THE LANDAU HAMILTONIAN WITH δ-POTENTIALS SUPPORTED ON CURVES

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ABSTRACT. The spectral properties of the singularly perturbed self-adjoint Landau Hamiltonian $\mathsf{A}_\alpha = (\mathrm{i}\nabla + \mathbf{A})^2 + \alpha\delta_\Sigma$ in $L^2(\mathbb{R}^2)$ with a δ-potential supported on a finite $C^{1,1}$-smooth curve Σ are studied. Here $\mathbf{A} = \frac{1}{2}B(-x_2, x_1)^\top$ is the vector potential, $B > 0$ is the strength of the homogeneous magnetic field, and $\alpha \in L^\infty(\Sigma)$ is a position-dependent real coefficient modeling the strength of the singular interaction on the curve Σ. After a general discussion of the qualitative spectral properties of A_α and its resolvent, one of the main objectives in the present paper is a local spectral analysis of A_α near the Landau levels $B(2q+1)$, $q \in \mathbb{N}_0$. Under various conditions on α it is shown that the perturbation smears the Landau levels into eigenvalue clusters, and the accumulation rate of the eigenvalues within these clusters is determined in terms of the capacity of the support of α. Furthermore, the use of Landau Hamiltonians with δ-perturbations as model operators for more realistic quantum systems is justified by showing that A_α can be approximated in the norm resolvent sense by a family of Landau Hamiltonians with suitably scaled regular potentials.

1. Introduction

Quantum motion in a geometrically complicated background is often modeled by networks of *leaky quantum wires*, which are mathematically described by Schrödinger operators with singular potentials supported on families of curves, see, e.g., the monograph [36, Chapter 10], the papers [11, 18, 32, 62, 83], and the references therein. Such models based on PDEs are mathematically more involved than the alternative concept of *quantum graphs* [15] based on ODEs, but have serious advantages from the physical point of view since they do not neglect quantum tunnelling between parts of the network. Although there is nowadays a comprehensive literature on spectral and scattering properties of Schrödinger operators with singular potentials, only few mathematical contributions are concerned with the influence of magnetic fields (see [35, 37, 38, 39, 52, 66]), despite the fact that applications of such fields, local or global, are an important area in modern physics. Magnetic Schrödinger operators with surface interactions appear, e.g., in the analysis of the non-linear Ginzburg-Landau equation, cf. [41, 74].

The present paper can be regarded as a first step towards a general treatment of Landau Hamiltonians with singular potentials supported on curves. Throughout this paper let the strength $B > 0$ of the homogeneous magnetic field be fixed, let the corresponding vector potential in the symmetric gauge be $\mathbf{A} := \frac{1}{2}B(-x_2, x_1)^\top$, and define the magnetic gradient by

$$\nabla_{\mathbf{A}} := \mathrm{i}\nabla + \mathbf{A}. \tag{1.1}$$

Our main goal is to construct a class of singular perturbations of the Landau Hamiltonian $\mathsf{A}_0 = \nabla_{\mathbf{A}}^2$ by δ-potentials supported on finite curves. We study the spectral properties of these singularly perturbed Landau Hamiltonians in detail and we justify their use as model operators for more realistic quantum systems by showing that they can be approximated in the norm resolvent sense by a family of Landau Hamiltonians with suitably scaled regular potentials. In order to explain our strategy and results more precisely, assume that Σ is the boundary of a compact $C^{1,1}$-domain, let $\alpha \in L^\infty(\Sigma)$ be a real function, consider the sesquilinear form

$$\mathfrak{a}_\alpha[f,g] = \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\mathbb{R}^2,\mathbb{C}^2)} + \big(\alpha f|_\Sigma, g|_\Sigma\big)_{L^2(\Sigma)}, \quad \operatorname{dom}\mathfrak{a}_\alpha = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2), \tag{1.2}$$

where $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : |\nabla_{\mathbf{A}} f| \in L^2(\mathbb{R}^2)\}$ is the magnetic Sobolev space, and denote the corresponding self-adjoint operator in $L^2(\mathbb{R}^2)$ by A_α. If δ_Σ denotes the δ-distribution supported on the curve Σ then on a formal level

$$\mathsf{A}_\alpha = \nabla_{\mathbf{A}}^2 + \alpha\delta_\Sigma = \mathsf{A}_0 + \alpha\delta_\Sigma. \tag{1.3}$$

Our approach to the spectral analysis of the Landau Hamiltonians with singular potentials is via abstract techniques from extension theory of symmetric operators. Here we shall use the notion of quasi boundary triples and their Weyl functions from [9, 10] to first determine the operator A_α associated to $\mathfrak{a}_\alpha$ and its domain via explicit interface conditions at Σ, see (4.1) for details. In particular, this leads to additional smoothness results for the functions in the domain of A_α, which do not follow directly from the quadratic form method. Furthermore, we obtain a Birman-Schwinger principle and the useful resolvent formula

$$(\mathsf{A}_\alpha - \lambda)^{-1} = (\mathsf{A}_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\overline{\lambda})^*, \tag{1.4}$$

where γ and M are the γ-field and Weyl function, respectively, corresponding to a suitable quasi boundary triple $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$. We refer the reader to Appendix A for a brief introduction to quasi boundary triples and Weyl functions, and here we only mention that $\gamma(\lambda)\colon L^2(\Sigma) \to L^2(\mathbb{R}^2)$ and $M(\lambda)\colon L^2(\Sigma) \to L^2(\Sigma)$ in (1.4) can also be viewed as (boundary) integral operators with the Green function of A_0 as integral kernel. The formula (1.4) can be seen as an interpretation of the formal equality (1.3): the resolvent difference is essentially reduced to the term $(1 + \alpha M(\lambda))^{-1}\alpha$, which is localized on the curve Σ and contains the main information on the spectrum of A_α. We remark that, roughly speaking, many of the above considerations can be directly extended to more general magnetic Schrödinger operators in arbitrary space dimension.

Our further investigations are based on a detailed analysis of the perturbation term

$$W_\lambda = -\gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\overline{\lambda})^* \tag{1.5}$$

in the resolvent formula (1.4). Since Σ is a compact curve, the Rellich-Kondrachov embedding theorem implies that W_λ is compact in $L^2(\mathbb{R}^2)$ and as an immediate consequence we conclude

$$\sigma_{\rm ess}(\mathsf{A}_\alpha) = \sigma_{\rm ess}(\mathsf{A}_0) = \sigma(\mathsf{A}_0) = \{B(2q+1)\colon q \in \mathbb{N}_0\},$$

where $\Lambda_q = B(2q+1)$, $q = 0, 1, 2, \dots$ are infinite dimensional eigenvalues of A_0, usually called Landau levels. It is well known that perturbations of the Landau Hamiltonian A_0 can generate accumulation of discrete eigenvalues to the Landau levels Λ_q. For additive perturbations of A_0 by an electric potential this was shown by G. Raikov in [72], see also [40, 57, 65, 69, 73, 77, 78]. More recently similar results were proved in [21, 20, 46, 67, 70] for Landau Hamiltonians on domains with Dirichlet, Neumann, and Robin boundary conditions; for closely related results in the three-dimensional situation we refer to [17, 23] and the references therein.

Our first main objective is to observe a similar phenomenon on the accumulation of discrete eigenvalues of A_α to the Landau levels Λ_q, and to prove singular value estimates and regularized summability properties of the discrete eigenvalues. From the physical point of view such a result provides one more illustration of how subtle the occurrence of the (exact) Landau levels in the spectrum of the magnetic Laplacian is. It requires the full translational invariance of the operator; once the latter is violated eigenvalues of a finite multiplicity may split off. This splitting is a generic effect and so is the fact that the eigenvalues arising in this way accumulate at the unperturbed level. To understand this phenomenon one has to realize that each Landau eigenspace is infinite-dimensional and, in particular, a translation of a fixed eigenfunction is again an eigenfunction related to the same Landau level. Furthermore, all these eigenfunctions have an exponential decay, hence the further an eigenfunction is concentrated from the local perturbation, the smaller the resulting shift in energy would be.

We point out again that our approach is not based on quadratic form techniques, but on extension theory methods and, in particular, a detailed analysis of the perturbation term W_λ in the resolvent formula (1.4). In fact, we are interested in the compression $P_q W_\lambda P_q$ of W_λ in (1.5) onto the eigenspace $\ker(\mathsf{A}_0 - \Lambda_q)$ of the unperturbed Landau Hamiltonian. The operators $P_q W_\lambda P_q$ are the analogues of the Toeplitz operators appearing in this connection in [22, 40, 70, 71, 72, 73, 78, 81, 84], and we note in this context that some of our observations rely on deep results in the theory of Toeplitz operators, and conversely that our approach and some of our considerations lead to new results for Toeplitz operators.

If the strength α in (1.2)–(1.3) is positive (negative) on Σ we show in Theorem 6.2 that the discrete spectrum of A_α accumulates to each Landau level Λ_q from above (below, respectively). Combining our technique with the constructions in [40, 70, 76], we obtain in Theorem 6.3 the same result for the lowest Landau level $\Lambda_0 = B$ under the weaker assumption that $\alpha \not\equiv 0$ is nonnegative (nonpositive), and in

Proposition 6.6 for the higher Landau levels assuming that $\operatorname{supp}\alpha$ contains a C^∞-smooth arc on which α is positive (negative, respectively). Relying on the analysis of $P_q W_\lambda P_q$, we also estimate the rate of the eigenvalue accumulation in Theorem 6.4. Although the upper bounds on the accumulation rate of the discrete eigenvalues hold also for sign-changing α it is a challenging open problem to show that the eigenvalue accumulation is indeed present in this situation. Furthermore, making use of the technique from [40, 70] we prove in Theorem 6.5 spectral asymptotics if $\operatorname{supp}\alpha$ is a C^∞-smooth arc Γ and α is uniformly positive (uniformly negative) in the interior of Γ. More precisely, if, e.g., $\alpha > 0$ inside the C^∞-smooth arc $\Gamma = \operatorname{supp}\alpha$ then the discrete eigenvalues (counted with multiplicities) of A_α in the interval $(\Lambda_q, \Lambda_q + B]$, $q \in \mathbb{N}_0$, form a sequence $\lambda_1^+(q) \geq \lambda_2^+(q) \geq \cdots \geq \Lambda_q$ with the asymptotic behaviour

$$\lim_{k\to\infty} \big(k!\,(\lambda_k^+(q) - \Lambda_q)\big)^{1/k} = \frac{B}{2}\big(\operatorname{Cap}(\Gamma)\big)^2, \tag{1.6}$$

where $\operatorname{Cap}(\Gamma)$ is the *logarithmic capacity* of Γ; cf. [40].

Besides the spectral analysis of the operators A_α in (1.3) our second main objective in this paper is to justify the use of such singular perturbations of the Landau Hamiltonian for more realistic model operators with regular potentials. The approximation problem of singular potentials by regular ones has been discussed in the absence of magnetic fields for δ-point interactions in great detail in the monograph [4], and for δ-surface interactions in [7, 33, 34] and [19, 43, 66, 68, 80], see also [5, 83] for more abstract approaches. We show in Theorem 4.5 and Corollary 4.6 that for real $\alpha \in L^\infty(\Sigma)$ the singular Landau Hamiltonian A_α can be approximated in the norm resolvent sense by a family of regular Landau Hamiltonians with potentials suitably scaled in the direction perpendicular to Σ. The choice of the approximating sequence of potentials is essentially the same as, e.g., in [7, 33, 34], but the technique of the proof is significantly different and more efficient.

Organization of the paper. Section 2 contains some preliminary material concerning the unperturbed Landau Hamiltonian, properties of Schatten-von Neumann ideals and some aspects of perturbation theory. In Subsection 2.4 we discuss a class of Toeplitz-like operators related to Landau Hamiltonians. In Section 3 we make use of the abstract concept of quasi boundary triples and their Weyl functions (see Appendix A for a brief introduction) in order to study Landau Hamiltonians with δ-potentials supported on curves. Using a suitable quasi boundary triple we show self-adjointness of A_α, provide qualitative spectral properties, and derive the Krein-type resolvent formula (1.4). The approximation of A_α by magnetic Schrödinger operators with scaled regular potentials is also discussed; the proof is technical and therefore outsourced to Appendix B. Section 5 is devoted to the spectral analysis of the compressed resolvent difference $P_q W_\lambda P_q$. Under various assumptions we obtain spectral estimates and spectral asymptotics for this operator, which lead to results on the eigenvalue clusters of A_α at Landau levels in Section 6.

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2. Preliminaries

In this section we provide useful notions and techniques that are needed in our analysis of magnetic Schrödinger operators with singular interactions. In Subsection 2.1 we recall the definition and some well-known properties of the Landau Hamiltonian, in Subsection 2.2 some facts on the Schatten-von Neumann ideals of compact operators are discussed, and in Subsections 2.3 and 2.4 we collect some results from perturbation theory and Toeplitz operators that will be needed in the main part of the paper. For the convenience of the reader we have often added short proofs to keep the paper self-contained.

2.1. **The Landau Hamiltonian.** In order to introduce the Landau Hamiltonian, that is, the unperturbed magnetic Schrödinger operator with homogeneous magnetic field, recall the definition of the magnetic gradient from (1.1) and define the first order L^2-based magnetic Sobolev space by

$$\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) := \big\{ f \in L^2(\mathbb{R}^2) \colon |\nabla_{\mathbf{A}} f| \in L^2(\mathbb{R}^2) \big\}, \tag{2.1}$$

which becomes a Hilbert space if it is endowed with the inner product

$$(f,g)_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)} := (f,g)_{L^2(\mathbb{R}^2)} + \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\mathbb{R}^2;\mathbb{C}^2)}, \quad f,g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2).$$

The space $C_0^\infty(\mathbb{R}^2)$ of smooth compactly supported functions is dense in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$, see, e.g., [60, Theorem 7.22]. Note that for $B = 0$ the space $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ coincides with the usual first order Sobolev space $H^1(\mathbb{R}^2)$; if $B \neq 0$ then still $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ coincide locally. The standard Sobolev spaces of order $s \in \mathbb{R}$ will be denoted in this paper by $H^s(\mathbb{R}^2)$.

Next consider the symmetric sesquilinear form

$$\mathfrak{a}_0[f,g] := \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\mathbb{R}^2;\mathbb{C}^2)}, \qquad \operatorname{dom} \mathfrak{a}_0 = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2), \tag{2.2}$$

and note that this form is densely defined, nonnegative, and closed in $L^2(\mathbb{R}^2)$. Hence it gives rise to a uniquely determined nonnegative self-adjoint operator A_0, which is given by

$$\mathsf{A}_0 f = \nabla^2_{\mathbf{A}} f, \qquad \operatorname{dom} \mathsf{A}_0 = \mathcal{H}^2_{\mathbf{A}}(\mathbb{R}^2) := \big\{ f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \colon \nabla^2_{\mathbf{A}} f \in L^2(\mathbb{R}^2) \big\}. \tag{2.3}$$

Note also that $C_0^\infty(\mathbb{R}^2)$ is a core for the sesquilinear form $\mathfrak{a}_0$ since $C_0^\infty(\mathbb{R}^2)$ is dense in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. The spectral properties and the Green function of the Landau Hamiltonian are recalled in the following proposition; cf. [51, §10.4.1], [53, §2.5.2], [66, Section 2], and [31].

Proposition 2.1. *Let A_0 be the Landau Hamiltonian in (2.3). Then*

$$\sigma(\mathsf{A}_0) = \sigma_{\rm ess}(\mathsf{A}_0) = \{B(2q+1)\colon q \in \mathbb{N}_0\},$$

i.e. the spectrum of A_0 consists only of the eigenvalues $\Lambda_q = B(2q+1)$, which are called Landau levels and have infinite multiplicity. If $\lambda \notin \sigma(\mathsf{A}_0)$, then the resolvent of A_0 is given by

$$((\mathsf{A}_0 - \lambda)^{-1} f)(x) = \int_{\mathbb{R}^2} G_\lambda(x,y) f(y) \mathsf{d}y, \qquad f \in L^2(\mathbb{R}^2),$$

with the Green function

$$G_\lambda(x,y) = \frac{1}{4\pi}\Phi_B(x,y)\Gamma\left(\frac{B-\lambda}{2B}\right) U\left(\frac{B-\lambda}{2B}, 1; \frac{B}{2}|x-y|^2\right), \tag{2.4}$$

where U is the irregular confluent hypergeometric function (see [1, §13.1]), Γ denotes the Euler gamma function and

$$\Phi_B(x,y) = \exp\left[-\frac{\mathsf{i}B}{2}(x_1y_2 - x_2y_1) - \frac{B}{4}|x-y|^2\right].$$

In the next proposition two well-known variants of the so-called diamagnetic inequality are provided, see, e.g. [6, Theorem 2.5] and [60, Theorem 7.21].

Proposition 2.2. *Let $-\Delta$ be the self-adjoint Laplace operator in $L^2(\mathbb{R}^2)$ defined on $H^2(\mathbb{R}^2)$. Then for $\beta > 0$, $\lambda < 0$, and $f \in L^2(\mathbb{R}^2)$ one has pointwise a.e. in $\mathbb{R}^2$*

$$\big|(\mathsf{A}_0 - \lambda)^{-\beta} f\big| \le (-\Delta - \lambda)^{-\beta}|f|. \tag{2.5}$$

Moreover, if $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$, then $|f|$ belongs to $H^1(\mathbb{R}^2)$ and one has pointwise a.e. in $\mathbb{R}^2$

$$\big|\nabla|f|\big| \le \big|\nabla_{\mathbf{A}} f\big|. \tag{2.6}$$

Proof. For the proof of (2.5) we follow ideas from [6, Theorem 2.5]. Recall that by [50, Proposition 3.3.5] the formula

$$(\mathsf{A} - \lambda)^{-\beta} f = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{\lambda t}\big(e^{-t\mathsf{A}} f\big) \mathsf{d}t, \qquad \lambda < 0,$$

holds for any self-adjoint nonnegative operator A acting in a Hilbert space $\mathcal{H}$ and for any $f \in \mathcal{H}$; here Γ denotes the Euler gamma function. Hence, the inequality

$$|e^{-t\mathsf{A}_0} f| \le e^{-t\Delta}|f|$$

pointwise a.e. in $\mathbb{R}^2$ (see, e.g., [26, eq. (1.8)]) yields

$$\begin{aligned}\left|(\mathsf{A}_0-\lambda)^{-\beta}f\right| &= \frac{1}{\Gamma(\beta)}\left|\int_0^\infty t^{\beta-1}e^{\lambda t}\big(e^{-t\mathsf{A}_0}f\big)\mathrm{d}t\right| \\ &\le \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{\lambda t}\big|e^{-t\mathsf{A}_0}f\big|\mathrm{d}t \\ &\le \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{\lambda t}\big(e^{-t\Delta}|f|\big)\mathrm{d}t = (-\Delta-\lambda)^{-\beta}|f|.\end{aligned}$$

The inequality (2.6) can be found in, e.g., [60, Theorem 7.21]. □

Using the diamagnetic inequality we can show that functions in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ have traces in $L^2(\Sigma)$. Here, and in the following, Σ is the boundary of a bounded $C^{1,1}$-domain $\Omega\subset\mathbb{R}^2$.

Corollary 2.3. *The mapping $C_0^\infty(\mathbb{R}^2)\ni f\mapsto f|_\Sigma$ can be extended by continuity to a bounded operator $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)\ni f\mapsto f|_\Sigma\in L^2(\Sigma)$. Moreover, for all $\varepsilon>0$ there exists $c(\varepsilon)>0$ such that*

$$\|f|_\Sigma\|^2_{L^2(\Sigma)}\le\varepsilon\|\nabla_{\mathbf{A}}f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)}+c(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^2)}$$

holds for all $f\in\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$.

