus [-1, 1]$*, the sequence*

$$F_n^{(2)}(z) := \int_{-1}^{1} e^{iH(k)z} d\mu_n(k), \qquad z \in \mathbb{C}, \tag{13}$$

is superoscillating with $\lim_{n\to\infty} F_n^{(2)}(z) = e^{iH(a)z}$ *convergent in* $\mathcal{A}_1(\mathbb{C})$.

Time Evolution of Superoscillations

Unlike von Neumann measurements of quantum systems, which cause the wave function to collapse and yield for bounded states an outcome corresponding to a function in L^2, weak measurements do not induce collapse. Instead, they extract only part of the information about the quantum state, generally resulting in a function that is not square integrable. Nevertheless, these measurements are still considered outcomes of a quantum measurement process. Aharonov's question was to study the evolution of these quantum states when evolved via the Schrödinger, the Dirac, or the Klein-Gordon equation. In quantum mechanics, he posed the central question: Does a superoscillatory function survive this time evolution under a given potential? In the Schrödinger setting, this leads to the study of the following evolution problem with superoscillatory initial datum:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= \Big(-\frac{\partial^2}{\partial x^2} + V(t,x)\Big)\Psi(t,x), \qquad && t > 0,\ x \in \mathbb{R}, \\ \Psi(0,x) &= F(x), && x \in \mathbb{R}. \end{aligned} \tag{14}$$

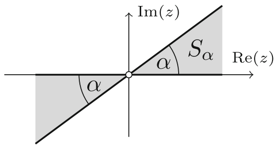
The investigation of this problem started with considering particular potentials where the solution of (14) was known explicitly. As to name the most important, there were the free particle ($V = 0$) considered in [1], the quantum harmonic oscillator in [12, 17, 18, 21], the electric field in [4], the centrifugal potential in [6, 12, 20, 21], the step potential in [8], and distributional potentials such as δ and δ' in [5, 9, 13]. Also examples in more than one dimension are treated, for example, in [34], the two-dimensional half-plane barrier is considered, and in [19] the three-dimensional uniform magnetic field.

Stepping away from considering explicit examples of potentials, it was in the paper [11], with a later refinement of the results and adding boundary condition in [33], where a general approach was provided, which avoids the explicit form of the solution, and only relies on qualitative properties of the corresponding Green's function. Note that the *Green's function* is a function $G : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, which only depends on the potential V, but not on the initial condition F, such that the solution Ψ of (14) admits the representation

$$\Psi(t, x) = \int_{\mathbb{R}} G(t, x, y)F(y)dy. \tag{15}$$

Since Green's functions as well as superoscillating functions are typically bounded but in general do not admit any decay at $\pm\infty$, the integral (15) does a priori not exist as an absolute convergent Lebesgue integral. For this reason, we need the following Fresnel integral technique to give meaning to the integral.

The idea of the *Fresnel integral technique* is to rotate the integration path $(-\infty, \infty)$ by some angle $\alpha > 0$ into the complex plane. Proceeding in this way, a fast oscillating factor of the form e^{iay^2}, which has to be present in the integrand, turns into a decaying Gaussian $e^{ia(ye^{i\alpha})^2}$ and takes care of the absolute integrability of the function under the integral. A necessary condition is of course that the involved integrand is holomorphic and satisfies some growth assumptions. More precisely, for every continuous function $f : S_\alpha \to \mathbb{C}$, which is holomorphic in the interior of the double sector



$$S_\alpha = \Big\{ z \in \mathbb{C} \setminus \{0\} \;\Big|\; \operatorname{Arg}(z) \in [0, \alpha] \cup [\pi, \pi + \alpha] \Big\},$$

and which is exponentially bounded as

$$|f(z)| \leq Ae^{B|z|}, \qquad z \in S_\alpha,$$

for some $A, B \geq 0$, we define the integral

$$\int_{\mathbb{R}} e^{iay^2} f(y)dy := e^{i\alpha} \int_{\mathbb{R}} e^{ia(ye^{i\alpha})^2} f(ye^{i\alpha})dy.$$

Since the decomposition (19) (see just below) and all the examples in Section Examples of Green's Functions demonstrate that, in various applications, the integrand in (15) indeed has this form, we define the wave function rigorously as

$$\Psi(t,x) := e^{i\alpha} \int_{\mathbb{R}} G(t,x,ye^{i\alpha}) F(ye^{i\alpha}) dy. \tag{16}$$

Let us now state the precise necessary assumptions on the Green's function, such that the wave function (16) is indeed a solution of the time-dependent Schrödinger equation (14), and more importantly that a sequence of superoscillating initial conditions $(F_n)_n$ leads to a sequence of solutions $(\Psi_n)_n$, which admits a similar superoscillatory frequency shift at all times $t > 0$. For the precise statement, see Theorem 2.

Assumption 1 *Let the Green's function* $G : (0,\infty)\times\mathbb{R}\times\mathbb{R} \to \mathbb{C}$ *extend in the third variable to the double sector* S_α*, for some* $\alpha > 0$*, such that for every fixed* $t > 0$*,* $x \in \mathbb{R}$ *the mapping* $G(t,x,\cdot)$ *is continuous on* S_α *and holomorphic on* $\operatorname{int}(S_\alpha)$*. Moreover, it will be assumed that G satisfies the following properties* i)–iii)*:*

i) *The function G is a solution of the time-dependent Schrödinger equation*

$$i\frac{\partial}{\partial t} G(t,x,y) = \Big(-\frac{\partial^2}{\partial x^2} + V(t,x) \Big) G(t,x,y), \qquad t > 0,\ x \in \mathbb{R},\ y \in S_\alpha, \tag{17}$$

with $V : (0,\infty)\times\mathbb{R} \to \mathbb{C}$ *the considered potential.*

ii) *For every* $x \in \mathbb{R}$*, there exists some* $x_0 > |x|$*, such that*

$$\lim_{t\to 0^+} \int_{-x_0}^{x_0} G(t,x,y)\varphi(y) dy = \varphi(x), \qquad \varphi \in \mathcal{C}^\infty([-x_0,x_0]). \tag{18}$$

iii) *There exists a function* $a : (0,\infty) \to \mathbb{R}$ *with* $a(t) > 0$ *and* $\lim_{t\to 0^+} a(t) = \infty$*, such that the function G decomposes into*

$$G(t,x,y) = e^{ia(t)(x-y)^2} \widetilde{G}(t,x,y), \qquad t > 0,\ x \in \mathbb{R},\ y \in S_\alpha. \tag{19}$$

Moreover, the function $\widetilde{G}$*, together with its spatial and time derivatives, has to satisfy certain exponential bounds of the form*

$$|\widetilde{G}(t,x,y)| \le A(t,x) e^{B_0(t,x)|y|}. \tag{20}$$

See [33, Assumption 3.1] *for more details.*

At this point, we want to mention that Pozzi and Wick proved a similar time persistence result of superoscillations, as the upcoming Theorem 2, but under a different list of assumptions. Indeed, in [32, Theorem 1.1], they impose assumptions on the moments of the Green's function and conclude a similar convergence as in (21).

Theorem 2 *Let $G : (0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be as in Assumption 1. Then, for every $F \in \mathcal{A}_1(\mathbb{C})$, the wave function $\Psi(t, x)$ in (16) is a solution of the time-dependent Schrödinger equation (14). Moreover, if the initial conditions $(F_n)_n \in \mathcal{A}_1(\mathbb{C})$ are superoscillating in the sense of Definition 2, then also the corresponding solutions converge as*

$$\lim_{n\to\infty} \Psi(t, x; F_n) = \Psi(t, x; e^{ia\,\cdot}), \tag{21}$$

for fixed $t > 0$ and uniformly on compact subsets of $\mathbb{R}$.