Proof. Let $\varepsilon>0$ and $f\in C_0^\infty(\mathbb{R}^2)$. It is well known that there exists a constant $c(\varepsilon)>0$ independent of f such that

$$\|f|_\Sigma\|^2_{L^2(\Sigma)}=\big\||f|\big|_\Sigma\big\|^2_{L^2(\Sigma)}\le\varepsilon\big\|\nabla|f|\big\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)}+c(\varepsilon)\big\||f|\big\|^2_{L^2(\mathbb{R}^2)}.$$

Using the diamagnetic inequality (2.6) we obtain

$$\|f|_\Sigma\|^2_{L^2(\Sigma)}\le\varepsilon\big\|\nabla_{\mathbf{A}}f\big\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)}+c(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^2)}.$$

Since $C_0^\infty(\mathbb{R}^2)$ is dense in the magnetic Sobolev space $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$, the claim follows. □

Remark 2.4. The trace map in Corollary 2.3 is even compact. To see this, choose a ball $\mathcal{B}\subset\mathbb{R}^2$ such that $\Sigma\subset\mathcal{B}$. Since the vector potential $\mathbf{A}$ belongs to $C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ it follows that the restriction operator $R:\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)\to H^1(\mathcal{B})$, $f\mapsto f|_{\mathcal{B}}$ is bounded. Furthermore, the trace map $\gamma_{\mathcal{B}}:H^1(\mathcal{B})\to H^{1/2}(\Sigma)$ is bounded (see, e.g. [64, Theorem 3.37]) and as the embedding $H^{1/2}(\Sigma)\hookrightarrow L^2(\Sigma)$ is compact we conclude that the trace map $\gamma_{\mathcal{B}}R:\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)\to L^2(\Sigma)$, $f\mapsto f|_\Sigma$, in Corollary 2.3 is compact.

Next we recall the definition of the Landau Hamiltonian on a domain Ω with Dirichlet boundary conditions. It is assumed here that Ω is either a bounded $C^{1,1}$-domain in $\mathbb{R}^2$ or the complement of a bounded $C^{1,1}$-domain; then the compact boundary $\Sigma:=\partial\Omega$ is a $C^{1,1}$-smooth curve. In analogy to (2.1) the first order L^2-based magnetic Sobolev space is defined by

$$\mathcal{H}^1_{\mathbf{A}}(\Omega):=\big\{f\in L^2(\Omega)\colon|\nabla_{\mathbf{A}}f|\in L^2(\Omega)\big\}$$

and is equipped with the Hilbert space inner product

$$(f,g)_{\mathcal{H}^1_{\mathbf{A}}(\Omega)} := (f,g)_{L^2(\Omega)} + \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\Omega;\mathbb{C}^2)}, \quad f,g \in \mathcal{H}^1_{\mathbf{A}}(\Omega).$$

Note that $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ coincides with $H^1(\Omega)$ if Ω is bounded or if $B = 0$; if $B \neq 0$ then still $\mathcal{H}^1_{\mathbf{A}}(\Omega)$ and $H^1(\Omega)$ coincide locally. The standard Sobolev spaces on Ω and the boundary Σ are denoted by $H^s(\Omega)$ and $H^t(\Sigma)$, respectively. The magnetic counterpart of the Sobolev space $H^1_0(\Omega)$ is defined as

$$\mathcal{H}^1_{\mathbf{A},0}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{\mathcal{H}^1_{\mathbf{A}}(\Omega)}}.$$

Now consider the symmetric sesquilinear form

$$\mathfrak{a}^\Omega_{\rm D}[f,g] := \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\Omega,\mathbb{C}^2)}, \qquad \operatorname{dom} \mathfrak{a}^\Omega_{\rm D} = \mathcal{H}^1_{\mathbf{A},0}(\Omega), \tag{2.7}$$

and observe that $\mathfrak{a}^\Omega_{\rm D}$ is nonnegative, closed, and densely defined in $L^2(\Omega)$. The nonnegative self-adjoint operator $\mathsf{A}^\Omega_{\rm D}$ corresponding to $\mathfrak{a}^\Omega_{\rm D}$ is the *Landau Hamiltonian on Ω with Dirichlet boundary conditions on Σ*. It is useful to note that for a bounded domain Ω the space $\mathcal{H}^1_{\mathbf{A},0}(\Omega) = H^1_0(\Omega)$ is compactly embedded in $L^2(\Omega)$ and hence

$$\sigma_{\rm ess}(\mathsf{A}^\Omega_{\rm D}) = \varnothing. \tag{2.8}$$

2.2. **Schatten von-Neumann ideals.** In this subsection we recall the definition and some properties of the Schatten-von Neumann ideals, which are used in the proofs of some of our main results. We partially follow the presentation in [11, 12], where further references can be found. A very useful result on the Schatten-von Neumann property of operators that map into Sobolev spaces $H^s(\Sigma)$ with $s > 0$ is provided in Proposition 2.5.

Let $\mathcal{H}, \mathcal{G}$, and $\mathcal{K}$ be separable Hilbert spaces. We denote the linear space of all bounded and everywhere defined operators from $\mathcal{H}$ into $\mathcal{G}$ by $\mathfrak{B}(\mathcal{H},\mathcal{G})$ and we write $\mathfrak{B}(\mathcal{H}) := \mathfrak{B}(\mathcal{H},\mathcal{H})$. We use the symbol $\mathfrak{S}_\infty(\mathcal{H},\mathcal{G})$ for the space of all compact operators from $\mathcal{H}$ to $\mathcal{G}$ and $\mathfrak{S}_\infty(\mathcal{H}) := \mathfrak{S}_\infty(\mathcal{H},\mathcal{H})$. The singular values $s_k(K)$, $k \in \mathbb{N}$, of $K \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{G})$ are the eigenvalues of the self-adjoint, nonnegative operator $(K^*K)^{1/2} \in \mathfrak{S}_\infty(\mathcal{H})$, which are ordered in a nonincreasing way with multiplicities taken into account. Note that $s_k(K) = s_k(K^*)$ for $k \in \mathbb{N}$. For $p > 0$ the Schatten-von Neumann ideal of order p is defined by

$$\mathfrak{S}_p(\mathcal{H},\mathcal{G}) := \left\{ K \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{G}) \colon \sum_{k=1}^\infty s_k(K)^p < \infty \right\}$$

and the weak Schatten-von Neumann ideal of order p is defined by

$$\mathfrak{S}_{p,\infty}(\mathcal{H},\mathcal{G}) := \left\{ K \in \mathfrak{S}_\infty(\mathcal{H},\mathcal{G}) \colon s_k(K) = \mathcal{O}(k^{-1/p}) \right\}.$$

The (weak) Schatten-von Neumann ideals are ordered in the sense that for $0 < p < q$ one has $\mathfrak{S}_p(\mathcal{H},\mathcal{G}) \subset \mathfrak{S}_q(\mathcal{H},\mathcal{G})$ and $\mathfrak{S}_{p,\infty}(\mathcal{H},\mathcal{G}) \subset \mathfrak{S}_{q,\infty}(\mathcal{H},\mathcal{G})$. Moreover, we have

$$\mathfrak{S}_p(\mathcal{H},\mathcal{G}) \subset \mathfrak{S}_{p,\infty}(\mathcal{H},\mathcal{G}) \qquad \text{and} \qquad \mathfrak{S}_{p,\infty}(\mathcal{H},\mathcal{G}) \subset \mathfrak{S}_q(\mathcal{H},\mathcal{G}).$$

The Schatten-von Neumann ideals are two-sided ideals, that is, for $K \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G})$ and $A \in \mathfrak{B}(\mathcal{H})$, $B \in \mathfrak{B}(\mathcal{G})$ one has $BKA \in \mathfrak{S}_p(\mathcal{H}, \mathcal{G})$. The analogous ideal property holds for the weak Schatten-von Neumann ideals. Eventually, if $p, q > 0$ and r are chosen such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then for $K_1 \in \mathfrak{S}_{p,\infty}(\mathcal{H}, \mathcal{G})$ and $K_2 \in \mathfrak{S}_{q,\infty}(\mathcal{G}, \mathcal{K})$ the product of these operators satisfies

$$K_2 K_1 \in \mathfrak{S}_{r,\infty}(\mathcal{H}, \mathcal{K}). \tag{2.9}$$

Finally, let $\Sigma \subset \mathbb{R}^2$ be the boundary of a sufficiently smooth bounded domain. It will be shown in the next proposition that operators with range in the Sobolev space $H^s(\Sigma)$ belong to certain weak Schatten-von Neumann ideals. In the special case that Σ is the boundary of a C^∞-domain this property is known; cf. [11, Lemma 2.11].

Proposition 2.5. *Let $k \in \mathbb{N}$ and let Σ be the boundary of a bounded $C^{k,1}$-domain $\Omega_{\rm i} \subset \mathbb{R}^2$. Let $\mathcal{H}$ be a separable Hilbert space and let $A \in \mathfrak{B}(\mathcal{H}, L^2(\Sigma))$ be such that $\operatorname{ran} A \subset H^{l/2}(\Sigma)$ for some $l \in \{1, \dots, 2k+1\}$. Then*

$$A \in \mathfrak{S}_{2/l,\infty}\big(\mathcal{H}, L^2(\Sigma)\big).$$

The proof of Proposition 2.5 uses a general result from [2] and some properties of the acoustic single layer potential for the Helmholtz equation $-\Delta + 1$, which will be briefly discussed for the convenience of the reader. Recall first from [85, Section 7.4] that the Green function for the differential expression $-\Delta + 1$ in $\mathbb{R}^2$ is given by $\frac{1}{2\pi} K_0(|\cdot|)$, where K_0 is the modified Bessel function of second kind and of order 0. It is well known that the boundary integral operator

$$(\mathcal{S}\varphi)(x) = \frac{1}{2\pi} \int_\Sigma K_0(|x-y|)\varphi(y) \mathrm{d}\sigma(y), \qquad x \in \Sigma, \tag{2.10}$$

gives rise to a bounded operator

$$\mathcal{S}_{-1/2} : H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma); \tag{2.11}$$

cf. [64, Theorem 6.11]. In the following lemma we provide some other useful properties of $\mathcal{S}$. The proof of (i) is inspired by the proof of [25, Theorem 3].

Lemma 2.6. *Let Σ be the boundary of a bounded $C^{k,1}$-domain $\Omega_{\rm i}$ with $k \geq 1$. Then the following holds.*

(i) *For all $s \in [-\frac{1}{2}, k - \frac{1}{2}]$ the restriction of $\mathcal{S}_{-1/2}$ in (2.11) onto $H^s(\Sigma)$ leads to a bijective bounded operator*

$$\mathcal{S}_s : H^s(\Sigma) \to H^{s+1}(\Sigma). \tag{2.12}$$

(ii) *The operator $\mathcal{S}_0 : L^2(\Sigma) \to H^1(\Sigma)$ in (2.12) can be viewed as nonnegative bounded self-adjoint operator in $L^2(\Sigma)$ with $\operatorname{ran} \mathcal{S}_0 = H^1(\Sigma)$. The square root $\mathcal{S}_0^{1/2}$ (defined via the functional calculus for self-adjoint operators) is a nonnegative bounded self-adjoint operator in $L^2(\Sigma)$ and also leads to a bijective bounded operator*

$$\mathcal{S}_0^{1/2} : L^2(\Sigma) \to H^{1/2}(\Sigma).$$

In particular, the operator $\mathcal{S}_0^{l/2} : L^2(\Sigma) \to H^{l/2}(\Sigma)$ *is bijective and bounded for all* $l \in \{1, \dots, 2k+1\}$.

Proof. (i) Note first that by [64, Theorem 7.1 and Theorem 7.2] the operator $\mathcal{S}_s$ in (2.12) is well defined as a linear map between the respective Sobolev spaces. Next, [56, Lemma 1.14(c)] (see also [62, Lemma 3.2]) implies $\ker \mathcal{S}_{-1/2} = \{0\}$ and hence also $\ker \mathcal{S}_s = \{0\}$ for all $s \in [-\frac{1}{2}, k - \frac{1}{2}]$. Moreover,

$$\mathcal{S}_s \in \mathfrak{B}(H^s(\Sigma), H^{s+1}(\Sigma)). \tag{2.13}$$

In fact, for $s = -\frac{1}{2}$ this is a consequence of [64, Theorem 6.11] and for $s > -\frac{1}{2}$ the closed graph theorem implies (2.13) after it has been shown that $\mathcal{S}_s$ is a closed operator. For this consider $(\varphi_n) \subset H^s(\Sigma)$ such that

$$\varphi_n \to \varphi \text{ in } H^s(\Sigma) \quad \text{and} \quad \mathcal{S}_s\varphi_n \to \psi \text{ in } H^{s+1}(\Sigma) \quad \text{as } n \to \infty.$$

Then $\varphi \in H^s(\Sigma) = \operatorname{dom} \mathcal{S}_s$, $\varphi_n \to \varphi$ in $H^{-1/2}(\Sigma)$ as $n \to \infty$, and as $\mathcal{S}_{-1/2} \in \mathfrak{B}(H^{-1/2}(\Sigma), H^{1/2}(\Sigma))$ we have $\mathcal{S}_s\varphi_n = \mathcal{S}_{-1/2}\varphi_n \to \mathcal{S}_{-1/2}\varphi$ in $H^{1/2}(\Sigma)$ for $n \to \infty$. On the other hand, since $H^{s+1}(\Sigma)$ is continuously embedded in $H^{1/2}(\Sigma)$ we also have $\mathcal{S}_{-1/2}\varphi_n = \mathcal{S}_s\varphi_n \to \psi$ in $H^{1/2}(\Sigma)$. Thus $\mathcal{S}_s\varphi = \mathcal{S}_{-1/2}\varphi = \psi$ and hence $\mathcal{S}_s$ is closed.

In order to verify that $\mathcal{S}_s$ in (2.12) is surjective for $s = j - 1/2$ and $j = \{0, 1, ..., k\}$, consider $\psi \in H^{j+1/2}(\Sigma)$. Then, in particular, $\psi \in H^{1/2}(\Sigma)$, and as $\mathcal{S}_{-1/2}$ is a Fredholm operator of index zero by [64, Theorem 7.6] and $\ker \mathcal{S}_{-1/2} = \{0\}$ it is clear that $\mathcal{S}_{-1/2}$ in (2.11) is bijective. Hence there exists a unique $\varphi \in H^{-1/2}(\Sigma)$ such that $\mathcal{S}_{-1/2}\varphi = \psi$. Eventually, it follows from [64, Theorem 7.16 (i)] that $\varphi \in H^{j-1/2}(\Sigma)$, so that $\mathcal{S}_{j-1/2}\varphi = \psi$. We have shown that the operators $\mathcal{S}_s$ in (2.12) for $s = j - 1/2$ and $j \in \{0, 1, ..., k\}$ are bijective. Now it follows from standard interpolation techniques that $\mathcal{S}_s \in \mathfrak{B}(H^s(\Sigma), H^{s+1}(\Sigma))$ is bijective for all $s \in [-\frac{1}{2}, k - \frac{1}{2}]$.

(ii) It is clear that $\mathcal{S}_0$ is a bounded operator in $L^2(\Sigma)$ with $\operatorname{ran} \mathcal{S}_0 = H^1(\Sigma)$. To see that $\mathcal{S}_0$ is nonnegative and self-adjoint in $L^2(\Sigma)$ let $\Omega_{\rm e} := \mathbb{R}^2 \setminus \overline{\Omega}_{\rm i}$ and decompose the functions $u \in L^2(\mathbb{R}^2)$ in the two components $u_j := u|_{\Omega_j}$, $j \in \{{\rm i}, {\rm e}\}$. For $\varphi \in L^2(\Sigma)$ there exists a unique $u \in H^1(\mathbb{R}^2)$ such that $-\Delta u_j + u_j = 0$, $j \in \{{\rm i}, {\rm e}\}$, and $\partial_\nu u_{\rm i}|_\Sigma - \partial_\nu u_{\rm e}|_\Sigma = \varphi$, and, moreover, one has $\mathcal{S}_0\varphi = u|_\Sigma$ (see, e.g., [11, Proposition 3.2 (ii) and Remark 3.3], where $\mathcal{S}_0 = \widetilde{M}(-1)$ in the notation of [11]). Hence, the first Green identity leads to

$$\begin{aligned}(\mathcal{S}_0\varphi, \varphi)_{L^2(\Sigma)} &= (u|_\Sigma, \partial_\nu u_{\rm i}|_\Sigma - \partial_\nu u_{\rm e}|_\Sigma)_{L^2(\Sigma)} \\ &= (u_{\rm i}, \Delta u_{\rm i})_{L^2(\Omega_{\rm i})} + (u_{\rm e}, \Delta u_{\rm e})_{L^2(\Omega_{\rm e})} + (\nabla u, \nabla u)_{L^2(\mathbb{R}^2;\mathbb{C}^2)} \\ &= (u, u)_{L^2(\mathbb{R}^2)} + (\nabla u, \nabla u)_{L^2(\mathbb{R}^2;\mathbb{C}^2)},\end{aligned}$$

which implies that $\mathcal{S}_0$ is a nonnegative self-adjoint operator in $L^2(\Sigma)$. Eventually, by the interpolation result [3, Theorem 3.2], which applies to $\mathcal{S}_0^{-1}$, we have $\operatorname{dom} \mathcal{S}_0^{-1/2} = H^{1/2}(\Sigma)$. Thus, we get $\operatorname{ran} \mathcal{S}_0^{1/2} = H^{1/2}(\Sigma)$ and $\mathcal{S}_0^{1/2}$ is a bijective bounded operator from $L^2(\Sigma)$ onto $H^{1/2}(\Sigma)$.

The last assertion is a direct consequence of (i) and (ii). In fact, for even l this follows from repeated applications of (i), whereas for odd l we use $\mathcal{S}_0^{l/2} = \mathcal{S}_0^{(l-1)/2}\mathcal{S}_0^{1/2}$, (ii) and repeated applications of (i). $\square$

Proof of Proposition 2.5. Assume that $\operatorname{ran} A \subset H^{l/2}(\Sigma)$ for some $l \in \{1, \dots, 2k+1\}$. It will be shown first that the operator $A_l \colon \mathcal{H} \to H^{l/2}(\Sigma)$, $A_l f = Af$, is continuous. In fact, consider a sequence $(f_n) \subset \mathcal{H}$ such that

$$f_n \to f \text{ in } \mathcal{H} \quad \text{and} \quad A_l f_n \to g \text{ in } H^{l/2}(\Sigma) \quad \text{as } n \to \infty.$$

Then $f \in \mathcal{H} = \operatorname{dom} A_l$ and as $A \in \mathfrak{B}(\mathcal{H}, L^2(\Sigma))$ we have $A_l f_n = A f_n \to Af$ in $L^2(\Sigma)$ for $n \to \infty$. On the other hand, since $H^{l/2}(\Sigma)$ is continuously embedded in $L^2(\Sigma)$ we also have $Af_n = A_l f_n \to g$ in $L^2(\Sigma)$. Thus, $A_l f = Af = g$ and hence A_l is closed and defined on all of $\mathcal{H}$. This implies $A_l \in \mathfrak{B}(\mathcal{H}, H^{l/2}(\Sigma))$.