Note that the convergence (21) does not say that the sequence of functions $\Psi(t, x; F_n)$ is superoscillating. First, the convergence is only uniformly on compact subsets and not in $\mathcal{A}_1(\mathbb{C})$, as required in (10). Secondly, the limit function $\Psi(t, x; e^{ia\,\cdot})$ will in general not be an exponential. So precisely speaking, the solution $\Psi(t, x; F_n)$ of the Schrödinger equation with superoscillatory initial datum does formally not belong to the class of superoscillatory functions anymore, although a certain frequency shift still appears.

It was first realized in [21] that the precise definition of superoscillations is for general potentials too narrow to persist in time. Hence, superoscillations were generalized to supershifts as a consequence. This notion basically ignores the oscillatory behavior of the involved exponential functions e^{ikz} and e^{iaz} and only considers the shift in the parameter k, i.e., that the sequence

$$\Psi(t, x; F_n) = \int_{-1}^{1} \Psi(t, x; e^{ik\,\cdot})\, d\mu_n(k)$$

only involves values $k \in [-1, 1]$, while the limit function $\Psi(t, x; e^{ia\,\cdot})$ is related to the frequency $a > 1$. This shift can also be interpreted as calculating the value of a function in the point $a > 1$ by only knowing the values in the bounded interval $[-1, 1]$, which is a characteristic property of holomorphic functions. It is a recent open problem to understand the precise connection between the supershift property and holomorphicity, and we will not summarize the present state of the art in this chapter. Let us only refer the interested reader to the publications [22, 23].

Examples of Green's Functions

In this final section, we will present some examples of explicit potentials, where the corresponding Green's function is known explicitly and satisfies Assumption 1. Hence, the time persistence of superoscillations result in Theorem 2 holds true. In particular, we will consider the repulsive centrifugal potential $V(t, x) = \lambda x^{-2}$ in Subsection Centrifugal Potential, the Pöschl-Teller potential $V(t, x) = -l(l + 1)\cosh^{-2}(x)$ in Subsection Pöschl-Teller Potential, as well as the distributional δ potential $V(t, x) = \alpha\delta(x)$ in Section δ Potential. More potentials, such as the electric field, the harmonic oscillator, the attractive centrifugal potential, the δ' potential, and general point interactions, can be found in [11, 33].

Centrifugal Potential

In this subsection, we consider the strongly singular centrifugal potential

$$V(t,x) = \frac{\lambda}{x^2}$$

of strength $\lambda > 0$. The restriction to $\lambda > 0$ is only for convenience and to avoid distinguishing different cases and technical difficulties. Indeed, also the case $\lambda < 0$ can be treated similarly, see [33, Section 5.1], and it is also possible to consider the combined centrifugal and harmonic oscillator potential $V(t,x) = \frac{\lambda}{x^2} + \omega x^2$, see [12]. The Green's function of the centrifugal potential $V(t,x) = \frac{\lambda}{x^2}$, $\lambda > 0$, can be explicitly determined by

$$G(t,x,y) = \frac{\Theta(xy)\sqrt{xy}}{2ti^{\nu+1}} e^{-\frac{x^2+y^2}{4it}} J_\nu\Big(\frac{xy}{2t}\Big), \qquad t > 0,\ x, y \in \mathbb{R} \setminus \{0\}, \tag{22}$$

where J_ν is the Bessel function with the value $\nu = \sqrt{\lambda + \frac{1}{4}}$. Note that due to the step function

$$\Theta(\xi) := \begin{cases} 0, & \xi < 0, \\ 1, & \xi > 0, \end{cases}$$

this Green's function vanishes for $xy < 0$, which can be justified by the singularity of the $\frac{1}{x^2}$ potential, which is too singular at $x = 0$ to allow any information exchange between the two halflines, see, for example, [24] for more explanation.

We will now check that the Green's function (22) satisfies Assumption 1. First, it is clear that it holomorphically extends to $\{z \in \mathbb{C} \mid \Re(z) \neq 0\}$, so in particular to any double sector S_α. Next we check the three points i)–iii) of Assumption 1:

i) Using the Bessel differential equation

$$\xi^2 J_\nu''(\xi) + \xi J_\nu'(\xi) + (\xi^2 - \nu^2) J_\nu(\xi) = 0,$$

it is straightforward to check that $G(t,x,y)$ satisfies the Schrödinger equation (17).

ii) The verification of the initial condition

$$\lim_{t \to 0^+} \int_{-x_0}^{x_0} G(t,x,y)\varphi(y)dy = \varphi(x),$$

for every $\varphi \in \mathcal{C}^\infty([-x_0, x_0])$, is somewhat tricky, so we do not want to provide a proof here, but rather refer to [33, Proposition 5.2].

iii) The important decomposition (19) of G into an exponential factor and a remainder $\widetilde{G}$ is given by

$$G(t, x, y) = e^{-\frac{(x-y)^2}{4it}} \widetilde{G}(t, x, y),$$

where

$$\widetilde{G}(t, x, y) = \frac{\Theta(xy)\sqrt{xy}}{2ti^{\nu+1}} e^{-\frac{xy}{2it}} J_\nu\Big(\frac{xy}{2t}\Big).$$

The bound (20) of the function $\widetilde{G}$ is now a consequence of the asymptotics

$$J_\nu(\xi) \sim \frac{\sqrt{2}}{\sqrt{\pi\xi}} \cos\Big(\xi - \frac{(2\nu+1)\pi}{4}\Big) \qquad \text{as } \xi \to \infty.$$

More details of this proof can be found in [33, Proposition 5.2].

Pöschl-Teller Potential

As the second example, we present the Pöschl-Teller potential

$$V(t, x) = -\frac{l(l+1)}{\cosh^2(x)}, \qquad l \in \mathbb{N}.$$

The Green's function of this potential is given by the expression

$$G(t, x, y) = \bigg(\frac{1}{2\sqrt{i\pi t}} + \sum_{m=1}^{l} \frac{m(l-m)!}{2(l+m)!} \times P_l^m(\tanh(x)) P_l^m(\tanh(y)) R\big(m^2 t, m(y-x)\big)\bigg) e^{-\frac{(y-x)^2}{4it}}, \tag{23}$$

where we used the associated Legendre polynomials P_l^m, as well as the function

$$R(t, \xi) := e^{\xi} \Lambda\Big(\frac{\xi}{2\sqrt{it}} - \sqrt{it}\Big) - e^{-\xi} \Lambda\Big(\frac{\xi}{2\sqrt{it}} + \sqrt{it}\Big), \qquad t > 0,\ \xi \in \mathbb{C}, \tag{24}$$

where Λ is the modification of the error function given by

$$\Lambda(z) := e^{z^2}(1 - \operatorname{erf}(z)). \tag{25}$$

This Green's function can, for example, be found in [31, Section 6.6.3]. Also for this Green's function, we check Assumption 1. First, it is clear that in the y-variable, the function $G(t, x, \cdot)$ holomorphically extends to $\mathbb{C} \setminus i\pi(\mathbb{Z} + \frac{1}{2})$, excluding the singularities $y = i\pi(n + \frac{1}{2})$, $n \in \mathbb{Z}$, of $\tanh(y)$. Next we verify the points i)–iii) of Assumption 1:

i) It can easily be checked that the function $R(t, \xi)$ admits the derivatives

$$\frac{\partial}{\partial \xi} R(t, \xi) = \frac{\xi}{2it} R(t, \xi) - \frac{2}{\sqrt{i\pi t}} \sinh(\xi),$$
$$\frac{\partial}{\partial t} R(t, \xi) = i\Big(1 + \frac{\xi^2}{4t^2}\Big) R(t, \xi) + \frac{\xi \sinh(\xi)}{t\sqrt{i\pi t}} + \frac{2i \cosh(\xi)}{\sqrt{i\pi t}}.$$

Using also the Legendre differential equation

$$(1 - \xi^2)(P_l^m(\xi))'' - 2\xi(P_l^m(\xi))' + \Big(l(l+1) - \frac{m^2}{1 - \xi^2}\Big) P_l^m(\xi) = 0,$$

it follows after some lengthy and tedious computations that the Green's function G in (23) is a solution of the Schrödinger equation (17).