Now consider the operator $\mathcal{S}_0$ in Lemma 2.6 as a nonnegative bounded self-adjoint operator in $L^2(\Sigma)$ and note that the integral kernel in (2.10) is the kernel of the polyhomogeneous pseudodifferential operator $(-\Delta + 1)^{-1}$, which is of order -2. Therefore, [2, Theorem 2.9] applies (for the class $\mathcal{P}^0$) and yields that

$$\mathcal{S}_0 \in \mathfrak{S}_{1,\infty}(L^2(\Sigma)).$$

Hence, the spectral theorem implies

$$\mathcal{S}_0^t \in \mathfrak{S}_{1/t,\infty}(L^2(\Sigma)), \qquad t > 0. \tag{2.14}$$

On the other hand, it follows from Lemma 2.10 that $\mathcal{S}_0^{l/2} \in \mathfrak{B}(L^2(\Sigma), H^{l/2}(\Sigma))$ is bijective and hence also $\mathcal{S}_0^{-l/2} \in \mathfrak{B}(H^{l/2}(\Sigma), L^2(\Sigma))$. Since

$$A = \mathcal{S}_0^{l/2}\mathcal{S}_0^{-l/2} A_l \quad \text{and} \quad \mathcal{S}_0^{-l/2} A_l \in \mathfrak{B}(\mathcal{H}, L^2(\Sigma))$$

we conclude from (2.14) with $t = l/2$ that $A \in \mathfrak{S}_{2/l,\infty}(\mathcal{H}, L^2(\Sigma))$. $\square$

2.3. **Compact perturbations of self-adjoint operators.** In this subsection we discuss some special results on compact perturbations. In the following let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let $\Lambda \in \mathbb{R}$ be an isolated eigenvalue of T of infinite multiplicity with the corresponding eigenprojection P_Λ. Furthermore, let $\tau_\pm > 0$ be such that

$$(\Lambda - 2\tau_-, \Lambda + 2\tau_+) \cap \sigma(T) = \{\Lambda\}.$$

For a self-adjoint operator W in $\mathcal{H}$ with corresponding spectral measure $E_W(\cdot)$ we denote by

$$W_+ = \int_0^\infty \lambda \, \mathrm{d}E_W(\lambda) \quad \text{and} \quad W_- = -\int_{-\infty}^0 \lambda \, \mathrm{d}E_W(\lambda) \tag{2.15}$$

the nonnegative and nonpositive part of W, respectively. Note that both W_+ and W_- are nonnegative self-adjoint operators in $\mathcal{H}$ and that the identities $W = W_+ -$

W_- and $|W| = W_+ + W_-$ hold. Now assume, in addition, that the self-adjoint operator W in $\mathcal{H}$ is compact and denote by

$$\mu_1^\pm \geq \mu_2^\pm \geq \mu_3^\pm \geq \cdots \geq 0$$

the eigenvalues of $P_\Lambda W_\pm P_\Lambda \geq 0$ in nonincreasing order with multiplicities taken into account and by

$$\lambda_1^- \leq \lambda_2^- \leq \cdots \leq \Lambda \leq \cdots \leq \lambda_2^+ \leq \lambda_1^+ \tag{2.16}$$

the eigenvalues of $T + W$ in the interval $(\Lambda - \tau_-, \Lambda + \tau_+)$. If there are only finitely many $\lambda_k^+ > \Lambda$ we set $\lambda_k^+ = \Lambda$ for all larger $k \in \mathbb{N}$, the same convention is used for λ_k^-. In the next proposition we state double-sided estimates of $\lambda_k^\pm$ in terms of $\mu_k^\pm$, assuming that either $W_- = 0$ or $W_+ = 0$.

Proposition 2.7. [70, Proposition 2.2] *Let T and $W = W_+ - W_-$ be as above. Then the following holds.*

(i) *If $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) = \infty$ and $W_- = 0$ then the eigenvalues of $T + W$ accumulate to Λ only from above and for $\varepsilon > 0$ there exists $\ell \in \mathbb{N}$ such that*

$$(1-\varepsilon)\mu_{k+\ell}^+ \leq \lambda_k^+ - \Lambda \leq (1+\varepsilon)\mu_{k-\ell}^+$$

for all $k \in \mathbb{N}$ sufficiently large.

(ii) *If $\operatorname{rank}(P_\Lambda W_- P_\Lambda) = \infty$ and $W_+ = 0$ then the eigenvalues of $T + W$ accumulate to Λ only from below and for $\varepsilon > 0$ there exists $\ell \in \mathbb{N}$ such that*

$$(1-\varepsilon)\mu_{k+\ell}^- \leq \Lambda - \lambda_k^- \leq (1+\varepsilon)\mu_{k-\ell}^-$$

for all $k \in \mathbb{N}$ sufficiently large.

Remark 2.8. If $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) < \infty$ or $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) < \infty$ in Proposition 2.7 then still the upper estimates

$$\lambda_k^+ - \Lambda \leq (1+\varepsilon)\mu_{k-\ell}^+ \quad \text{or} \quad \Lambda - \lambda_k^- \leq (1+\varepsilon)\mu_{k-\ell}^-,$$

respectively, for $k \in \mathbb{N}$ sufficiently large remain valid. This follows from the proof of [70, Proposition 2.2].

In the following, we denote by $\mathcal{N}_\mathcal{I}(A)$ the number of eigenvalues of a self-adjoint operator A in an interval $\mathcal{I} \subset \mathbb{R} \setminus \sigma_{\rm ess}(A)$ counted with multiplicities. The next standard perturbation lemma will be useful. We state it for the convenience of the reader.

Lemma 2.9. [16, §9.3, Theorem 3 and §9.4, Lemma 3] *Let $C, D \in \mathfrak{B}(\mathcal{H})$ be self-adjoint operators such that $V := D - C$ is compact with $\sigma(V) \subseteq [v_-, v_+]$. Let $\mathcal{I} = (c_-, c_+) \subset \mathbb{R}$ be an interval satisfying $\mathcal{I} \cap \sigma_{\rm ess}(C) = \varnothing$. Then the following hold.*

(i) *If $\operatorname{rank} V = r < \infty$, then $\mathcal{N}_\mathcal{I}(C) \leq \mathcal{N}_\mathcal{I}(D) + r$.*

(ii) *If $\mathcal{I}' := (c_- + v_-, c_+ + v_+) \cap \sigma_{\rm ess}(C) = \varnothing$, then $\mathcal{N}_\mathcal{I}(C) \leq \mathcal{N}_{\mathcal{I}'}(D)$.*

The next proposition complements Proposition 2.7 and Remark 2.8. If the definiteness assumption on W is dropped then one still obtains one-sided estimates on $\lambda_k^+ - \Lambda$ and $\Lambda - \lambda_k^-$.

Proposition 2.10. *Let T and $W = W_+ - W_-$ be as above. Then the following holds.*

(i) *For $\varepsilon > 0$ there exists $\ell \in \mathbb{N}$ such that*
$$\lambda_k^+ - \Lambda \leq (1+\varepsilon)\mu_{k-\ell}^+$$
for all $k \in \mathbb{N}$ sufficiently large.

(ii) *For $\varepsilon > 0$ there exists $\ell \in \mathbb{N}$ such that*
$$\Lambda - \lambda_k^- \leq (1+\varepsilon)\mu_{k-\ell}^-$$
for all $k \in \mathbb{N}$ sufficiently large.

Proof. It suffices to prove item (i); the proof of (ii) is analogous. Moreover, it is no restriction to assume $\Lambda = 0$. Throughout the proof we denote the eigenvalues in the interval $[0, \tau_+)$ of the operator $S_U = T + U$ with a generic compact self-adjoint perturbation U by

$$\lambda_1^+(S_U) \geq \lambda_2^+(S_U) \geq \lambda_3^+(S_U) \geq \cdots \geq 0, \tag{2.17}$$

which are repeated with multiplicities taken into account.

Let us fix $\varepsilon > 0$. Since W_- is compact and nonnegative, it can be decomposed as $W_- = F_- + R_-$, where $\operatorname{rank} F_- = r_0 < \infty$ and the operator R_- satisfies $\sigma(R_-) \subseteq [0, \tau_+]$. Hence, the operator $S_W = T + W$ can be written as
$$S_W = T + W_+ - F_- - R_-.$$

If $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) = \infty$ then Proposition 2.7 (i) applies for the operator $S_{W_+} = T + W_+$ and yields

$$\lambda_k^+(S_{W_+}) \leq (1+\varepsilon)\mu_{k-\ell_0}^+ \tag{2.18}$$

for some $\ell_0 \in \mathbb{N}$ and all $k \in \mathbb{N}$ sufficiently large; in the case $\operatorname{rank}(P_\Lambda W_+ P_\Lambda) < \infty$ the estimate (2.18) follows from Remark 2.8. Since the rank of F_- is finite, Lemma 2.9 (i) with $C = S_{W_+}$ and $D = S_{W_+ - F_-}$ and (2.18) imply

$$\lambda_k^+(S_{W_+ - F_-}) \leq \lambda_{k-r_0}^+(S_{W_+}) \leq (1+\varepsilon)\mu_{k-\ell_1}^+ \tag{2.19}$$

for $\ell_1 := \ell_0 + r_0$ and all $k \in \mathbb{N}$ sufficiently large. Further, we set

$$r_1 := \mathcal{N}_{[\tau_+, 2\tau_+)}(S_{W_+ - F_-}) \in \mathbb{N}_0. \tag{2.20}$$

Note that the operator S_W can be decomposed as $S_W = S_{W_+ - F_-} - R_-$. Now we apply Lemma 2.9 (ii) with $C = S_W$, $D = S_{W_+ - F_-}$, $V = R_-$, $[v_-, v_+] = [0, \tau_+]$ and $\mathcal{I} = (t, \tau_+)$ for $t \in (0, \tau_+)$, and conclude together with (2.20) that
$$\mathcal{N}_{(t,\tau_+)}(S_W) \leq \mathcal{N}_{(t,2\tau_+)}(S_{W_+ - F_-}) = \mathcal{N}_{(t,\tau_+)}(S_{W_+ - F_-}) + r_1.$$

Since we only consider eigenvalues in the interval $[0, \tau_+)$ (see (2.16) and (2.17)) this estimate and (2.19) with $\ell := \ell_1 + r_1$ lead to

$$\lambda_k^+(S_W) \leq \lambda_{k-r_1}^+(S_{W_+-F_-}) \leq (1+\varepsilon)\mu_{k-\ell}^+$$

for all $k \in \mathbb{N}$ sufficiently large. $\square$

The last proposition of this subsection characterizes the total variation of the discrete spectrum under a trace class perturbation.

Proposition 2.11. [28, Corollary 5.1.2] *Let $C, D \in \mathfrak{B}(\mathcal{H})$ be self-adjoint operators such that $D - C \in \mathfrak{S}_1(\mathcal{H})$. Then*

$$\sum_{\lambda \in \sigma_{\rm disc}(C)} \operatorname{dist}(\lambda, \sigma(D)) < \infty.$$

The above proposition is a variant of an older theorem by T. Kato [55, Theorem II]. In this form, the statement is particularly convenient to apply for perturbed Landau Hamiltonians.

2.4. **A class of Toeplitz-type operators.** In this subsection we define and recall some well-known properties of Toeplitz-type operators related to Landau Hamiltonians. In the following let Σ be the boundary of a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^2$ and let $\Gamma \subset \Sigma$ be a closed subset of Σ. Note that Γ and Σ are both compact subsets of $\mathbb{R}^2$. In particular, Γ can be a subarc of Σ with two endpoints, a union of finitely many such subarcs, or coincide with Σ. The latter three geometric settings are of particular importance for our considerations. In fact, in our applications Γ is typically the essential support of the strength $\alpha \in L^\infty(\Sigma)$ of the δ-interaction for the Hamiltonian A_α. Recall that the (essential) support of α is a closed subset of Σ uniquely defined by

$$\operatorname{supp}\alpha := \Sigma \setminus \bigcup \{\sigma \subset \Sigma \colon \sigma \text{ is open and } \alpha = 0 \text{ a.e. in } \sigma\};$$

cf. [60, Section 1.5]. We introduce the Hilbert space $L^2(\Gamma)$ with the usual inner product $(\cdot,\cdot)_{L^2(\Gamma)}$, defined by means of the natural arc-length measure on Σ restricted to Γ. We denote by $|\Gamma|$ the arc-length measure of Γ, that is, the length of Γ. Corollary 2.3 implies that the trace mapping $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2) \ni u \mapsto u|_\Gamma \in L^2(\Gamma)$ is well defined and bounded.

We denote by $P_q \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $q \in \mathbb{N}_0$, the orthogonal projection onto the spectral subspace corresponding to the eigenvalue $\Lambda_q = B(2q+1)$ of the Landau Hamiltonian A_0; cf. Proposition 2.1. Following the lines of [70, Section 4], we introduce a family of Toeplitz-type operators, which correspond to the formal product $P_q \delta_\Gamma P_q$.

Proposition 2.12. *For all $q \in \mathbb{N}_0$ the symmetric sesquilinear form*

$$\mathfrak{t}_q^\Gamma[f,g] := \big((P_q f)|_\Gamma, (P_q g)|_\Gamma\big)_{L^2(\Gamma)}, \quad \operatorname{dom}\mathfrak{t}_q^\Gamma = L^2(\mathbb{R}^2), \tag{2.21}$$

is well defined and bounded.

Proof. Note that for any $f \in L^2(\mathbb{R}^2)$ we have

$$\mathfrak{t}_q^\Gamma[f] = \|(P_q f)|_\Gamma\|^2_{L^2(\Gamma)} \leq \|(P_q f)|_\Sigma\|^2_{L^2(\Sigma)} \leq \varepsilon \|\nabla_{\mathbf{A}} P_q f\|^2_{L^2(\mathbb{R}^2)} + c(\varepsilon)\|P_q f\|^2_{L^2(\mathbb{R}^2)}$$

with $\varepsilon > 0$ and $c(\varepsilon) > 0$ by Corollary 2.3. Using (2.2) and the first representation theorem we find

$$\|\nabla_{\mathbf{A}} P_q f\|^2_{L^2(\mathbb{R}^2)} = \mathfrak{a}_0[P_q f, P_q f] = (\mathsf{A}_0 P_q f, P_q f)_{L^2(\mathbb{R}^2)} = \Lambda_q \|P_q f\|^2_{L^2(\mathbb{R}^2)},$$

and hence $\mathfrak{t}_q^\Gamma[f] \leq c'(\varepsilon)\|P_q f\|^2_{L^2(\mathbb{R}^2)}$ for some $c'(\varepsilon) > 0$. This implies that the symmetric sesquilinear form $\mathfrak{t}_q$ is well defined and bounded. □

The Toeplitz-type operators we are interested in can now be defined.

Definition 2.13. *For $q \in \mathbb{N}_0$ the bounded self-adjoint operator in $L^2(\mathbb{R}^2)$ associated with the form $\mathfrak{t}_q^\Gamma$ in (2.21) is denoted by T_q^Γ.*

Note that $T_q^\Gamma = T_q^{\Gamma'}$ for closed subsets $\Gamma, \Gamma' \subset \Sigma$ that satisfy $|(\Gamma \setminus \Gamma') \cup (\Gamma' \setminus \Gamma)| = 0$ and that $T_q^\Gamma = 0$ if $|\Gamma| = 0$. Certain fundamental spectral properties of such Toeplitz-type operators were obtained in [40, 70]. The operators T_q^Γ can be viewed as variants of a better studied class of Toeplitz operators $P_q V P_q$, where $V\colon \mathbb{R}^2 \to \mathbb{R}$ is a regular function [40, 70, 71, 78]. Very roughly speaking in our considerations the δ-distribution supported on Γ plays the role of V. Before we provide some properties of T_q^Γ which are essential for our considerations we first introduce a notion from potential theory, see [59, §II.4], [82, Appendix A.VIII], and [44, §III.1].

Definition 2.14. *The* logarithmic energy *of a measure $\mu \geq 0$ on $\mathbb{R}^2$ is given by*

$$I(\mu) := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln \frac{1}{|x-y|} \mathsf{d}\mu(x)\mathsf{d}\mu(y).$$

The logarithmic capacity *of a compact set $\mathcal{K} \subset \mathbb{R}^2$ is defined by*

$$\operatorname{Cap}(\mathcal{K}) := \sup\left\{e^{-I(\mu)}\colon \mu \geq 0 \textit{ measure on } \mathbb{R}^2,\ \operatorname{supp}\mu \subset \mathcal{K},\ \mu(\mathcal{K}) = 1\right\}.$$

It is well known (see, e.g., [44, § III]) that the supremum in the definition of the logarithmic capacity is in fact a maximum. This maximum is attained by the so-called *equilibrium measure*. In the next proposition we collect some useful properties of the logarithmic capacity.

Proposition 2.15. [44, §III] *Let $\mathcal{K}, \mathcal{L} \subset \mathbb{R}^2$ be compact sets, let $\eta > 0$ and consider the compact set $U_\eta(\mathcal{K}) := \{x \in \mathbb{R}^2\colon \operatorname{dist}(x, \mathcal{K}) \leq \eta\}$. Then the following holds.*

(i) $\operatorname{Cap}(\mathcal{K}) \leq \operatorname{Cap}(\mathcal{L})$ *if* $\mathcal{K} \subset \mathcal{L}$.

(ii) $\operatorname{Cap}(U_\eta(\mathcal{K})) \to \operatorname{Cap}(\mathcal{K})$ *as* $\eta \to 0^+$.

Using the notion of logarithmic capacity of Γ one gets an asymptotic upper bound on the singular values of T_q^Γ and even exact asymptotics for them, provided that Γ is smooth. Note that the singular values of T_q^Γ coincide with its eigenvalues since

T_q^Γ is a self-adjoint nonnegative operator. Item (i) in the next proposition can be seen as consequence of [70, Proposition 4.1 (i)]. For the convenience of the reader we provide a short proof. Item (ii) coincides with [70, Proposition 4.1 (ii)].

Proposition 2.16. *Let $\Gamma \subset \Sigma$ be a closed subset with $|\Gamma| > 0$. Then the self-adjoint Toeplitz-type operator T_q^Γ, $q \in \mathbb{N}_0$, in Definition 2.13 is compact and its singular values satisfy:*

(i) $\limsup_{k\to\infty} \big(k!\, s_k(T_q^\Gamma)\big)^{1/k} \le \frac{B}{2}\big(\mathrm{Cap}\,(\Gamma)\big)^2$;

(ii) $\lim_{k\to\infty} \big(k!\, s_k(T_q^\Gamma)\big)^{1/k} = \frac{B}{2}\big(\mathrm{Cap}\,(\Gamma)\big)^2$ *if, in addition, Γ is a C^∞-smooth arc with two endpoints. In particular, the operator T_q^Γ is of infinite rank.*

Proof. (i) Denote by $U_\eta := U_\eta(\Gamma) \subset \mathbb{R}^2$ the η-neighborhood of Γ for $\eta > 0$ as in Proposition 2.15 and fix a cut-off function $\omega \in C_0^\infty(\mathbb{R}^2)$, $0 \le \omega \le 1$, such that $\omega \equiv 1$ on Γ and $\omega \equiv 0$ on $\mathbb{R}^2 \setminus U_\eta$.

For $f \in L^2(\mathbb{R}^2)$ the function $\omega P_q f$ belongs to $\mathrm{dom}\, \mathsf{A}_0$ and by Corollary 2.3 we have

$$\begin{aligned} \mathfrak{t}_q^\Gamma[f] = \|(P_q f)|_\Gamma\|^2_{L^2(\Gamma)} &= \|(\omega P_q f)|_\Gamma\|^2_{L^2(\Gamma)} \\ &\le \varepsilon \|\nabla_{\mathbf{A}} \omega P_q f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + c(\varepsilon)\|\omega P_q f\|^2_{L^2(\mathbb{R}^2)} \\ &\le \varepsilon \|\nabla_{\mathbf{A}} \omega P_q f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + c(\varepsilon)\|P_q f\|^2_{L^2(U_\eta)} \end{aligned} \tag{2.22}$$

for $\varepsilon > 0$ and suitable $c(\varepsilon) > 0$. For $f \in L^2(\mathbb{R}^2)$ it follows from [74, Proposition 4.2] that

$$\begin{aligned} \|\nabla_{\mathbf{A}} \omega P_q f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} &= (\mathsf{A}_0 P_q f, \omega^2 P_q f)_{L^2(\mathbb{R}^2)} + (|\nabla\omega|^2 P_q f, P_q f)_{L^2(\mathbb{R}^2)} \\ &= \Lambda_q(\omega^2 P_q f, P_q f)_{L^2(\mathbb{R}^2)} + (|\nabla\omega|^2 P_q f, P_q f)_{L^2(\mathbb{R}^2)} \\ &\le c'\|P_q f\|^2_{L^2(U_\eta)}, \end{aligned} \tag{2.23}$$

where we have also used that the supports of ω^2 and $|\nabla\omega|^2$ are contained in U_η and $c' > 0$ is some constant. Hence, if χ_η denotes the characteristic function of U_η we conclude from (2.22) and (2.23) the operator inequality

$$T_q^\Gamma \le c'' P_q \chi_\eta P_q, \qquad c'' = \varepsilon c' + c(\varepsilon).$$

Using [70, Proposition 4.1 (i)] we obtain that

$$\limsup_{k\to\infty} \big(k!\, s_k(T_q^\Gamma)\big)^{1/k} \le \limsup_{k\to\infty} \big(k!\, s_k(P_q\chi_\eta P_q)\big)^{1/k} = \frac{B}{2}\big(\mathrm{Cap}\,(U_\eta)\big)^2.$$

Finally, the desired inequality follows from Proposition 2.15 (ii) upon passing to the limit $\eta \to 0^+$.