ii) For the initial condition (18), we note that from the asymptotics

$$R(t, \xi) = \frac{4 \sinh(\xi) \sqrt{it}}{\xi \sqrt{\pi}} \qquad \text{as } t \to 0^+,$$

it follows that

$$\lim_{t \to 0^+} \sqrt{4t}\, G(t, x, x) = \frac{1}{\sqrt{i\pi}}. \tag{26}$$

It is then the result [33, Corollary 3.4] that shows that this limit is sufficient to satisfy the initial condition (18).

iii) The decomposition (19) of the Green's function is satisfied with the function

$$\widetilde{G}(t, x, y) = \frac{1}{2\sqrt{i\pi t}} + \sum_{m=1}^{l} \frac{m(l-m)!}{2(l+m)!} \times P_l^m(\tanh(x)) P_l^m(\tanh(y)) R\big(m^2 t, m(y - x)\big).$$

Since the double sector S_α has a positive distance from the singularities $y = i\pi(n + \frac{1}{2})$, $n \in \mathbb{Z}$, of $\tanh(y)$, we can estimate the associated Legendre polynomials by

$$|P_l^m(\tanh(y))| \le A_l^m e^{B_l^m |z|}, \qquad z \in S_\alpha.$$

Due to the properties [5, Lemma 2.1] of the function Λ, we can also estimate (24) by

$$|R(t,\xi)| \le 2\Lambda\Big(-\frac{\sqrt{t}}{\sqrt{2}}\Big)e^{|\Re(\xi)|}, \qquad t>0,\ \xi\in\mathbb{C}.$$

With these estimates, we then conclude the exponential bound (20) of the function $\widetilde{G}(t,x,y)$.

δ Potential

The interest in the delta potential and its variations relies on the fact that these potentials approximate short-range interactions and are often easier to handle compared to more physically realistic potentials, and at the same time serve as a good approximation of the physical model under consideration. Distributional potentials such as the δ potential $V(x)=\lambda\delta(x)$, of strength $\lambda\in\mathbb{R}$, manifest themselves by some vanishing classical potential $V(x)=0$ for every $x\in\mathbb{R}\setminus\{0\}$, but with a transmission condition at $x=0$. The particular transmission condition of the δ potential is given by the continuity of the wave function and a jump in the normal derivative

$$\begin{aligned}\Psi(t,0^+)&=\Psi(t,0^-),\\ \frac{\partial}{\partial x}\Psi(t,0^+)-\frac{\partial}{\partial x}\Psi(t,0^-)&=\lambda\Psi(t,0).\end{aligned}$$

The Green's function of the δ potential is given by

$$G(t,x,y)=\frac{1}{2\sqrt{i\pi t}}e^{-\frac{(x-y)^2}{4it}}-\frac{\lambda}{4}\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}}+\frac{\lambda\sqrt{it}}{2}\Big)e^{-\frac{(|x|+|y|)^2}{4it}},$$

where the function Λ is the same as in (25). A detailed study of this potential in the direction of time evolution of superoscillations is done in [5, 9, 13]. In this chapter, we will check that also this Green's function satisfies the points i)–iii) of Assumption 1:

i) It is an easy explicit task to verify that the Green's function satisfies the Schrödinger equation as well as the transmission condition

$$\begin{aligned}i\frac{\partial}{\partial t}G(t,x,y)&=-\frac{\partial^2}{\partial x^2}G(t,x,y), && t>0,\ x\in\mathbb{R}\setminus\{0\},\\ G(t,0^+,y)&=G(t,0^-,y), && t>0,\\ \frac{\partial}{\partial x}G(t,0^+,y)-\frac{\partial}{\partial x}G(t,0^-,y)&=\lambda G(t,0,y), && t>0.\end{aligned}$$

ii) For the initial condition (18), one can show that

$$\lim_{t\to 0^+}\sqrt{4t}\,G(t,x,x)=\frac{1}{\sqrt{i\pi}}-\lim_{t\to 0^+}\sqrt{t}\,\frac{\lambda}{2}\Lambda\Big(\frac{|x|}{\sqrt{it}}+\frac{\lambda\sqrt{it}}{2}\Big)e^{-\frac{|x|^2}{it}}=\frac{1}{\sqrt{i\pi}}.$$