The asymptotics in (ii) are shown in [70, Proposition 4.1 (ii)]. $\square$

It is a priori not clear that the rank of the Toeplitz-type operator T_q^Γ is infinite without extra regularity assumption on Γ. However, for $q = 0$ this claim can be deduced from a result by D. Luecking in [61] (see also its extension in [76]). To this aim, we define $\Psi(z) := \frac{1}{4}B|z|^2$ and consider the *Segal-Bargmann* (or *Fock*) space of analytic functions

$$\mathcal{F}^2 := \{f : \mathbb{C} \to \mathbb{C} : f \text{ is analytic}, e^{-\Psi} f \in L^2(\mathbb{C})\}.$$

Using an identification of $\mathbb{C}$ with $\mathbb{R}^2$ it was shown in [70, Section 4.2] that the multiplication operator

$$U : \mathcal{F}^2 \to L^2(\mathbb{R}^2), \qquad Uf := e^{-\Psi} f, \tag{2.24}$$

is unitary from the closed subspace $\mathcal{F}^2$ of $L^2(\mathbb{C}; e^{-2\Psi}\mathsf{d}z)$ onto the closed subspace $\operatorname{ran} P_0 = \ker(\mathsf{A}_0 - \Lambda_0)$ of $L^2(\mathbb{R}^2)$. Using this equivalence it follows easily that the rank of T_0^Γ is infinite.

Proposition 2.17. *Let $\Gamma \subset \Sigma$ be a closed subset with $|\Gamma| > 0$. Then the self-adjoint Toeplitz-type operator T_0^Γ $(q = 0)$ in Definition 2.13 has infinite rank.*

Proof. For $f, g \in \mathcal{F}^2$ we have

$$\left(U^* T_0^\Gamma U f, g\right)_{\mathcal{F}^2} = \left(T_0^\Gamma e^{-\Psi} f, e^{-\Psi} g\right)_{L^2(\mathbb{R}^2)} = \mathfrak{t}_0^\Gamma[e^{-\Psi} f, e^{-\Psi} g]$$

and since $e^{-\Psi} f, e^{-\Psi} g \in \operatorname{ran} P_0$ it follows that

$$\left(U^* T_0^\Gamma U f, g\right)_{\mathcal{F}^2} = \left((e^{-\Psi} f)|_\Gamma, (e^{-\Psi} g)|_\Gamma\right)_{L^2(\Gamma)} = \int_\Gamma e^{-2\Psi(z)} f(z)\overline{g(z)}\mathsf{d}\sigma(z). \tag{2.25}$$

If we define the compactly supported measure μ in $\mathbb{R}^2$ by

$$G \mapsto \mu(G) := \int_{G\cap\Gamma} \exp\bigl(-2\Psi(z)\bigr)\mathsf{d}\sigma(z), \qquad G \subset \mathbb{C} \simeq \mathbb{R}^2,$$

we can rewrite (2.25) as

$$\left(U^* T_0^\Gamma U f, g\right)_{\mathcal{F}^2} = \int_{\mathbb{C}} f(z)\overline{g(z)}\mathsf{d}\mu(z). \tag{2.26}$$

These considerations show that T_0^Γ is unitarily equivalent via U in (2.24) to the classical Toeplitz operator $T_\mu^{\mathcal{F}}$ on $\mathcal{F}^2$ defined via the symmetric sesquilinear form on the right hand side in (2.26). Note that the measure μ can not be represented as a sum of finitely many point measures. Therefore, the result by D. Luecking in [61, Theorem A (Restated)] shows that the operator $T_\mu^{\mathcal{F}}$, and hence also T_0^Γ, is of infinite rank. ☐

Later in this paper we show for the case $\Gamma = \Sigma$ in Corollary 5.4 that the rank of T_q^Σ is infinite for all $q \in \mathbb{N}$ with $C^{1,1}$-smooth Σ using a technique rather different from the one in [40, 70]. In this context we remark that one can go beyond $C^{1,1}$-smoothness up to a Lipschitz boundary by a small modification of the method.

3. A quasi boundary triple for Landau Hamiltonians

In this section we construct a quasi boundary triple which is suitable to define and study Landau Hamiltonians with δ-perturbations supported on $C^{1,1}$-curves. The notion of quasi boundary triples and their Weyl functions is recalled in Appendix A. From now on we shall assume that the following hypothesis holds.

Hypothesis 3.1. *Let Ω_i be a bounded $C^{1,1}$-domain with the boundary $\Sigma := \partial\Omega_\mathrm{i}$ and let $\Omega_\mathrm{e} := \mathbb{R}^2 \setminus \overline{\Omega_\mathrm{i}}$. The unit normal vector field pointing outward of Ω_i (and hence inward of Ω_e) will be denoted by ν.*

In the following, $\partial_\nu = \nu \cdot \nabla$ and $\partial_\nu^{\mathbf{A}} = -\mathrm{i}\nu \cdot \nabla_{\mathbf{A}} = \partial_\nu - \mathrm{i}\nu \cdot \mathbf{A}$ stand for the normal derivative and the magnetic normal derivative with respect to the normal vector ν pointing outward of Ω_i. Further, we set

$$\mathcal{D}_\mathrm{i} = H^{3/2}_\Delta(\Omega_\mathrm{i}) := \big\{ f_\mathrm{i} \in H^{3/2}(\Omega_\mathrm{i}) \colon \Delta f_\mathrm{i} \in L^2(\Omega_\mathrm{i}) \big\},$$

where the Laplacian is understood in the distributional sense. Recall that the Dirichlet and Neumann trace maps

$$\mathcal{D}_\mathrm{i} \ni f \mapsto f|_\Sigma \in H^1(\Sigma) \qquad \text{and} \qquad \mathcal{D}_\mathrm{i} \ni f \mapsto \partial_\nu f|_\Sigma \in L^2(\Sigma)$$

are bounded and surjective; cf. [45, Lemma 3.1 and 3.2]. Note that the spaces $H^{3/2}_\Delta$ appear also in [11] in the treatment of non-magnetic Schrödinger operators with δ-interactions.

In the next lemma we provide variants of the first and second Green identity in the present situation.

Lemma 3.2. *For $f_\mathrm{i}, g_\mathrm{i} \in \mathcal{D}_\mathrm{i}$ one has $\nabla^2_{\mathbf{A}} f_\mathrm{i}, \nabla^2_{\mathbf{A}} g_\mathrm{i} \in L^2(\Omega_\mathrm{i})$ and the following holds.*

(i) $(\nabla^2_{\mathbf{A}} f_\mathrm{i}, g_\mathrm{i})_{L^2(\Omega_\mathrm{i})} = (\nabla_{\mathbf{A}} f_\mathrm{i}, \nabla_{\mathbf{A}} g_\mathrm{i})_{L^2(\Omega_\mathrm{i};\mathbb{C}^2)} - (\partial_\nu^{\mathbf{A}} f_\mathrm{i}|_\Sigma, g_\mathrm{i}|_\Sigma)_{L^2(\Sigma)}$.

(ii) $(\nabla^2_{\mathbf{A}} f_\mathrm{i}, g_\mathrm{i})_{L^2(\Omega_\mathrm{i})} - (f_\mathrm{i}, \nabla^2_{\mathbf{A}} g_\mathrm{i})_{L^2(\Omega_\mathrm{i})} = (f_\mathrm{i}|_\Sigma, \partial_\nu^{\mathbf{A}} g_\mathrm{i}|_\Sigma)_{L^2(\Sigma)} - (\partial_\nu^{\mathbf{A}} f_\mathrm{i}|_\Sigma, g_\mathrm{i}|_\Sigma)_{L^2(\Sigma)}$.

Proof. For $f_\mathrm{i} \in \mathcal{D}_\mathrm{i}$ and all $h_\mathrm{i} \in C_0^\infty(\Omega_\mathrm{i})$ one has

$$\begin{aligned}\big(f_\mathrm{i}, \nabla^2_{\mathbf{A}} h_\mathrm{i}\big)_{L^2(\Omega_\mathrm{i})} &= \big(f_\mathrm{i}, (-\Delta + 2\mathrm{i}\mathbf{A}\cdot\nabla + \mathbf{A}^2) h_\mathrm{i}\big)_{L^2(\Omega_\mathrm{i})} \\ &= \big(-\Delta f_\mathrm{i}, h_\mathrm{i}\big)_{L^2(\Omega_\mathrm{i})} + \big((2\mathrm{i}\mathbf{A}\cdot\nabla + \mathbf{A}^2) f_\mathrm{i}, h_\mathrm{i}\big)_{L^2(\Omega_\mathrm{i})},\end{aligned}$$

where $\nabla \cdot \mathbf{A} = 0$ and also $\mathcal{H}^1_{\mathbf{A}}(\Omega_\mathrm{i}) = H^1(\Omega_\mathrm{i})$ were used. This shows

$$\nabla^2_{\mathbf{A}} f_\mathrm{i} = -\Delta f_\mathrm{i} + (2\mathrm{i}\mathbf{A}\cdot\nabla + \mathbf{A}^2) f_\mathrm{i} \in L^2(\Omega_\mathrm{i}).$$

It follows from the divergence theorem and the particular form of $\mathbf{A}$ that

$$\mathcal{B}[f_\mathrm{i}, g_\mathrm{i}] := (\mathrm{i}\nabla f_\mathrm{i}, \mathbf{A} g_\mathrm{i})_{L^2(\Omega_\mathrm{i};\mathbb{C}^2)} - (\mathbf{A} f_\mathrm{i}, \mathrm{i}\nabla g_\mathrm{i})_{L^2(\Omega_\mathrm{i};\mathbb{C}^2)} = (\mathrm{i}(\nu\cdot\mathbf{A} f_\mathrm{i})|_\Sigma, g_\mathrm{i}|_\Sigma)_{L^2(\Sigma)}$$

holds for $f_{\rm i}, g_{\rm i} \in \mathcal{D}_{\rm i}$. Now a simple computation

$$\begin{aligned}(\nabla_{\mathbf{A}} f_{\rm i}, &\nabla_{\mathbf{A}} g_{\rm i})_{L^2(\Omega_{\rm i};\mathbb{C}^2)} - (\nabla^2_{\mathbf{A}} f_{\rm i}, g_{\rm i})_{L^2(\Omega_{\rm i})} \\ &= \big[(\nabla f_{\rm i}, \nabla g_{\rm i})_{L^2(\Omega_{\rm i};\mathbb{C}^2)} - \big(-\Delta f_{\rm i}, g_{\rm i}\big)_{L^2(\Omega_{\rm i})}\big] - \mathcal{B}[f_{\rm i}, g_{\rm i}] \\ &= (\partial_\nu f_{\rm i}|_\Sigma, g_{\rm i}|_\Sigma)_{L^2(\Sigma)} - ({\rm i}(\nu\cdot \mathbf{A} f_{\rm i})|_\Sigma, g_{\rm i}|_\Sigma)_{L^2(\Sigma)} = (\partial^{\mathbf{A}}_\nu f_{\rm i}|_\Sigma, g_{\rm i}|_\Sigma)_{L^2(\Sigma)}\end{aligned}$$

yields the identity in (i). The identity in (ii) follows from (i). □

In order to define an appropriate counterpart of the space $\mathcal{D}_{\rm i}$ on the exterior domain $\Omega_{\rm e}$ one has to pay some attention to the properties of the functions in a neighborhood of ∞. This leads to the following construction. Fix some bounded open set K such that $\overline{\Omega_{\rm i}} \subset K$ and define

$$\mathcal{D}_{\rm e} := \big\{f_{\rm e} \in \mathcal{H}^1_{\mathbf{A}}(\Omega_{\rm e})\colon \nabla^2_{\mathbf{A}} f_{\rm e} \in L^2(\Omega_{\rm e}), f_{\rm e} \upharpoonright (K\cap\Omega_{\rm e}) \in H^{3/2}_\Delta(K\cap\Omega_{\rm e})\big\},$$

where $H^{3/2}_\Delta(K\cap\Omega_{\rm e}) := \{h \in H^{3/2}(K\cap\Omega_{\rm e})\colon \Delta h \in L^2(K\cap\Omega_{\rm e})\}$. Using [45, Lemma 3.1 and 3.2] one checks that the Dirichlet and Neumann trace maps

$$\mathcal{D}_{\rm e} \ni f \mapsto f|_\Sigma \in H^1(\Sigma) \qquad \text{and} \qquad \mathcal{D}_{\rm e} \ni f \mapsto \partial_\nu f|_\Sigma \in L^2(\Sigma)$$

are bounded and surjective.

In the same way as in Lemma 3.2 one obtains the following statements. Observe that ν is pointing inwards in $\Omega_{\rm e}$, which leads to different signs compared to Lemma 3.2.

Lemma 3.3. *For $f_{\rm e}, g_{\rm e} \in \mathcal{D}_{\rm e}$ the following holds.*

(i) $(\nabla^2_{\mathbf{A}} f_{\rm e}, g_{\rm e})_{L^2(\Omega_{\rm e})} = (\nabla_{\mathbf{A}} f_{\rm e}, \nabla_{\mathbf{A}} g_{\rm e})_{L^2(\Omega_{\rm e};\mathbb{C}^2)} + (\partial^{\mathbf{A}}_\nu f_{\rm e}|_\Sigma, g_{\rm e}|_\Sigma)_{L^2(\Sigma)}$.

(ii) $(\nabla^2_{\mathbf{A}} f_{\rm e}, g_{\rm e})_{L^2(\Omega_{\rm e})} - (f_{\rm e}, \nabla^2_{\mathbf{A}} g_{\rm e})_{L^2(\Omega_{\rm e})} = -(f_{\rm e}|_\Sigma, \partial^{\mathbf{A}}_\nu g_{\rm e}|_\Sigma)_{L^2(\Sigma)} + (\partial^{\mathbf{A}}_\nu f_{\rm e}|_\Sigma, g_{\rm e}|_\Sigma)_{L^2(\Sigma)}$.

Next, we introduce the operator T acting in $L^2(\mathbb{R}^2)$ by

$$Tf := \nabla^2_{\mathbf{A}} f_{\rm i} \oplus \nabla^2_{\mathbf{A}} f_{\rm e}, \quad \operatorname{dom} T := \big\{f = f_{\rm i}\oplus f_{\rm e} \in \mathcal{D}_{\rm i}\oplus\mathcal{D}_{\rm e}\colon f_{\rm i}|_\Sigma = f_{\rm e}|_\Sigma\big\},$$

and the trace mappings $\Gamma_0, \Gamma_1\colon \operatorname{dom} T \to L^2(\Sigma)$ by

$$\Gamma_0 f := \partial^{\mathbf{A}}_\nu f_{\rm i}|_\Sigma - \partial^{\mathbf{A}}_\nu f_{\rm e}|_\Sigma = \partial_\nu f_{\rm i}|_\Sigma - \partial_\nu f_{\rm e}|_\Sigma \qquad \text{and} \qquad \Gamma_1 f := f|_\Sigma. \tag{3.1}$$

Then we have the following result, which is important for our further investigations in the next section.

Theorem 3.4. *Let T be as above and define*

$$S := \mathsf{A}_0 \upharpoonright \big\{f \in \mathcal{H}^2_{\mathbf{A}}(\mathbb{R}^2)\colon f|_\Sigma = 0\big\}.$$

Then S is a densely defined, closed, symmetric operator and $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $T \subset S^*$. Moreover, $T \upharpoonright \ker\Gamma_0$ coincides with the Landau Hamiltonian A_0 and $\operatorname{ran}\Gamma_0 = L^2(\Sigma)$.

Proof. We apply Theorem A.2 to prove the claim. Using that the traces of $f_\mathrm{i}, f_\mathrm{e}$ and $g_\mathrm{i}, g_\mathrm{e}$ coincide on Σ for $f, g \in \operatorname{dom} T$, we get from Lemma 3.2 (ii) and Lemma 3.3 (ii) that

$$\begin{aligned}
&(Tf, g)_{L^2(\mathbb{R}^2)} - (f, Tg)_{L^2(\mathbb{R}^2)} \\
&\quad = (\nabla^2_{\mathbf{A}} f_\mathrm{i}, g_\mathrm{i})_{L^2(\Omega_\mathrm{i})} - (f_\mathrm{i}, \nabla^2_{\mathbf{A}} g_\mathrm{i})_{L^2(\Omega_\mathrm{i})} + (\nabla^2_{\mathbf{A}} f_\mathrm{e}, g_\mathrm{e})_{L^2(\Omega_\mathrm{e})} - (f_\mathrm{e}, \nabla^2_{\mathbf{A}} g_\mathrm{e})_{L^2(\Omega_\mathrm{e})} \\
&\quad = (f_\mathrm{i}|_\Sigma, \partial_\nu^{\mathbf{A}} g_\mathrm{i}|_\Sigma)_{L^2(\Sigma)} - (\partial_\nu^{\mathbf{A}} f_\mathrm{i}|_\Sigma, g_\mathrm{i}|_\Sigma)_{L^2(\Sigma)} - (f_\mathrm{e}|_\Sigma, \partial_\nu^{\mathbf{A}} g_\mathrm{e}|_\Sigma)_{L^2(\Sigma)} + (\partial_\nu^{\mathbf{A}} f_\mathrm{e}|_\Sigma, g_\mathrm{e}|_\Sigma)_{L^2(\Sigma)} \\
&\quad = (f|_\Sigma, \partial_\nu^{\mathbf{A}} g_\mathrm{i}|_\Sigma - \partial_\nu^{\mathbf{A}} g_\mathrm{e}|_\Sigma)_{L^2(\Sigma)} - (\partial_\nu^{\mathbf{A}} f_\mathrm{i}|_\Sigma - \partial_\nu^{\mathbf{A}} f_\mathrm{e}|_\Sigma, g|_\Sigma)_{L^2(\Sigma)} \\
&\quad = (\Gamma_1 f, \Gamma_0 g)_{L^2(\Sigma)} - (\Gamma_0 f, \Gamma_1 g)_{L^2(\Sigma)},
\end{aligned}$$

that is, the Green identity holds.

Next, it follows from the Green identity that the operator $T \upharpoonright \ker \Gamma_0$ is symmetric in $L^2(\mathbb{R}^2)$. It is easy to see that the self-adjoint Landau Hamiltonian A_0 is contained in $T \upharpoonright \ker \Gamma_0$ and consequently $\mathsf{A}_0 = T \upharpoonright \ker \Gamma_0$. Furthermore, let $\chi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function which is identically equal to one in a neighborhood of Ω_i and set $\chi_\mathrm{e} = \chi|_{\Omega_\mathrm{e}}$. Then the space

$$\left\{ \begin{pmatrix} f_\mathrm{i} \\ \chi_\mathrm{e} f_\mathrm{e} \end{pmatrix} : f_\mathrm{i} \in H^2(\Omega_\mathrm{i}),\ f_\mathrm{e} \in H^2(\Omega_\mathrm{e}), f_\mathrm{e}|_\Sigma = f_\mathrm{i}|_\Sigma \right\}$$

is contained in $\operatorname{dom} T$. Thus, it follows from the properties of the trace mappings in [63, Theorem 3] that

$$H^{1/2}(\Sigma) \times H^{3/2}(\Sigma) \subset \operatorname{ran} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix},$$

i.e. $\operatorname{ran}(\Gamma_0, \Gamma_1)^\top$ is dense in $L^2(\Sigma) \times L^2(\Sigma)$. Furthermore, since $C_0^\infty(\mathbb{R}^2 \setminus \Sigma) \subset \operatorname{dom} S$, it is clear that also $\ker(\Gamma_0, \Gamma_1)^\top = \operatorname{dom} S$ is dense in $L^2(\mathbb{R}^2)$.

Finally, to show that Γ_0 is surjective we use the single layer potential $\mathsf{SL}\colon L^2(\Sigma) \to L^2(\mathbb{R}^2)$ associated to Σ and the Helmholtz equation $-\Delta + 1$; cf. [64, Chapter 6]. To be more precise, for $\varphi \in L^2(\Sigma)$ define the function $f := \widetilde{\chi}\mathsf{SL}\varphi$, where $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ is a cutoff function such that $\chi \equiv 1$ in a neighborhood of Σ. Then using the properties of the single layer potential from [64, Theorem 6.11 and Theorem 6.13] we see that f belongs to $\operatorname{dom} T$ and $\Gamma_0 f = \varphi$. Now Theorem A.2 leads to the assertions. $\square$

In the next step we compute the γ-field and the Weyl function associated to the quasi boundary triple $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ from Theorem 3.4. Recall that G_λ in (2.4) is the integral kernel of the resolvent of the Landau Hamiltonian.

Proposition 3.5. *Let $\lambda \in \rho(\mathsf{A}_0)$ and let G_λ be given by (2.4). Then the values of the γ-field $\gamma(\lambda)$ and of the Weyl function $M(\lambda)$ satisfy the following.*

(i) *The operator $\gamma(\lambda) \in \mathfrak{B}(L^2(\Sigma), L^2(\mathbb{R}^2))$ is given by*

$$\gamma(\lambda)\varphi(x) = \int_\Sigma G_\lambda(x, y)\varphi(y) \mathrm{d}\sigma(y), \quad \varphi \in L^2(\Sigma),\ x \in \mathbb{R}^2,$$

and belongs to the weak Schatten-von Neumann ideal $\mathfrak{S}_{2/3,\infty}(L^2(\Sigma), L^2(\mathbb{R}^2))$.

(ii) *The adjoint operator* $\gamma(\lambda)^* \in \mathfrak{B}(L^2(\mathbb{R}^2), L^2(\Sigma))$ *is given by*

$$\gamma(\lambda)^* f(x) = \int_{\mathbb{R}^2} G_{\overline{\lambda}}(x,y) f(y) \mathsf{d}y, \quad f \in L^2(\mathbb{R}^2),\, x \in \Sigma,$$

and belongs to the weak Schatten-von Neumann ideal $\mathfrak{S}_{2/3,\infty}(L^2(\mathbb{R}^2), L^2(\Sigma))$.

(iii) *The operator* $M(\lambda) \in \mathfrak{B}(L^2(\Sigma))$ *is given by*

$$M(\lambda)\varphi(x) = \int_{\Sigma} G_\lambda(x,y)\varphi(y) \mathsf{d}\sigma(y), \quad \varphi \in L^2(\Sigma),\, x \in \Sigma,$$

and belongs to the weak Schatten-von Neumann ideal $\mathfrak{S}_{1,\infty}(L^2(\Sigma))$.

In particular, the operators $\gamma(\lambda)$, $\gamma(\lambda)^*$, *and* $M(\lambda)$ *are compact.*

Proof. First, we verify statement (ii). Since $\gamma(\lambda)^* = \Gamma_1(\mathsf{A}_0 - \overline{\lambda})^{-1}$, the representation of $\gamma(\lambda)^*$ follows directly from the form of the resolvent of A_0 in Proposition 2.1. Moreover, as $\operatorname{ran}(\mathsf{A}_0 - \overline{\lambda})^{-1} = \operatorname{dom}\mathsf{A}_0 = \mathcal{H}^2_{\mathbf{A}}(\mathbb{R}^2)$, and since this space coincides locally with $H^2(\mathbb{R}^2)$, we conclude from the boundedness of Σ and the mapping properties of the trace map that $\operatorname{ran}\gamma(\lambda)^* = \Gamma_1(H^2(\mathbb{R}^2)) = H^{3/2}(\Sigma)$. Therefore, Proposition 2.5 with $k=1$ and $l=3$ shows $\gamma(\lambda)^* \in \mathfrak{S}_{2/3,\infty}(L^2(\mathbb{R}^2), L^2(\Sigma))$.

The claim of item (i) follows from (ii) by taking adjoints, as $\overline{G_{\overline{\lambda}}}(y,x) = G_\lambda(x,y)$ and $\operatorname{dom}\gamma(\lambda) = \operatorname{ran}\Gamma_0 = L^2(\Sigma)$.

Finally, the representation of the Weyl function follows immediately from $M(\lambda) = \Gamma_1\gamma(\lambda)$ and item (i). In particular, since $\operatorname{ran} M(\lambda) \subset \operatorname{ran}\Gamma_1 \subset H^1(\Sigma)$ we conclude from Proposition 2.5 with $k=1$ and $l=2$ that $M(\lambda) \in \mathfrak{S}_{1,\infty}(L^2(\Sigma))$. □

Next, we provide a useful estimate on the decay of the Weyl function M, which is an application of Theorem A.5 for the quasi boundary triple in Theorem 3.4. Recall that $\min\sigma(\mathsf{A}_0) = B \geq 0$; cf. Proposition 2.1.

Proposition 3.6. *For all* $\varepsilon \in (0, \frac{1}{2})$ *and all* $w_0 < B$ *there exists a constant* $D > 0$ *such that*

$$\|M(\lambda)\| \leq \frac{D}{|\lambda - B|^{1/2-\varepsilon}}, \qquad \lambda < w_0.$$

Proof. Let $w_0 < B$ and fix $\lambda < w_0$. We check that the operator $\Gamma_1(\mathsf{A}_0 - \lambda)^{-\beta}$ is bounded and everywhere defined for $\beta = \frac{1}{4} + \frac{\varepsilon}{2}$. In fact, let $-\Delta$ be the free Laplacian defined on $H^2(\mathbb{R}^2)$ and let $f \in L^2(\mathbb{R}^2)$. Using the diamagnetic inequality (2.5), the trace theorem and the boundedness of $(-\Delta - \lambda)^{-\beta}\colon L^2(\mathbb{R}^2) \to H^{2\beta}(\mathbb{R}^2)$ we find constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \big\|\Gamma_1(\mathsf{A}_0 - \lambda)^{-\beta} f\big\|^2_{L^2(\Sigma)} &= \int_\Sigma \big|(\mathsf{A}_0 - \lambda)^{-\beta} f\big|^2 \mathsf{d}\sigma \leq \int_\Sigma \big|(-\Delta - \lambda)^{-\beta}|f|\big|^2 \mathsf{d}\sigma \\ &= \big\|\big((-\Delta - \lambda)^{-\beta}|f|\big)\big|_\Sigma\big\|^2_{L^2(\Sigma)} \leq C_1 \big\|(-\Delta - \lambda)^{-\beta}|f|\big\|^2_{H^{2\beta}(\mathbb{R}^2)} \\ &\leq C_2 \|f\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Hence $\Gamma_1(\mathsf{A}_0 - \lambda)^{-\beta}$ is bounded. Now Theorem A.5 leads to the assertion. □

Finally, we provide an auxiliary lemma which is essential in the proof of Proposition 4.9. Recall that $\mathsf{A}_{\rm D}^{\Omega_{\rm i}}$ denotes the Landau Hamiltonian in $\Omega_{\rm i}$ with Dirichlet boundary conditions, which was defined via the quadratic form in (2.7). Since $\Omega_{\rm i}$ is bounded one has $\sigma_{\rm ess}(\mathsf{A}_{\rm D}^{\Omega}) = \varnothing$; cf. (2.8).

Lemma 3.7. *For any $q \in \mathbb{N}_0$ one has*

$$\dim\ker(S - \Lambda_q) \leq \dim\ker(\mathsf{A}_{\rm D}^{\Omega_{\rm i}} - \Lambda_q)$$

and, in particular, the space $\ker(S - \Lambda_q)$ is finite-dimensional.

Proof. Assume that $\dim\ker(\mathsf{A}_{\rm D}^{\Omega_{\rm i}} - \Lambda_q) = k$ for some $k \in \mathbb{N}_0$ and suppose that $h_1, \dots, h_{k+1} \in \ker(S - \Lambda_q)$ are linearly independent. Set $h_j^{\rm i} = h_j|_{\Omega_{\rm i}}$ and $h_j^{\rm e} = h_j|_{\Omega_{\rm e}}$ for $j = 1, 2, \dots, k+1$. It is clear that $h_1^{\rm i}, \dots, h_{k+1}^{\rm i} \in \ker(\mathsf{A}_{\rm D}^{\Omega_{\rm i}} - \Lambda_q)$ and hence we conclude without loss of generality that there exist $\beta_1, \dots, \beta_k \in \mathbb{C}$ such that

$$h_{k+1}^{\rm i} = \sum_{j=1}^{k} \beta_j h_j^{\rm i}. \tag{3.2}$$

Note that also $h_1^{\rm e}, \dots, h_{k+1}^{\rm e} \in \ker(\mathsf{A}_{\rm D}^{\Omega_{\rm e}} - \Lambda_q)$ and as $h_1, \dots, h_{k+1} \in \operatorname{dom} S$ it follows that

$$\partial_\nu h_j^{\rm e}|_\Sigma = \partial_\nu h_j^{\rm i}|_\Sigma, \qquad j = 1, \dots, k+1.$$

Now observe that for the function

$$g^{\rm e} := h_{k+1}^{\rm e} - \sum_{j=1}^{k} \beta_j h_j^{\rm e} \in \ker(\mathsf{A}_{\rm D}^{\Omega_{\rm e}} - \Lambda_q)$$

one has by (3.2)

$$\partial_\nu g^{\rm e}|_\Sigma = \partial_\nu h_{k+1}^{\rm e}|_\Sigma - \sum_{j=1}^{k} \beta_j \partial_\nu h_j^{\rm e}|_\Sigma = \partial_\nu h_{k+1}^{\rm i}|_\Sigma - \sum_{j=1}^{k} \beta_j \partial_\nu h_j^{\rm i}|_\Sigma = 0$$

and hence unique continuation [86] (see also the proof of Proposition 2.5 in [14]) yields $g^{\rm e} = 0$. But this implies

$$h_{k+1}^{\rm e} = \sum_{j=1}^{k} \beta_j h_j^{\rm e}$$

and together with (3.2) we conclude

$$h_{k+1} = \sum_{j=1}^{k} \beta_j h_j;$$

a contradiction, since by assumption the functions $h_1, \dots, h_{k+1}$ are linearly independent. □

4. Landau Hamiltonians with singular potentials

In this section we define and study the Landau Hamiltonian A_α with a δ-potential supported on Σ with a position-dependent real strength $\alpha \in L^\infty(\Sigma)$. We shall use the quasi boundary triple $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ from Theorem 3.4 and its γ-field and Weyl function to derive various properties for the operator A_α and its resolvent. As in the previous section we assume that Hypothesis 3.1 holds.

4.1. Definition of A_α, self-adjointness, and qualitative spectral properties.

Let us start with the rigorous definition of A_α.

Definition 4.1. *Let $\alpha \in L^\infty(\Sigma)$ be a real function. The Landau Hamiltonian with δ-potential of strength α supported on Σ is defined as the operator $\mathsf{A}_\alpha := T \upharpoonright \ker(\Gamma_0 + \alpha\Gamma_1)$ in $L^2(\mathbb{R}^2)$, or, more explicitly*

$$
\begin{aligned}
\mathsf{A}_\alpha f &:= (\nabla^2_{\mathbf{A}} f_{\mathrm{i}}) \oplus (\nabla^2_{\mathbf{A}} f_{\mathrm{e}}) \\
\operatorname{dom} \mathsf{A}_\alpha &:= \big\{ f = f_{\mathrm{i}} \oplus f_{\mathrm{e}} \in \mathcal{D}_{\mathrm{i}} \oplus \mathcal{D}_{\mathrm{e}} \colon f_{\mathrm{i}}|_\Sigma = f_{\mathrm{e}}|_\Sigma, \partial_\nu f_{\mathrm{e}}|_\Sigma - \partial_\nu f_{\mathrm{i}}|_\Sigma = \alpha f|_\Sigma \big\}.
\end{aligned} \tag{4.1}
$$

Note that the jump of the normal derivatives $\partial_\nu f_{\mathrm{e}}|_\Sigma - \partial_\nu f_{\mathrm{i}}|_\Sigma$ in (4.1) can also be replaced by the jump of the magnetic normal derivatives $\partial^{\mathbf{A}}_\nu f_{\mathrm{e}}|_\Sigma - \partial^{\mathbf{A}}_\nu f_{\mathrm{i}}|_\Sigma$; cf. (3.1).

In the next theorem we prove that A_α is self-adjoint, obtain a version of the Birman-Schwinger principle, and derive a Krein-type resolvent formula, which also implies that the resolvent difference of A_α and A_0 is compact. Moreover, we estimate the decay of the singular values for this resolvent difference. As a direct consequence, we obtain a characterisation of the essential spectrum for A_α.

Theorem 4.2. *Let $\{L^2(\Sigma), \Gamma_0, \Gamma_1\}$ be the quasi boundary triple from Theorem 3.4 with $\mathsf{A}_0 = T \upharpoonright \ker \Gamma_0$, γ-field γ and Weyl function M. Let $\alpha \in L^\infty(\Sigma)$ be real and let A_α be as in Definition 4.1. Then the following assertions hold.*

(i) *A_α is a self-adjoint operator in $L^2(\mathbb{R}^2)$.*

(ii) *$\lambda \notin \sigma(\mathsf{A}_0)$ is an eigenvalue of A_α if and only if $-1 \in \sigma_{\mathrm{p}}(\alpha M(\lambda))$.*

(iii) *For all $\lambda \in \rho(\mathsf{A}_\alpha) \cap \rho(\mathsf{A}_0)$ one has $(1 + \alpha M(\lambda))^{-1} \in \mathfrak{B}(L^2(\Sigma))$ and*

$$
(\mathsf{A}_\alpha - \lambda)^{-1} - (\mathsf{A}_0 - \lambda)^{-1} = -\gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1} \alpha \gamma(\overline{\lambda})^*. \tag{4.2}
$$

(iv) *For all $\lambda \in \rho(\mathsf{A}_\alpha) \cap \rho(\mathsf{A}_0)$ the singular values s_k of the resolvent difference (4.2) are in $\mathcal{O}(k^{-3})$ and, in particular, the operator (4.2) is in $\mathfrak{S}_p(L^2(\mathbb{R}^2))$ for all $p > \frac{1}{3}$.*

(v) *$\sigma_{\mathrm{ess}}(\mathsf{A}_\alpha) = \sigma_{\mathrm{ess}}(\mathsf{A}_0) = \sigma(\mathsf{A}_0) = \{B(2q+1) \colon q \in \mathbb{N}_0\}$.*

Proof. Items (i)–(iii) follow from Corollary A.4 with $B = -\alpha$. In fact, we have $\|\alpha M(\lambda_0)\| < 1$ for $\lambda_0 < 0$ with sufficiently large absolute value using $\alpha \in L^\infty(\Sigma)$ and Proposition 3.6. To prove (iv) note that $(1 + \alpha M(\lambda))^{-1}\alpha \in \mathfrak{B}(L^2(\Sigma))$. By Proposition 3.5 we have $\gamma(\lambda) \in \mathfrak{S}_{2/3,\infty}(L^2(\Sigma), L^2(\mathbb{R}^2))$ and $\gamma(\overline{\lambda})^* \in \mathfrak{S}_{2/3,\infty}(L^2(\mathbb{R}^2), L^2(\Sigma))$, and together with (2.9) this implies (iv). Finally, (v) is an immediate consequence of (iv) and well-known perturbation results. ☐

Remark 4.3. The estimate of the singular values in Theorem 4.2 (iv) is known to be sharp in the absence of a magnetic field (that is, $B = 0$) if both Σ and α are C^∞-smooth; cf. [11, Theorem C (i)] and [8, Theorem 5.1]. The magnetic case is new in this setting. A similar estimate for the magnetic Robin Laplacian on an exterior domain is contained in [47, Lemma 2.2 and Remark 2.4].

In the following proposition we show that A_α can also be defined as the self-adjoint operator corresponding to the quadratic form $\mathfrak{a}_\alpha$ in (1.2); cf. [66].

Proposition 4.4. *The symmetric sesquilinear form* $\mathfrak{a}_\alpha$

$$\mathfrak{a}_\alpha[f,g] = \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \big(\alpha f|_\Sigma, g|_\Sigma\big)_{L^2(\Sigma)}, \quad \operatorname{dom}\mathfrak{a}_\alpha = \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2), \tag{4.3}$$

is densely defined, closed, bounded from below, and $C_0^\infty(\mathbb{R}^2)$ *is a core for* $\mathfrak{a}_\alpha$*. The corresponding self-adjoint operator coincides with* A_α *in Definition 4.1 and, in particular, the operator* A_α *is bounded from below and satisfies* $\min\sigma(\mathsf{A}_\alpha) \le \min\sigma(\mathsf{A}_0) = B$.

Proof. Recall first that the form $\mathfrak{a}_0$ corresponding to the Landau Hamiltonian in (2.2) is densely defined, nonnegative, closed, and $C_0^\infty(\mathbb{R}^2)$ is a core for $\mathfrak{a}_0$. Consider the form

$$\mathfrak{b}_\alpha[f,g] := \int_\Sigma \alpha f|_\Sigma \overline{g|_\Sigma}\, \mathrm{d}\sigma, \qquad \operatorname{dom}\mathfrak{b}_\alpha := \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2),$$

and note that $\mathfrak{b}_\alpha$ is well defined by Corollary 2.3. It is clear that $\mathfrak{a}_\alpha = \mathfrak{a}_0 + \mathfrak{b}_\alpha$ is densely defined. Choose $\varepsilon > 0$ such that $\varepsilon\|\alpha\|_{L^\infty(\Sigma)} < 1$. Then by Corollary 2.3

$$\begin{aligned} \big|\mathfrak{b}_\alpha[f]\big| &\le \int_\Sigma |\alpha||f|_\Sigma|^2 \mathrm{d}\sigma \le \|\alpha\|_{L^\infty(\Sigma)}\|f\|^2_{L^2(\Sigma)} \\ &\le \varepsilon\|\alpha\|_{L^\infty(\Sigma)}\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + c(\varepsilon)\|\alpha\|_{L^\infty(\Sigma)}\|f\|^2_{L^2(\mathbb{R}^2)} \end{aligned} \tag{4.4}$$

holds for all $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. Therefore, $\mathfrak{b}_\alpha$ is form bounded with respect to $\mathfrak{a}_0$ with form bound less than one and hence the KLMN theorem (see [75, Theorem X.17] or [54, §6 Theorem 1.33 and Theorem 2.1]) implies that $\mathfrak{a}_\alpha$ is closed, bounded from below, and $C_0^\infty(\mathbb{R}^2)$ is a core of $\mathfrak{a}_\alpha$.

In order to show that the corresponding self-adjoint operator coincides with A_α let $f \in \operatorname{dom}\mathsf{A}_\alpha \subset \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ and $g \in C_0^\infty(\mathbb{R}^2)$. Then

$$\alpha f|_\Sigma = \partial_\nu f_{\rm e}|_\Sigma - \partial_\nu f_{\rm i}|_\Sigma = \partial_\nu^{\mathbf{A}} f_{\rm e}|_\Sigma - \partial_\nu^{\mathbf{A}} f_{\rm i}|_\Sigma$$

and hence it follows from Lemma 3.2 and Lemma 3.3 that

$$(\mathsf{A}_\alpha f, g)_{L^2(\mathbb{R}^2)} = \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \big(\partial_\nu^{\mathbf{A}} f_{\rm e}|_\Sigma - \partial_\nu^{\mathbf{A}} f_{\rm i}|_\Sigma, g|_\Sigma\big)_{L^2(\Sigma)} = \mathfrak{a}_\alpha[f,g].$$

Since $C_0^\infty(\mathbb{R}^2)$ is a core for $\mathfrak{a}_\alpha$ it follows from the first representation theorem [54, §6 Theorem 2.1] that the self-adjoint operator A_α is contained in the self-adjoint operator representing the form $\mathfrak{a}_\alpha$, and hence both coincide. This also implies that

A_α is bounded from below (with the same lower bound as the form $\mathfrak{a}_\alpha$) and the inequality $\min \sigma(\mathsf{A}_\alpha) \le \min \sigma(\mathsf{A}_0) = B$ follows from Proposition 2.1. $\square$

For later use we note here a simple consequence of Proposition 4.4: it follows from (4.4) that there are constants C_1, C_2 with $C_1 \in (0,1)$ such that

$$\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} = \mathfrak{a}_\alpha[f] - \mathfrak{b}_\alpha[f] \le \mathfrak{a}_\alpha[f] + C_1\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + C_2\|f\|^2_{L^2(\mathbb{R}^2)}$$

holds for all $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$, where $\mathfrak{b}_\alpha$ is defined as in the proof above. Hence, there exist constants $c_1, c_2 > 0$ such that

$$\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} \le c_1 \mathfrak{a}_\alpha[f] + c_2\|f\|^2_{L^2(\mathbb{R}^2)}, \quad f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2). \tag{4.5}$$

4.2. **Approximation of A_α by Landau Hamiltonians with regular potentials.** Before we proceed further with the spectral analysis of A_α, we show that this operator can be regarded as the limit of a family of Landau Hamiltonians with squeezed regular potentials which are supported in a small neighborhood of the interaction support Σ. This justifies A_α as an idealized model for Landau Hamiltonians with regular potentials localized in a neighborhood of Σ.