Analogously to (26), it then follows from [33, Corollary 3.4] that the initial condition (18) is satisfied.

iii) The function $\widetilde{G}$ in the decomposition (19) is for the δ potential given by

$$\widetilde{G}(t,x,y)=\frac{1}{2\sqrt{i\pi t}}-\frac{\lambda}{4}\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}}+\frac{\lambda\sqrt{it}}{2}\Big)e^{-\frac{xy+|xy|}{2it}}.$$

Using the estimate

$$\Big|\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}}+\frac{\lambda\sqrt{it}}{2}\Big)\Big|\leq\Lambda\Big(\frac{\lambda\sqrt{t}}{2\sqrt{2}}\Big),$$

the exponential bound (20) of $\widetilde{G}(t,x,y)$ is clearly satisfied.

Acknowledgments The research of Jussi Behrndt is supported by the Austrian Science Fund (FWF) 10.55776/P 33568-N. Fabrizio Colombo is supported by MUR grant Dipartimento di Eccellenza 2023-2027. Peter Schlosser was funded by the Austrian Science Fund (FWF) under Grant No. J 4685-N and by the European Union–NextGenerationEU.

Competing Interest Declaration The author(s) has no competing interests to declare that are relevant to the content of this manuscript.

References

1. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: On the Cauchy problem for the Schrödinger equation with superoscillatory initial data. J. Math. Pures Appl. **99**, 165–173 (2013)
2. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Superoscillating sequences as solutions of generalized Schrödinger equations. J. Math. Pur. Appl. **103**, 522–534 (2015)
3. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Superoscillating sequences in several variables. J. Four. Anal. Appl. **22**, 751–767 (2016)
4. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: Evolution of superoscillatory initial data in several variables in uniform electric field. J. Phys. A Math. Theor. **50**, 185201 (2017)
5. Aharonov, Y., Behrndt, J., Colombo, F., Schlosser, P.: Schrödinger evolution of superoscillations with δ- and δ'-potentials. Quant. Stud. Math. Found. **7**, 293–305 (2020)
6. Aharonov, Y., Colombo, F., Struppa, D.C., Tollaksen, J.: Schrödinger evolution of superoscillations under different potentials. Quant. Stud. Math. Found. **5**, 485–504 (2018)
7. Aharonov, Y., Sabadini, I., Tollaksen, J., Yger, A.: Classes of superoscillating functions. Quant. Stud. Math. Found. **5**, 439-454 (2018)
8. Aharonov, Y., Colombo, F., Sabadini, I., Struppa, D.C., Tollaksen, J.: How superoscillating tunneling waves can overcome the step potential. Ann. Phys. **414**, 168088 (2020)