In order to avoid complicated notation and technical difficulties we discuss the case that the bounded $C^{1,1}$-domain $\Omega_{\rm i}$ is simply connected, so that the boundary $\Sigma = \partial\Omega_{\rm i}$ is given by one regular, closed $C^{1,1}$-curve in $\mathbb{R}^2$ without self-intersections. The more general case can be treated in a similar way. For $\varepsilon > 0$ we define

$$\Sigma_\varepsilon := \big\{ x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma, t \in (-\varepsilon, \varepsilon) \big\}.$$

Since Σ is a closed and bounded $C^{1,1}$-curve, there exists some $\beta > 0$ such that the mapping

$$\Sigma \times (-\varepsilon, \varepsilon) \ni (x_\Sigma, t) \mapsto x_\Sigma + t\nu(x_\Sigma) \in \Sigma_\varepsilon \tag{4.6}$$

is bijective for all $\varepsilon \in (0, \beta)$, cf. [42, Section 3] and [58, Section 1.2]. Choose a fixed real $V \in L^\infty(\mathbb{R}^2)$ which is supported in Σ_β and define the squeezed potentials $V_\varepsilon \in L^\infty(\mathbb{R}^2)$ by

$$V_\varepsilon(x) := \begin{cases} \frac{\beta}{\varepsilon} V\big(x_\Sigma + \frac{\beta}{\varepsilon} t\nu(x_\Sigma)\big), & \text{if } x = x_\Sigma + t\nu(x_\Sigma) \in \Sigma_\varepsilon, \\ 0, & \text{if } x \notin \Sigma_\varepsilon. \end{cases} \tag{4.7}$$

Note that the function V_ε is supported in Σ_ε by definition. We introduce for $\varepsilon \in (0, \beta)$ in $L^2(\mathbb{R}^2)$ the operator

$$\mathsf{H}_\varepsilon f := \mathsf{A}_0 f + V_\varepsilon f, \qquad \operatorname{dom} \mathsf{H}_\varepsilon = \operatorname{dom} \mathsf{A}_0 = \mathcal{H}^2_{\mathbf{A}}(\mathbb{R}^2), \tag{4.8}$$

which is self-adjoint, since A_0 is self-adjoint and V_ε is real and bounded.

The following theorem contains the result that H_ε converges in the norm resolvent sense to A_α; we would like to point out that the interaction strength α of the limit operator is some suitable mean value of the potential V along the normal direction, see (4.9) below. Our proof uses a method which differs from the one in [7, 33, 34]. In these papers explicit cumbersome calculations involving the Green function associated to the free Laplacian played an important role. Here, we avoid using the Green function of the Landau Hamiltonian and work more efficiently with the quadratic forms corresponding to A_0 and H_ε, and estimates that follow from the diamagnetic inequality. Since this proof is of more technical nature we postpone it to Appendix B.

Theorem 4.5. *Let $V \in L^\infty(\mathbb{R}^2)$ be real and supported in Σ_β, let $\varepsilon \in (0, \beta)$ and V_ε be as in (4.7), let H_ε be given by (4.8), and define $\alpha \in L^\infty(\Sigma)$ by*

$$\alpha(x_\Sigma) := \int_{-\beta}^{\beta} V(x_\Sigma + t\nu(x_\Sigma))\mathrm{d}t, \qquad x_\Sigma \in \Sigma. \tag{4.9}$$

Then for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there exists a constant $c > 0$ (depending on λ) such that

$$\left\|(\mathsf{H}_\varepsilon - \lambda)^{-1} - (\mathsf{A}_\alpha - \lambda)^{-1}\right\| \leq c\sqrt{\varepsilon}.$$

In particular, H_ε converges in the norm resolvent sense to A_α as $\varepsilon \to 0$.

In the following corollary we show a converse of Theorem 4.5: given an $\alpha \in L^\infty(\Sigma)$ there is a potential V such that the corresponding operators H_ε converge to A_α.

Corollary 4.6. *Let $\alpha \in L^\infty(\Sigma)$ be real and define almost everywhere in $\mathbb{R}^2$ the function*

$$V(x) := \begin{cases} \frac{1}{2\beta}\alpha(x_\Sigma), & \text{if } x = x_\Sigma + t\nu(x_\Sigma) \in \Sigma_\beta, \\ 0, & \text{if } x \notin \Sigma_\beta, \end{cases}$$

and for $\varepsilon \in (0, \beta)$ the scaled potentials V_ε by (4.7). Then the operators H_ε in (4.8) satisfy

$$\left\|(\mathsf{H}_\varepsilon - \lambda)^{-1} - (\mathsf{A}_\alpha - \lambda)^{-1}\right\| \leq c\sqrt{\varepsilon}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

for some constant $c > 0$ (depending on λ). In particular, H_ε converges in the norm resolvent sense to A_α as $\varepsilon \to 0$.

4.3. **Analysis of the resolvent difference of A_α and A_0.** In this subsection we investigate the resolvent difference

$$W_\lambda := -\gamma(\lambda)\big(1 + \alpha M(\lambda)\big)^{-1}\alpha\gamma(\overline{\lambda})^*, \quad \lambda \in \rho(\mathsf{A}_\alpha) \cap \rho(\mathsf{A}_0), \tag{4.10}$$

in (4.2) in more detail. First of all we show a useful variant of Krein's resolvent formula for A_α in which the operator of multiplication with the strength of interaction α is represented as a product $\alpha = \alpha_2\alpha_1$ of two bounded operators α_1 and α_2.

Lemma 4.7. *Let $\alpha \in L^\infty(\Sigma)$ be real and let A_α be as in Definition 4.1. Let $\mathcal{H}$ be a Hilbert space and let $\alpha_1 \colon L^2(\Sigma) \to \mathcal{H}$ and $\alpha_2 \colon \mathcal{H} \to L^2(\Sigma)$ be bounded operators such that the multiplication operator with α fulfils $\alpha = \alpha_2\alpha_1$. For all $\lambda \in \rho(\mathsf{A}_\alpha) \cap \rho(\mathsf{A}_0)$ one has $(1 + \alpha_1 M(\lambda)\alpha_2)^{-1} \in \mathfrak{B}(\mathcal{H})$ and*

$$(\mathsf{A}_\alpha - \lambda)^{-1} = (\mathsf{A}_0 - \lambda)^{-1} - \gamma(\lambda)\alpha_2\big(1 + \alpha_1 M(\lambda)\alpha_2\big)^{-1}\alpha_1\gamma(\overline{\lambda})^*. \tag{4.11}$$

Proof. Consider first $\lambda \in (-\infty, \lambda_0)$, where $\lambda_0 < 0$ is chosen such that

$$\|\alpha_1\| \cdot \|\alpha_2\| \cdot \|M(\lambda)\| < 1, \quad \lambda \in (-\infty, \lambda_0).$$

Note that such λ_0 exists by Proposition 3.6. Then $(1 + \alpha_2\alpha_1 M(\lambda))^{-1} \in \mathfrak{B}(L^2(\Sigma))$, $(1 + \alpha_1 M(\lambda)\alpha_2)^{-1} \in \mathfrak{B}(\mathcal{H})$, and a direct calculation shows that

$$\begin{aligned}
&\big(1 + \alpha_2\alpha_1 M(\lambda)\big)^{-1}\alpha_2 - \alpha_2\big(1 + \alpha_1 M(\lambda)\alpha_2\big)^{-1} \\
&= \big(1 + \alpha_2\alpha_1 M(\lambda)\big)^{-1}\Big[\alpha_2\big(1 + \alpha_1 M(\lambda)\alpha_2\big) - \big(1 + \alpha_2\alpha_1 M(\lambda)\big)\alpha_2\Big]\big(1 + \alpha_1 M(\lambda)\alpha_2\big)^{-1} \\
&= 0
\end{aligned}$$

holds for all $\lambda \in (-\infty, \lambda_0)$. Hence, it follows from Theorem 4.2 and $\alpha = \alpha_2\alpha_1$ that

$$\begin{aligned}
(\mathsf{A}_\alpha - \lambda)^{-1} &= (\mathsf{A}_0 - \lambda)^{-1} - \gamma(\lambda)\big(1 + \alpha_2\alpha_1 M(\lambda)\big)^{-1}\alpha_2\alpha_1\gamma(\overline{\lambda})^* \\
&= (\mathsf{A}_0 - \lambda)^{-1} - \gamma(\lambda)\alpha_2\big(1 + \alpha_1 M(\lambda)\alpha_2\big)^{-1}\alpha_1\gamma(\overline{\lambda})^*
\end{aligned}$$

which is (4.11). Finally, we note that for arbitrary $\lambda \in \rho(\mathsf{A}_\alpha) \cap \rho(\mathsf{A}_0)$ the formula (4.11) follows from an analytic continuation argument. □

Next we provide sign properties of the perturbation term W_λ.

Lemma 4.8. *Let $\lambda_0 < \min \sigma(\mathsf{A}_\alpha)$. If $\alpha \in L^\infty(\Sigma)$ is such that $\alpha(x) \geq 0$ ($\alpha(x) \leq 0$) for a.e. $x \in \Sigma$ then W_{λ_0} is a nonpositive (nonnegative, respectively) self-adjoint operator in $L^2(\mathbb{R}^2)$.*

Proof. Let $\mathfrak{a}_0$ and $\mathfrak{a}_\alpha$ be the sesquilinear forms corresponding to A_0 and A_α in (2.2) and in (4.1), respectively. For a nonnegative function α and all $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ one has $\mathfrak{a}_0[f] \leq \mathfrak{a}_\alpha[f]$ and hence by [54, §6 Theorem 2.21] the inequality

$$(\mathsf{A}_\alpha - \lambda_0)^{-1} \leq (\mathsf{A}_0 - \lambda_0)^{-1}$$

holds for $\lambda_0 < \min \sigma(\mathsf{A}_\alpha)$. Now (4.2) implies that W_{λ_0} is nonpositive. The same argument applies for nonpositive α. □

Recall that P_q denotes the orthogonal projection onto the infinite dimensional eigenspace $\ker(\mathsf{A}_0 - \Lambda_q)$ corresponding to the Landau level Λ_q, $q \in \mathbb{N}_0$. Now it will be shown that for sign-definite functions α the compression $P_q W_\lambda P_q$ of the perturbation term W_λ in (1.5) onto $\ker(\mathsf{A}_0 - \Lambda_q)$ is a compact operator which has infinite rank.

Proposition 4.9. *Assume that $\alpha \in L^\infty(\Sigma)$ and that either $\alpha > 0$ a.e. or $\alpha < 0$ a.e. on Σ. Then there exists $\lambda_0 \in \rho(\mathsf{A}_\alpha) \cap \rho(\mathsf{A}_0) \cap (-\infty, 0)$ such that the compact operator $P_q W_{\lambda_0} P_q$ has infinite rank.*

Proof. We discuss the case $\alpha(x) > 0$ for a.e. $x \in \Sigma$. According to Proposition 3.6 we can choose $\lambda_0 \in (-\infty, 0)$ such that $\|\sqrt{\alpha} M(\lambda_0)\sqrt{\alpha}\| < 1$. Using Lemma 4.7 we see that $-P_q W_{\lambda_0} P_q$ can be written in the form

$$-P_q W_{\lambda_0} P_q = P_q \gamma(\lambda_0)\sqrt{\alpha}\big(1 + \sqrt{\alpha} M(\lambda_0)\sqrt{\alpha}\big)^{-1}\sqrt{\alpha}\gamma(\lambda_0)^* P_q \tag{4.12}$$

and W_{λ_0} is compact in $L^2(\Sigma)$ by Theorem 4.2 (iv). It remains to show that (4.12) has infinite rank. For this we define

$$C := \big(1 + \sqrt{\alpha} M(\lambda_0)\sqrt{\alpha}\big)^{-1} \qquad \text{and} \qquad D := \sqrt{\alpha} C \sqrt{\alpha}.$$

In the present situation C is a nonnegative self-adjoint operator in $L^2(\Sigma)$ such that $0 \in \rho(C)$ and the operators D and $\sqrt{D}$ are both nonnegative and self-adjoint in $L^2(\Sigma)$. We claim that $0 \notin \sigma_{\rm p}(D)$ and hence also $0 \notin \sigma_{\rm p}(\sqrt{D})$. In fact, $D\varphi = 0$ for some $\varphi \in L^2(\Sigma)$ implies

$$\begin{aligned}|(C\sqrt{\alpha}\varphi, \psi)_{L^2(\Sigma)}|^2 &\leq (C\sqrt{\alpha}\varphi, \sqrt{\alpha}\varphi)_{L^2(\Sigma)}(C\psi, \psi)_{L^2(\Sigma)} \\ &= (D\varphi, \varphi)_{L^2(\Sigma)}(C\psi, \psi)_{L^2(\Sigma)} = 0\end{aligned}$$

for all $\psi \in L^2(\Sigma)$ and hence $C\sqrt{\alpha}\varphi = 0$. As $0 \in \rho(C)$ it follows that $\sqrt{\alpha}\varphi = 0$ and the assumption $\alpha(x) > 0$ for a.e. $x \in \Sigma$ yields $\varphi = 0$. Therefore, $0 \notin \sigma_{\rm p}(D)$ and $0 \notin \sigma_{\rm p}(\sqrt{D})$. In particular, $\operatorname{ran} \sqrt{D}$ is dense in $L^2(\Sigma)$.

Next we claim that

$$\operatorname{ran}\big(P_q \gamma(\lambda_0)\sqrt{D}\big) \quad \text{is dense in} \quad \ker(\mathsf{A}_0 - \Lambda_q) \ominus \ker(S - \Lambda_q), \tag{4.13}$$

and we recall that the latter space is infinite dimensional by Lemma 3.7 and $\dim \ker(\mathsf{A}_0 - \Lambda_q) = \infty$. For (4.13) assume that $h \in \ker(\mathsf{A}_0 - \Lambda_q) \ominus \ker(S - \Lambda_q)$ satisfies

$$(P_q \gamma(\lambda_0)\sqrt{D}\varphi, h)_{L^2(\mathbb{R}^2)} = 0 \qquad \text{for all} \quad \varphi \in L^2(\Sigma).$$

Using (A.1) one obtains

$$\begin{aligned} 0 &= (P_q \gamma(\lambda_0)\sqrt{D}\varphi, h)_{L^2(\mathbb{R}^2)} = (\sqrt{D}\varphi, \gamma(\lambda_0)^* h)_{L^2(\Sigma)} \\ &= \big(\sqrt{D}\varphi, \Gamma_1(\mathsf{A}_0 - \lambda_0)^{-1} h\big)_{L^2(\Sigma)} = \frac{1}{\Lambda_q - \lambda_0}(\sqrt{D}\varphi, \Gamma_1 h)_{L^2(\Sigma)} \end{aligned}$$

for all $\varphi \in L^2(\Sigma)$. Since $\operatorname{ran} \sqrt{D}$ is dense in $L^2(\Sigma)$ this implies $\Gamma_1 h = 0$. Furthermore, since $h \in \operatorname{dom} \mathsf{A}_0$ also $\Gamma_0 h = 0$. Therefore $h \in \operatorname{dom} S \cap \ker(\mathsf{A}_0 - \Lambda_q)$ and hence $h \in \ker(S - \Lambda_q)$. By assumption $h \in \ker(\mathsf{A}_0 - \Lambda_q) \ominus \ker(S - \Lambda_q)$ and thus $h = 0$, that is, (4.13) holds.

Now observe that the operator in (4.12) can be written in the form

$$-P_q W_{\lambda_0} P_q = P_q \gamma(\lambda_0) D \gamma(\lambda_0)^* P_q = RR^*, \tag{4.14}$$

where $R = P_q \gamma(\lambda_0)\sqrt{D}$. Since $\ker RR^* = \ker R^*$ it follows that

$$\overline{\operatorname{ran} RR^*} = \overline{\operatorname{ran} R}$$

and $\overline{\operatorname{ran} R}$ is infinite dimensional by (4.13). Hence the same is true for $\overline{\operatorname{ran} RR^*}$ and also for $\operatorname{ran} RR^*$. Taking into account (4.14) the assertion follows. □

5. Estimates and asymptotics for the singular values of $P_q W_\lambda P_q$

In this section we continue our study of the resolvent difference (4.2) of the unperturbed Landau Hamiltonian A_0 and the Landau Hamiltonian A_α with a δ-potential supported on Σ. In the following we fix some $\lambda_0 < \min\{0, \min \sigma(\mathsf{A}_\alpha)\}$ such that $\|\alpha\|_\infty \|M(\lambda_0)\| < 1$, which is possible due to Proposition 3.6. For convenience we use the notation $W := W_{\lambda_0}$ for the resolvent difference, that is,

$$W = (\mathsf{A}_\alpha - \lambda_0)^{-1} - (\mathsf{A}_0 - \lambda_0)^{-1} = -\gamma(\lambda_0)\big(1 + \alpha M(\lambda_0)\big)^{-1}\alpha\gamma(\lambda_0)^*; \tag{5.1}$$

cf. (4.10). As before we write $W = W_+ - W_-$, where $W_+ \geq 0$ is the nonnegative part of W and by $W_- \geq 0$ is the nonpositive part of W; cf. (2.15). The orthogonal projection on the eigenspace $\ker(\mathsf{A}_0 - \Lambda_q)$, $q \in \mathbb{N}_0$, is denoted by P_q. The goal is to obtain asymptotic estimates and sharp spectral asymptotics for the singular values of the operators $P_q W_\pm P_q$ and $P_q |W| P_q$, under different sign conditions on α and smoothness conditions on Σ. This section is split in two subsections dealing with the $C^{1,1}$-case and the C^∞-case, respectively.

5.1. $C^{1,1}$-smooth Σ.

In this subsection it is assumed that Σ is the boundary of a bounded $C^{1,1}$-domain $\Omega_{\rm i}$; cf. Hypothesis 3.1. In the first proposition we consider the compression $P_q|W|P_q$ of $|W|$ onto $\ker(\mathsf{A}_0 - \Lambda_q)$ and estimate this operator by the Toeplitz-type operator in Definition 2.13. To prove the lower bound we require sign-definite functions α.