9. Aharonov, Y., Behrndt, J., Colombo, F., Schlosser, P.: Green's function for the Schrödinger equation with a generalized point interaction and stability of superoscillations. J. Diff. Eq. **277**, 153–190 (2021)
10. Aharonov, Y., Colombo, F., Sabadini, I., Shushi, T., Struppa, D.C., Tollaksen, J.: A new method to generate superoscillating functions and supershifts. Proc. Roy. Soc. A **477**, 20210020 (2021)
11. Aharonov, Y., Behrndt, J., Colombo, F., Schlosser, P.: A unified approach to Schrödinger evolution of superoscillations and supershifts. J. Evol. Eq. **22**, Art. No. 22 (2022)
12. Alpay, D., Colombo, F., Sabadini, I., Struppa, D.C.: Aharonov-Berry superoscillations in the radial harmonic oscillator potential. Quant. Stud. Math. Found. **7**, 269–283 (2020)
13. Behrndt, J., Colombo, F., Schlosser, P.: Evolution of Aharonov-Berry superoscillations in Dirac δ-potential. Quant. Stud. Math. Found. **6**, 279–293 (2019)
14. Behrndt, J., Colombo, F., Schlosser, P., Struppa, D.C.: Integral representation of superoscillations via Borel measures and their convergence. Trans. Amer. Math. Soc. **376**, 6315–6340 (2023)
15. Berry, M.V.: Faster than Fourier. In: Quantum coherence and reality; in celebration of the 60th Birthday of Yakir Aharonov, Anandan, J.S., Safko, J.L. (eds.), pp. 55–65. World Scientific, Singapore (1994)
16. Berry, M.V.: Representing superoscillations and narrow Gaussians with elementary functions. Milan J. Math. **84**, 217–230 (2016)
17. Buniy, R., Colombo, F., Sabadini, I., Struppa, D.C.: Quantum harmonic oscillator with superoscillating initial datum. J. Math. Phys. **55**, 113511 (2014)
18. Bussell, M., Strange, P.: Evolution of superoscillations in the quantum mechanical harmonic oscillator. Eur. J. Phys. **36**, 065028 (2015)
19. Colombo, F., Gantner, J., Struppa, D.C.: Evolution of superoscillations for Schrödinger equation in a uniform magnetic field. J. Math. Phys. **58**, 092103 (2017)
20. Colombo, F., Gantner, J., Struppa, D.C.: Evolution by Schrödinger equation of Aharonov-Berry superoscillations in centrifugal potential. Proc. R. Soc. A **475**, 20180390 (2019)
21. Colombo, F., Sabadini, I., Struppa, D.C., Yger, A.: Superoscillating sequences and supershifts for families of generalized functions. Compl. Anal. Op. Th. **16**, Art. No. 34 (2022)
22. Colombo, F., Sabadini, I., Struppa, D.C., Yger, A.: Analyticity and supershift with irregular sampling. Compl. Anal. Synerg. **10**, Paper No. 10 (2024)
23. Colombo, F., Sabadini, I., Struppa, D.C., Yger, A.: Analyticity and supershift with regular sampling. Compl. Anal. Synerg. **11**, Paper No. 6 (2025)
24. Essin, A.M., Griffiths, D.J.: Quantum mechanics of the $\frac{1}{x^2}$ potential. Am. J. Phys. **74**, 109–117 (2006)
25. Ferreira, P., Kempf, A.: The energy expense of superoscillations. In: 2002 11th European Signal Processing Conference, pp. 1–4 (2002)
26. Ferreira, P., Kempf, A.: Superoscillations: faster than the Nyquist rate. IEEE Trans. Sign. Proc. **54**, 3732–3740 (2006)
27. Ferreira, P., Lee, D.: Superoscillations of prescribed amplitude and derivative. IEEE Trans. Sign. Proc. Lett. **62**, 3371–3378 (2014)
28. Ferreira, P., Lee, D.: Superoscillations with optimal numerical stability. IEEE Sign. Proc. Lett. **21**, 1443–1447 (2014)
29. Ferreira, P., Lee, D.: Superoscillations with optimum energy concentration. IEEE Sign. Proc. Lett. **62**, 4857–4867 (2014)
30. Ferreira, P., Kempf, A., Reis, M.: Construction of Aharonov-Berry's superoscillations. J. Phys. A Math. Theor. **40**, 5141–5147 (2007)
31. Grosche, C., Steiner, F.: Handbook of Feynmann path integrals. Springer tracts in modern physics, vol. 145. Springer, Berlin (1998)
32. Pozzi, E., Wick, B.: Persistence of superoscillations under the Schrödinger equation. Evol. Eq. Control Theory **11**, 869–894 (2022)
33. Schlosser, P.: Time evolution of superoscillations for the Schrödinger equation on $R \setminus \{0\}$. Quant. Stud. Math. Found. **9**, 343–366 (2022)
34. Schlosser, P.: Infinite order differential operators associated with superoscillations in the half-plane barrier. Compl. Anal. Op. Th. **18**, Art. No. 110 (2024)