Proposition 5.1. *Let $\alpha \in L^\infty(\Sigma)$ be real with $\Gamma := \operatorname{supp} \alpha$, assume $|\Gamma| > 0$, and let the resolvent difference W be as in (5.1). Let T_q^Γ be the self-adjoint Toeplitz-type operator as in Definition 2.13. Then the following holds.*

- (i) *$P_q|W|P_q \leq c T_q^\Gamma$ and $P_q W_\pm P_q \leq c_\pm T_q^\Gamma$ for some $c, c_\pm > 0$.*
- (ii) *If α is nonnegative (nonpositive) on Γ and uniformly positive (uniformly negative, respectively) on a closed subset $\Gamma' \subset \Gamma$ such that $|\Gamma'| > 0$ then $P_q|W|P_q \geq c' T_q^{\Gamma'}$ for some $c' > 0$.*

Proof. We start with a preliminary observation. Let $\chi_{\Gamma_\star} \colon \Sigma \to [0,1]$ be the characteristic function of some closed subset $\Gamma_\star \subset \Sigma$ with $|\Gamma_\star| > 0$ and consider the bounded operator $D_{\Gamma_\star} := P_q \gamma(\lambda_0) \chi_{\Gamma_\star} \gamma(\lambda_0)^* P_q$. For $f \in L^2(\mathbb{R}^2)$ we find

$$\begin{aligned}\big(D_{\Gamma_\star} f, f\big)_{L^2(\mathbb{R}^2)} &= \big(\chi_{\Gamma_\star}\gamma(\lambda_0)^* P_q f, \chi_{\Gamma_\star}\gamma(\lambda_0)^* P_q f\big)_{L^2(\Sigma)} \\ &= \frac{\|(P_q f)|_{\Gamma_\star}\|^2_{L^2(\Gamma_\star)}}{(\Lambda_q - \lambda_0)^2} = \frac{\mathfrak{t}_q^{\Gamma_\star}[f]}{(\Lambda_q - \lambda_0)^2},\end{aligned}$$

where (A.1) and $\Gamma_1 f = f|_\Sigma$ were used in the second equality. Hence, $D_{\Gamma_\star}$ and the Toeplitz-type operator $T_q^{\Gamma_\star}$ are related via

$$D_{\Gamma_\star} = \frac{T_q^{\Gamma_\star}}{(\Lambda_q - \lambda_0)^2}. \tag{5.2}$$

(i) We prove the claim for W_+. The proof for W_- is analogous and the estimates for W_+ and W_- also imply the estimate for $|W| = W_+ + W_-$. Consider the mappings

$$\alpha_1\colon L^2(\Sigma) \to L^2(\Gamma), \qquad \alpha_1\phi := (\alpha\phi)|_\Gamma,$$

$$\alpha_2\colon L^2(\Gamma) \to L^2(\Sigma), \qquad \alpha_2\psi := \begin{cases} \psi & \text{on } \Gamma, \\ 0 & \text{on } \Sigma \setminus \Gamma. \end{cases}$$

It is not difficult to see that the product $\alpha_2\alpha_1$ coincides with multiplication operator with α. Hence, Krein's formula in Lemma 4.7 implies that the resolvent difference in (5.1) can be expressed as

$$W = \gamma(\lambda_0) C \gamma(\lambda_0)^*, \tag{5.3}$$

where

$$C := -\alpha_2\big(1 + \alpha_1 M(\lambda_0)\alpha_2\big)^{-1}\alpha_1 \in \mathfrak{B}(L^2(\Sigma))$$

is self-adjoint (since W in (5.3) is self-adjoint). The nonnegative part C_+ of C can be estimated by $C_+ \leq \|C\|$ in the operator sense. For the nonnegative part W_+ of W we have

$$W_+ = \gamma(\lambda_0) C_+ \gamma(\lambda_0)^* = \gamma(\lambda_0)\chi_\Gamma C_+ \chi_\Gamma \gamma(\lambda_0)^*$$

and from

$$\big(P_q W_+ P_q f, f\big)_{L^2(\mathbb{R}^2)} = \big(C_+ \chi_\Gamma \gamma(\lambda_0)^* P_q f, \chi_\Gamma \gamma(\lambda_0)^* P_q f\big)_{L^2(\mathbb{R}^2)} \leq \|C\| (D_\Gamma f, f)_{L^2(\mathbb{R}^2)}$$

we obtain $P_q W_+ P_q \leq \|C\| D_\Gamma$. Hence, using (5.2) we find

$$P_q W_+ P_q \leq \frac{\|C\|}{(\Lambda_q - \lambda_0)^2} T_q^\Gamma,$$

and the estimate for W_+ in (i) follows with $c_+ := \frac{\|C\|}{(\Lambda_q - \lambda_0)^2}$.

(ii) We prove the claim for nonnegative α. Suppose that α (as well as $\sqrt{\alpha}$) is nonnegative on Γ and uniformly positive on $\Gamma' \subset \Gamma$. Then Krein's formula in Lemma 4.7 with the mappings

$$\alpha_1\colon L^2(\Sigma) \to L^2(\Gamma), \qquad \alpha_1\phi := (\sqrt{\alpha}\phi)|_\Gamma,$$

$$\alpha_2\colon L^2(\Gamma) \to L^2(\Sigma), \qquad \alpha_2\phi := \begin{cases} \sqrt{\alpha}\phi & \text{on } \Gamma, \\ 0 & \text{on } \Sigma \setminus \Gamma, \end{cases}$$

shows

$$W = -\gamma(\lambda_0)\alpha_2 \widehat{C} \alpha_1 \gamma(\lambda_0)^*,$$

where the middle-term

$$\widehat{C} := \big(1 + \alpha_1 M(\lambda_0)\alpha_2\big)^{-1} \in \mathfrak{B}(L^2(\Gamma))$$

is self-adjoint and uniformly positive in $L^2(\Gamma)$. Hence, the operator W is nonpositive. Thus, we obtain from (5.2) in the same way as in the proof of (i) that

$$\begin{aligned} P_q |W| P_q &\geq \big(\inf \sigma(\widehat{C})\big) \cdot P_q \gamma(\lambda_0) \chi_{\Gamma'} \alpha \chi_{\Gamma'} \gamma(\lambda_0)^* P_q \\ &\geq \big(\inf \sigma(\widehat{C})\big)\big(\inf_{x \in \Gamma'} \alpha(x)\big) \cdot P_q \gamma(\lambda_0) \chi_{\Gamma'} \gamma(\lambda_0)^* P_q \geq c' T_q^{\Gamma'}, \end{aligned}$$

with

$$c' = \frac{\inf \sigma(\widehat{C})}{(\Lambda_q - \lambda_0)^2} \cdot \inf_{x \in \Gamma'} \alpha(x) > 0.$$

This proves the inequality in (ii). ☐

Now we formulate three corollaries of the above proposition. The first one follows from the upper bound on $P_q|W|P_q$ from Proposition 5.1 (i) and the spectral estimate for T_q^Γ in Proposition 2.16 (i).

Corollary 5.2. *Let $\alpha \in L^\infty(\Sigma)$ be real with $\Gamma = \operatorname{supp}\alpha$, assume $|\Gamma| > 0$, and let the resolvent difference W be as in (5.1). Then the singular values of the operator $P_q|W|P_q$, $q \in \mathbb{N}_0$, satisfy*

$$\limsup_{k\to\infty} \big(k!\, s_k(P_q|W|P_q)\big)^{1/k} \le \frac{B}{2}\big(\operatorname{Cap}(\Gamma)\big)^2.$$

In particular, the singular values of the operator $P_q W_\pm P_q$, $q \in \mathbb{N}_0$, satisfy

$$\limsup_{k\to\infty} \big(k!\, s_k(P_q W_\pm P_q)\big)^{1/k} \le \frac{B}{2}\big(\operatorname{Cap}(\Gamma)\big)^2.$$

We remark that in the present $C^{1,1}$-setting the lower bound in Proposition 5.1 (ii) in the case of a sign-definite α can not be used directly to conclude a lower bound on the singular values for $P_q W P_q$ since the estimate in Proposition 2.16 (i) is only one-sided. However, the situation is better for the lowest Landau level Λ_0. In fact, Proposition 5.1 (ii) and Proposition 2.17 imply the next corollary.

Corollary 5.3. *Consider the resolvent difference W in (5.1) and assume that $\alpha \not\equiv 0$ is either nonnegative or nonpositive. Then the rank of $P_0 W P_0$ is infinite.*

Proof. Assume that α is nonnegative and $\alpha \not\equiv 0$. Then there exists $\varepsilon > 0$ and $S_\varepsilon \subset \Gamma$ measurable such that $\alpha(x) \ge \varepsilon$ for a.e. $x \in S_\varepsilon$. Hence there is also a closed subset $K \subset S_\varepsilon$ such that $|K| > 0$ and $\alpha > \varepsilon$ on K. Now Proposition 5.1 (ii) and Proposition 2.17 lead to the statement. ☐

In Proposition 4.9 it was shown that for positive (or negative) $\alpha \in L^\infty(\Sigma)$ the rank of $P_q|W|P_q$, $q \in \mathbb{N}_0$, is infinite. This observation leads to an interesting consequence for Toeplitz-type operators.

Corollary 5.4. *The rank of the self-adjoint Toeplitz-type operator T_q^Σ, $q \in \mathbb{N}_0$, in Definition 2.13 is infinite.*

Proof. Consider the self-adjoint operator $\mathsf{A}_\alpha = \mathsf{A}_1$ with $\alpha \equiv 1$. Fix $\lambda_0 < 0$ such that $\|M(\lambda_0)\| < 1$ and note that the resolvent difference W in (5.1) is nonpositive by Lemma 4.8. By Proposition 4.9 the rank of $P_q W P_q = P_q W_- P_q$ is infinite for all $q \in \mathbb{N}_0$. Since $P_q W_- P_q \le c T_q^\Sigma$ by Proposition 5.1 (i) the rank of T_q^Σ is infinite as well. ☐

5.2. C^∞**-smooth setting.** Now we pass to the discussion of the C^∞-smooth setting. Here, we are able to get more precise results. In the formulation of the next theorem, and also later on, we denote by $B_\varepsilon(x) \subset \mathbb{R}^2$ the disc of radius $\varepsilon > 0$ centered at $x \in \mathbb{R}^2$.

Theorem 5.5. *Let $\alpha \in L^\infty(\Sigma)$ be real, assume that $\Gamma = \operatorname{supp}\alpha$ is a C^∞-smooth arc and that α is nonnegative (nonpositive) on Γ and uniformly positive (uniformly negative, respectively) on the truncated arc $\Gamma_\varepsilon := \{x \in \Gamma \colon B_\varepsilon(x) \cap \Sigma \subset \Gamma\}$ for all $\varepsilon > 0$ sufficiently small. Let the resolvent difference W be as in (5.1). Then the singular values of the operator $P_q|W|P_q$, $q \in \mathbb{N}_0$, satisfy*

$$\lim_{k\to\infty} \big(k!\, s_k(P_q|W|P_q)\big)^{1/k} = \frac{B}{2}\big(\mathrm{Cap}(\Gamma)\big)^2.$$

Proof. By Corollary 5.2 we get

$$\limsup_{k\to\infty} \big(k!\, s_k(P_q|W|P_q)\big)^{1/k} \le \frac{B}{2}\left(\mathrm{Cap}\,(\Gamma)\right)^2$$

and for $\varepsilon > 0$ we conclude from Proposition 5.1 (ii) and Proposition 2.16 (ii) that

$$\liminf_{k\to\infty} \big(k!\, s_k(P_q|W|P_q)\big)^{1/k} \ge \frac{B}{2}\left(\mathrm{Cap}\,(\Gamma_\varepsilon)\right)^2.$$

Hence, the claim of the theorem follows from $\liminf_{\varepsilon\to 0+} \mathrm{Cap}\,(\Gamma_\varepsilon) = \mathrm{Cap}\,(\Gamma)$. In fact, by Proposition 2.15 (i) we know that $\mathrm{Cap}\,(\Gamma_\varepsilon) \le \mathrm{Cap}\,(\Gamma)$ since $\Gamma_\varepsilon \subset \Gamma$. For the other inequality consider the equilibrium measure μ for Γ. It is no restriction to assume that μ has no point masses, as otherwise $I(\mu) = \infty$ and hence $\mathrm{Cap}(\Gamma) = 0$, which is a trivial case. First, it follows from the dominated convergence theorem that $\mu(\Gamma_\varepsilon) \to 1$, as $\varepsilon \to 0$. Hence, for $\varepsilon > 0$ the measure μ_ε acting on Borel sets $\mathcal{M} \subset \mathbb{R}^2$ as

$$\mu_\varepsilon(\mathcal{M}) := \frac{1}{\mu(\Gamma_\varepsilon)}\mu(\mathcal{M} \cap \Gamma_\varepsilon)$$

is well defined and clearly, $\mu_\varepsilon \ge 0$, $\operatorname{supp}\mu_\varepsilon = \Gamma_\varepsilon$, and $\mu_\varepsilon(\Gamma_\varepsilon) = 1$. Another application of the dominated convergence theorem yields

$$I(\mu_\varepsilon) = \frac{1}{\mu(\Gamma_\varepsilon)^2}\int_{\Gamma_\varepsilon}\int_{\Gamma_\varepsilon} \ln\frac{1}{|x-y|}\mathsf{d}\mu(x)\mathsf{d}\mu(y) \to \int_\Gamma\int_\Gamma \ln\frac{1}{|x-y|}\mathsf{d}\mu(x)\mathsf{d}\mu(y) = I(\mu)$$

as $\varepsilon \to 0$, which shows that $\liminf_{\varepsilon\to 0+} \mathrm{Cap}(\Gamma_\varepsilon) \ge \mathrm{Cap}(\Gamma)$. □

Remark 5.6. For a continuous interaction strength α the assumption in Theorem 5.5 means that α is allowed to vanish at the endpoints and has to be positive in the interior of Γ. In particular, α does not have to be uniformly positive on Γ.

Under slightly weaker assumptions on α we conclude the following lower bound on the singular values $P_q|W|P_q$ from Proposition 5.1 (ii) and Proposition 2.16 (ii).

Proposition 5.7. *Let $\alpha \in L^\infty(\Sigma)$ be real, assume that there exists a C^∞-smooth subarc $\Gamma' \subset \operatorname{supp}\alpha$ with two endpoints, $|\Gamma'| > 0$, and that α is nonnegative (nonpositive) on Σ*

and uniformly positive (uniformly negative, respectively) on Γ'. Let the resolvent difference W be as in (5.1). Then the singular values of the operator $P_q|W|P_q$, $q \in \mathbb{N}_0$, satisfy

$$\liminf_{k\to\infty} \big(k!\, s_k(P_q|W|P_q)\big)^{1/k} \geq \frac{B}{2}\big(\mathrm{Cap}(\Gamma')\big)^2.$$

6. Eigenvalue clustering at Landau levels

In this section we prove results on the local spectral properties of the perturbed Landau Hamiltonian of A_α. Throughout this section we fix some λ_0 such that

$$\lambda_0 < \min\{0, \min \sigma(\mathsf{A}_\alpha)\}.$$

We note first that for sign-definite interaction strengths α accumulation of the eigenvalues from one side to each Landau level can be excluded. This is a direct consequence of well-known perturbation results.

Proposition 6.1. *Assume that $\alpha \in L^\infty(\Sigma)$ is real. Then the following holds.*

(i) *If α is nonnegative, then there is no accumulation of eigenvalues of A_α from below to the Landau levels Λ_q, $q \in \mathbb{N}_0$.*
(ii) *If α is nonpositive, then there is no accumulation of eigenvalues of A_α from above to the Landau levels Λ_q, $q \in \mathbb{N}_0$.*

Proof. We prove only (i); the proof of (ii) is analogous. Recall that

$$(\mathsf{A}_\alpha - \lambda_0)^{-1} - (\mathsf{A}_0 - \lambda_0)^{-1} = -\gamma(\lambda_0)\big(1 + \alpha M(\lambda_0)\big)^{-1}\alpha\gamma(\lambda_0)^* \leq 0$$

by Lemma 4.8 and hence the eigenvalues of $(\mathsf{A}_\alpha - \lambda_0)^{-1}$ do not accumulate from above to the eigenvalues $(\Lambda_q - \lambda_0)^{-1}$ of $(\mathsf{A}_0 - \lambda_0)^{-1}$; cf. [16, Chapter 9, §4, Theorem 7]. Therefore, the eigenvalues of A_α do not accumulate to Λ_q from below. □

If α is either positive or negative on Σ one always has accumulation of eigenvalues to each Landau level.

Theorem 6.2. *Assume that $\alpha \in L^\infty(\Sigma)$ is real. Then the following holds.*

(i) *If $\alpha > 0$ a.e. on Σ, then the eigenvalues of A_α accumulate from above to Λ_q, $q \in \mathbb{N}_0$.*
(ii) *If $\alpha < 0$ a.e. on Σ, then the eigenvalues of A_α accumulate from below to Λ_q, $q \in \mathbb{N}_0$.*

Proof. We prove only (i). Recall that by Lemma 4.8 the perturbation term in (5.1) is a nonpositive operator. It follows from Proposition 4.9 that the rank of P_qWP_q is infinite. Hence, Proposition 2.7 implies that the eigenvalues of $(\mathsf{A}_\alpha - \lambda_0)^{-1}$ accumulate from below to the eigenvalues $(\Lambda_q - \lambda_0)^{-1}$ of $(\mathsf{A}_0 - \lambda_0)^{-1}$. Therefore, the eigenvalues of A_α accumulate from above to each Landau level Λ_q. □

For the lowest Landau level $\Lambda_0 = B$, it is not necessary to assume that α is positive or negative on all of Σ. The proof of the next theorem is the same as the proof of Theorem 6.2, but in order to conclude that the rank of $P_0 W P_0$ is infinite one uses Corollary 5.3.

Theorem 6.3. *Assume that $\alpha \in L^\infty(\Sigma)$ is real and $\alpha \not\equiv 0$. Then the following holds.*

(i) *If α is nonnegative, then the eigenvalues of A_α accumulate from above to Λ_0.*

(ii) *If α is nonpositive, then the eigenvalues of A_α accumulate from below to Λ_0.*

In order to formulate our results on the rate of accumulation of the eigenvalues of A_α to the Landau levels the following notation is convenient:

$$\begin{aligned} q = 0: &\qquad I_0^- := (-\infty, \Lambda_0), &\qquad I_0^+ := (\Lambda_0, \Lambda_0 + B],\\ q \geq 1: &\qquad I_q^- := (\Lambda_q - B, \Lambda_q), &\qquad I_q^+ := (\Lambda_q, \Lambda_q + B]. \end{aligned}$$

Note that

$$\mathbb{R} = \bigcup_{q=0}^{\infty} I_q^- \cup \bigcup_{q=0}^{\infty} I_q^+ \cup \bigcup_{q=0}^{\infty} \{\Lambda_q\}.$$

In the first theorem the $C^{1,1}$-smooth case is considered. We obtain regularized summability of the discrete spectrum of A_α over all clusters and an asymptotic spectral estimate within each cluster. We point out that these results are true for sign-changing α.

Theorem 6.4. *Let $\{\lambda_k^\pm(q)\}_k$, $q \in \mathbb{N}_0$, be the eigenvalues of A_α lying in the interval $I_q^\pm$, ordered in such a way that the distance from Λ_q is nonincreasing and with multiplicities taken into account. Then the following holds.*

(i) $\sum_{q=0}^\infty \frac{1}{(2q+1)^2} \left(\sum_k \left|\lambda_k^+(q) - \Lambda_q\right| + \sum_k \left|\lambda_k^-(q) - \Lambda_q\right|\right) < \infty.$

(ii) $\limsup_{k\to\infty} \left(k! \, |\lambda_k^\pm(q) - \Lambda_q|\right)^{1/k} \leq \frac{B}{2} \left(\mathrm{Cap}\,(\Gamma)\right)^2.$

Proof. (i) By Theorem 4.2 (iv) the resolvent difference W in (5.1) belongs to the Schatten-von Neumann class $\mathfrak{S}_p(L^2(\mathbb{R}^2))$ for all $p > \frac{1}{3}$ and, in particular, for $p = 1$. Again we use that the spectrum of $D := (\mathsf{A}_0 - \lambda_0)^{-1}$ consists of the infinite dimensional eigenvalues $\{(\Lambda_q - \lambda_0)^{-1}\}_{q\in\mathbb{N}_0}$. Recall also that $\lambda_0 < \min\{0, \min\sigma(\mathsf{A}_\alpha)\}$. One verifies that there exists $c_\pm, c_0 > 0$ such that for all $q \in \mathbb{N}_0$ we have

$$\begin{aligned} \mathfrak{d}_k^+(q) :=& \,\mathrm{dist}\left(\frac{1}{\lambda_k^+(q) - \lambda_0}, \sigma(D)\right)\\ =& \min\left\{\frac{1}{\lambda_k^+(q) - \lambda_0} - \frac{1}{\Lambda_{q+1} - \lambda_0}, \frac{1}{\Lambda_q - \lambda_0} - \frac{1}{\lambda_k^+(q) - \lambda_0}\right\}\\ \geq& \frac{c_+(\lambda_k^+(q) - \Lambda_q)}{\Lambda_q^2}, \end{aligned}$$

and for all $q \in \mathbb{N}$

$$\begin{aligned}
\mathfrak{d}_k^-(q) &:= \operatorname{dist}\left(\frac{1}{\lambda_k^-(q)-\lambda_0}, \sigma(D)\right) \\
&= \min\left\{\frac{1}{\lambda_k^-(q)-\lambda_0} - \frac{1}{\Lambda_q-\lambda_0}, \frac{1}{\Lambda_{q-1}-\lambda_0} - \frac{1}{\lambda_k^-(q)-\lambda_0}\right\} \\
&\geq \frac{c_-(\Lambda_q-\lambda_k^-(q))}{\Lambda_q^2},
\end{aligned}$$

and for $q = 0$

$$\mathfrak{d}_k^-(0) := \operatorname{dist}\left(\frac{1}{\lambda_k^-(0)-\lambda_0}, \sigma(D)\right) = \frac{\Lambda_0-\lambda_k^-(0)}{(\Lambda_0-\lambda_0)(\lambda_k^-(0)-\lambda_0)} \geq \frac{c_0(\Lambda_0-\lambda_k^-(0))}{\Lambda_0^2}.$$

Hence, we get with $C = (\mathsf{A}_\alpha - \lambda)^{-1}$

$$\begin{aligned}
\sum_{\lambda\in\sigma_{\mathrm{disc}}(C)} \operatorname{dist}(\lambda, \sigma(D)) &= \sum_{q=0}^{\infty}\sum_k \left(\mathfrak{d}_k^+(q) + \mathfrak{d}_k^-(q)\right) \\
&\geq \sum_{q=0}^{\infty} \frac{c}{B^2(2q+1)^2} \sum_k \left(\left|\lambda_k^+(q)-\Lambda_q\right| + \left|\lambda_k^-(q)-\Lambda_q\right|\right)
\end{aligned}$$

and the claim follows from Proposition 2.11.

(ii) We shall use Proposition 2.10 with

(6.1a) $W = W_{\lambda_0}$ in (5.1), $T = (\mathsf{A}_0 - \lambda_0)^{-1}$, $\Lambda = \dfrac{1}{\Lambda_q - \lambda_0}$,

(6.1b) $P_\Lambda = P_q$, $\varepsilon = \dfrac{1}{2}$, $\tau_\pm = \pm\dfrac{1}{2}\left[\dfrac{1}{\Lambda_q \mp B - \lambda_0} - \dfrac{1}{\Lambda_q - \lambda_0}\right]$.

Note that the eigenvalues of $T + W$ in the interval $(\Lambda - 2\tau_-, \Lambda + 2\tau_+)$ are given by

$$\frac{1}{\lambda_1^+(q)-\lambda_0} \leq \frac{1}{\lambda_2^+(q)-\lambda_0} \leq \cdots \leq \Lambda \leq \cdots \leq \frac{1}{\lambda_2^-(q)-\lambda_0} \leq \frac{1}{\lambda_1^-(q)-\lambda_0}.$$

We conclude from Proposition 2.10 that there exists a constant $\ell = \ell(q) \in \mathbb{N}$ such that

$$\left|\frac{1}{\lambda_k^\pm(q)-\lambda_0} - \frac{1}{\Lambda_q-\lambda_0}\right| \leq \frac{3}{2} s_{k-\ell}(P_q W_\mp P_q),$$

for all $k \in \mathbb{N}$ large enough. Using Corollary 5.2 we find

$$\begin{aligned}
&\limsup_{k\to\infty} \left(k!\,|\lambda_k^\pm(q) - \Lambda_q|\right)^{1/k} \\
&\quad= \limsup_{k\to\infty} (\lambda_k^\pm(q)-\lambda_0)^{1/k}(\Lambda_q-\lambda_0)^{1/k}\left(k!\left|\frac{1}{\lambda_k^\pm(q)-\lambda_0} - \frac{1}{\Lambda_q-\lambda_0}\right|\right)^{1/k} \\
&\quad\leq \limsup_{k\to\infty} \left(k!\, s_{k-\ell}(P_q W_\mp P_q)\right)^{1/k}, \\
&\quad= \limsup_{k\to\infty} \left(k!\, s_k(P_q W_\mp P_q)\right)^{1/k} \leq \frac{B}{2}\left(\operatorname{Cap}(\Gamma)\right)^2,
\end{aligned}$$

where we have used $\lim_{k\to\infty} a^{\frac{1}{k}} = 1$ for $a > 0$ and $\limsup_{k\to\infty}(k!\,\xi_{k\pm\ell})^{1/k} = \limsup_{k\to\infty}(k!\,\xi_k)^{1/k}$ for any nonincreasing nonnegative sequence $\{\xi_k\}_k$; cf. [70, Section 2.2]. $\square$

Now we present a result on the local spectral asymptotics for A_α within each cluster; here we rely on Theorem 5.5 and hence we have to assume that $\operatorname{supp}\alpha$ is C^∞-smooth. We remark that the eigenvalue asymptotics in the following theorem comply with [40, Remark 2 and Theorem 2].

Theorem 6.5. *Let $\alpha \in L^\infty(\Sigma)$ be real, assume that $\Gamma = \operatorname{supp}\alpha$ is a C^∞-smooth arc and that α is nonnegative (nonpositive) on Γ and uniformly positive (uniformly negative, respectively) on the truncated arc $\Gamma_\varepsilon := \{x \in \Gamma \colon B_\varepsilon(x)\cap\Sigma \subset \Gamma\}$ for all $\varepsilon > 0$ sufficiently small. Let $\{\lambda_k(q)\}_k$, $q \in \mathbb{N}_0$, be the eigenvalues of A_α lying in the interval I_q^+ (I_q^-, respectively). Then*

$$\lim_{k\to\infty}\left(k!|\lambda_k(q) - \Lambda_q|\right)^{1/k} = \frac{B}{2}\big(\mathrm{Cap}\,(\Gamma)\big)^2$$

and, in particular, the eigenvalues of A_α accumulate to Λ_q from above (from below, respectively) for all $q \in \mathbb{N}_0$.

Proof. We discuss the case $\alpha \ge 0$. By Theorem 6.2 the eigenvalues of A_α accumulate to Λ_q from above and there is no accumulation from below. It follows from Theorem 5.5 that $\operatorname{rank} P_qWP_q = \infty$. Using Proposition 2.7 with W, T, Λ, P_Λ, ε, and $\tau_\pm$ as in (6.1) we obtain that there exists a constant $\ell = \ell(q) \in \mathbb{N}$ such that

$$\frac{1}{2}s_{k+\ell}(P_qWP_q) \le \left|\frac{1}{\lambda_k(q)-\lambda} - \frac{1}{\Lambda_q - \lambda}\right| \le \frac{3}{2}s_{k-\ell}(P_qWP_q)$$

for all $k \in \mathbb{N}$ sufficiently large. These estimates and the asymptotics of the singular values of P_qWP_q in Theorem 5.5 yield the claim in the same way as in the proof of Theorem 6.4 (ii). $\square$

Mimicking the proof of the above theorem, but using Proposition 5.7 instead of Theorem 5.5 we get an asymptotic lower bound within each cluster under relaxed assumptions on α and Γ.

Proposition 6.6. *Let $\alpha \in L^\infty(\Sigma)$ be real, assume that there exists a C^∞-smooth subarc $\Gamma' \subset \operatorname{supp}\alpha$ with two endpoints, $|\Gamma'| > 0$, and that α is nonnegative (nonpositive) on Σ and uniformly positive (uniformly negative, respectively) on Γ'. Let $\{\lambda_k(q)\}_k$, $q \in \mathbb{N}_0$, be the eigenvalues of A_α lying in the interval I_q^+ (I_q^-, respectively). Then*

$$\lim_{k\to\infty}\left(k!|\lambda_k(q) - \Lambda_q|\right)^{1/k} \ge \frac{B}{2}\left(\mathrm{Cap}\,(\Gamma')\right)^2$$

and, in particular, the eigenvalues of A_α accumulate to Λ_q from above (from below, respectively) for all $q \in \mathbb{N}_0$.

The above proposition applies to several additional cases of interest. E.g., α can be a nonnegative or nonpositive function which is continuous (and does not vanish identically), or $\operatorname{supp}\alpha$ may consist of finitely many disjoint arcs. In both situations one can choose a C^∞-smooth subarc $\Gamma' \subset \operatorname{supp}\alpha$ with two endpoints, such that $|\Gamma'| > 0$ and α uniformly positive (or uniformly negative) on Γ'. Moreover, Proposition 6.6 can also be applied if the support of α is not C^∞-smooth itself but contains a C^∞-smooth subarc with two endpoints on which α is uniformly positive (or uniformly negative).

Appendix A. Quasi boundary triples and their Weyl functions

In this appendix we provide a brief introduction to the abstract notion of quasi boundary triples and their Weyl functions from extension theory of symmetric operators. For more details and complete proofs we refer the reader to [9, 10].

In the following let $\mathcal{H}$ be a Hilbert space and assume that S is a densely defined closed symmetric operator in $\mathcal{H}$.

Definition A.1. *Assume that T is a linear operator in $\mathcal{H}$ such that $\overline{T} = S^*$. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is called a* quasi boundary triple *for $T \subset S^*$ if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings such that the following holds:*

(i) *The abstract Green identity*

$$(Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}$$

is valid for all $f, g \in \operatorname{dom} T$.

(ii) *The map $\Gamma = (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ has dense range.*

(iii) *The operator $A_0 := T \upharpoonright \ker \Gamma_0$ is self-adjoint in $\mathcal{H}$.*

We recall that a quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for $T \subset S^*$ exists if and only if the deficiency indices $n_\pm(S) = \dim\ker(S^* \mp i)$ coincide, in which case one has $\dim \mathcal{G} = n_\pm(S)$. We also note that for a quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for $T \subset S^*$ one automatically has

$$\operatorname{dom} S = \ker \Gamma_0 \cap \ker \Gamma_1$$

and that the extension $A_1 := T \upharpoonright \ker \Gamma_1$ of S is symmetric in $\mathcal{H}$ but in general not closed or self-adjoint. Furthermore, if $\dim \mathcal{G} = n_\pm(S)$ is finite then T and S^* coincide, the abstract Green identity in Definition A.1 (i) holds for all $f, g \in \operatorname{dom} S^*$ and the map $\Gamma = (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} S^* \to \mathcal{G} \times \mathcal{G}$ in Definition A.1 (i) is surjective. A triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ with these two properties is an *ordinary boundary triple* in the sense of [24, 29, 48, 79]. Also recall the notion of *generalized boundary triples*: If $\overline{T} = S^*$ and a triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ with linear mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ satisfies (i) and (iii) in Definition A.1 and instead of (ii) the stronger condition $\operatorname{ran}\Gamma_0 = \mathcal{G}$ then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is said to be a *generalized boundary triple*; cf. [30, Definition 6.1 and Lemma 6.1 (3)].

When determining a quasi boundary triple it is often nontrivial to prove that the operator T satisfies $\overline{T} = S^*$. The following theorem from [9, Theorem 2.3] offers a way to circumvent this problem. Theorem A.2 is applied in proof of Theorem 3.4.

Theorem A.2. *Let $\mathcal{H}$ and $\mathcal{G}$ be Hilbert spaces, let T be a linear operator in $\mathcal{H}$ and assume that there are linear mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ such that the following holds:*

(i) *For all $f, g \in \operatorname{dom} T$ one has*

$$(Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}.$$

(ii) *The kernel and range of $\Gamma = (\Gamma_0, \Gamma_1)^\top : \operatorname{dom} T \to \mathcal{G} \times \mathcal{G}$ are dense in $\mathcal{H}$ and $\mathcal{G} \times \mathcal{G}$, respectively.*

(iii) *The operator $T \upharpoonright \ker \Gamma_0$ contains a self-adjoint operator A_0.*

Then

$$S := T \upharpoonright \big(\ker \Gamma_0 \cap \ker \Gamma_1\big)$$

is a densely defined closed symmetric operator in $\mathcal{H}$ and $\overline{T} = S^*$. Moreover, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a quasi boundary triple for $T \subset S^*$ such that $A_0 = T \upharpoonright \ker \Gamma_0$.

Next the γ-field and Weyl function corresponding to a quasi boundary triple will be introduced; formally the definitions are the same as for ordinary and generalized boundary triples, see [29, 30]. In the following let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ and consider the self-adjoint operator $A_0 = T \upharpoonright \ker \Gamma_0$. It is not difficult to verify that for all $\lambda \in \rho(A_0)$ the following direct sum decomposition of $\operatorname{dom} T$ is valid:

$$\operatorname{dom} T = \operatorname{dom} A_0 \dot{+} \ker(T - \lambda) = \ker \Gamma_0 \dot{+} \ker(T - \lambda), \qquad \lambda \in \rho(A_0).$$

Therefore the restriction $\Gamma_0 \upharpoonright \ker(T - \lambda)$ is invertible for all $\lambda \in \rho(A_0)$ and we define the γ-field corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ as the operator function

$$\lambda \mapsto \gamma(\lambda) := \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1}$$

defined on $\rho(A_0)$. It is clear that the values $\gamma(\lambda)$ of the γ-field are densely defined linear operators from $\mathcal{G}$ into $\mathcal{H}$ with $\operatorname{dom} \gamma(\lambda) = \operatorname{ran} \Gamma_0$ and $\operatorname{ran} \gamma(\lambda) = \ker(T - \lambda)$. It can be shown that $\gamma(\lambda)$ is a bounded operator for all $\lambda \in \rho(A_0)$ and hence admits a closure $\overline{\gamma(\lambda)} \in \mathfrak{B}(\mathcal{G}, \mathcal{H})$. The function $\lambda \mapsto \overline{\gamma(\lambda)} \in \mathfrak{B}(\mathcal{G}, \mathcal{H})$ is holomorphic on $\rho(A_0)$. For the adjoint operators one verifies as a consequence of the abstract Green identity the relation

$$\gamma(\lambda)^* = \Gamma_1 (A_0 - \overline{\lambda})^{-1} \in \mathfrak{B}(\mathcal{H}, \mathcal{G}), \qquad \lambda \in \rho(A_0). \tag{A.1}$$

For more properties and detailed proofs we refer the reader to [9, Proposition 2.6] and [10, Proposition 6.13]. An important analytic object associated with the quasi boundary triple $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is the Weyl function M. It is defined on $\rho(A_0)$ by

$$\lambda \mapsto M(\lambda) = \Gamma_1 \big(\Gamma_0 \upharpoonright \ker(T - \lambda)\big)^{-1},$$

and it is clear from the definition that $M(\lambda)$, $\lambda \in \rho(A_0)$, is a densely defined linear operator in $\mathcal{G}$ with $\operatorname{dom} M(\lambda) = \operatorname{ran} \Gamma_0$ and $\operatorname{ran} M(\lambda) \subset \operatorname{ran} \Gamma_1$. In contrast to ordinary and generalized boundary triples the values $M(\lambda)$ of the Weyl function can be unbounded and non-closed operators in $\mathcal{G}$. However, one has the relation

$$M(\overline{\lambda}) \subset M(\lambda)^*, \qquad \lambda \in \rho(A_0),$$

and hence $M(\lambda)$ is a closable operator in $\mathcal{G}$. Furthermore, the Weyl function and γ-field are connected via

$$M(\lambda) - M(\mu)^* = (\lambda - \overline{\mu})\gamma(\mu)^*\gamma(\lambda), \qquad \lambda, \mu \in \rho(A_0);$$

cf. [9, Proposition 2.6] and [10, Proposition 6.14] for more details. For the present paper the special case that $\operatorname{ran} \Gamma_0 = \mathcal{G}$ holds is of particular interest. In this situation one has $\operatorname{dom} \gamma(\lambda) = \operatorname{dom} M(\lambda) = \mathcal{G}$ and it follows, in particular, that the values $M(\lambda)$ of the Weyl function are bounded operators in $\mathcal{G}$.

In the following our interest will be in restrictions of T defined by

$$A_{[B]}f = Tf, \qquad \operatorname{dom} A_{[B]} = \big\{ f \in \operatorname{dom} T : \Gamma_0 f = B\Gamma_1 f \big\}, \tag{A.2}$$

where B is a linear operator in $\mathcal{G}$. If B is not defined on the whole space $\mathcal{G}$ the boundary condition in (A.2) is understood for only those $f \in \operatorname{dom} T$ such that $\Gamma_1 f \in \operatorname{dom} B$. Typically the interest is to conclude from qualitative properties of B qualitative properties of $A_{[B]}$. In the present situation we will focus on self-adjointness. Suppose first that B is a symmetric operator in $\mathcal{G}$. Then it follows together with the abstract Green identity in Definition A.1 (i) that for $f, g \in \operatorname{dom} A_{[B]}$ we have

$$\begin{aligned}
(A_{[B]}f, g)_{\mathcal{H}} - (f, A_{[B]}g)_{\mathcal{H}} &= (Tf, g)_{\mathcal{H}} - (f, Tg)_{\mathcal{H}} \\
&= (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}} \\
&= (\Gamma_1 f, B\Gamma_1 g)_{\mathcal{G}} - (B\Gamma_1 f, \Gamma_1 g)_{\mathcal{G}} \\
&= 0
\end{aligned}$$

and therefore the operator $A_{[B]}$ is symmetric in $\mathcal{H}$. However, self-adjointness of B in $\mathcal{G}$ does not automatically imply that $A_{[B]}$ is self-adjoint in $\mathcal{H}$. In fact, this conclusion does not even hold for bounded self-adjoint operators B and hence one has to impose additional conditions. Such conditions may involve mapping properties of the Weyl function, the parameter B, or the boundary mappings Γ_0 and Γ_1. In this context we recall [13, Corollary 4.4] and a special case of it below. For more general boundary conditions we refer the reader to [13].

Theorem A.3. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M. Let $B \in \mathfrak{B}(\mathcal{G})$ be a self-adjoint operator and assume that for some $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ the following conditions hold:*

(i) $1 \in \rho(B\overline{M(\lambda_0)})$;
(ii) $B(\operatorname{ran} \overline{M(\lambda_0)}) \subset \operatorname{ran} \Gamma_0$;
(iii) $B(\operatorname{ran} \Gamma_1) \subset \operatorname{ran} \Gamma_0$ *or* $\lambda_0 \in \rho(A_1)$.

Then the operator $A_{[B]}$ in (A.2) is a self-adjoint extension of S in $\mathcal{H}$ such that $\lambda_0 \in \rho(A_{[B]})$. Furthermore, $\lambda \in \rho(A_0)$ is an eigenvalue of $A_{[B]}$ if and only if $1 \in \sigma_{\rm p}(BM(\lambda))$, for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$ one has $(1 - BM(\lambda))^{-1} \in \mathfrak{B}(\mathcal{G})$ and

$$(A_{[B]} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(1 - BM(\lambda)\big)^{-1} B\gamma(\overline{\lambda})^*.$$

For our purposes it is convenient to state the following special case of Theorem A.3, where the quasi boundary triple is even a generalized boundary triple, that is, we require $\operatorname{ran}\Gamma_0 = \mathcal{G}$. In this situation it is clear that (ii) and (iii) in Theorem A.3 hold and $\overline{M(\lambda_0)} = M(\lambda_0) \in \mathfrak{B}(\mathcal{G})$.

Corollary A.4. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding γ-field γ and Weyl function M, and assume, in addition, that $\operatorname{ran}\Gamma_0 = \mathcal{G}$. Let $B \in \mathfrak{B}(\mathcal{G})$ be a self-adjoint operator and assume that $1 \in \rho(BM(\lambda_0))$ for some $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$. Then the operator $A_{[B]}$ in (A.2) is a self-adjoint extension of S in $\mathcal{H}$ such that $\lambda_0 \in \rho(A_{[B]})$. Furthermore, $\lambda \in \rho(A_0)$ is an eigenvalue of $A_{[B]}$ if and only if $1 \in \sigma_{\rm p}(BM(\lambda))$, for all $\lambda \in \rho(A_{[B]}) \cap \rho(A_0)$ one has $(1 - BM(\lambda))^{-1} \in \mathfrak{B}(\mathcal{G})$ and*

$$(A_{[B]} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(1 - BM(\lambda)\big)^{-1} B\gamma(\overline{\lambda})^*.$$

A typical way to satisfy the condition $1 \in \rho(BM(\lambda_0))$ in Corollary A.4 (or Theorem A.3) is to prove that $\|M(\lambda_0)\| \to 0$ for $\lambda_0 \to -\infty$ if A_0 is bounded from below. The next result contains a useful sufficient condition for the decay of the Weyl function along the negative half-line. Theorem A.5 is a special case of [13, Theorem 6.1], where in a more general setting the decay of the Weyl functions in different sectors of $\mathbb{C}$ is discussed.

Theorem A.5. *Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a quasi boundary triple for $T \subset S^*$ with corresponding Weyl function M, assume that $\operatorname{ran}\Gamma_0 = \mathcal{G}$, that A_0 is bounded from below and that*

$$\Gamma_1 |A_0 - \mu|^{-\beta} : \mathcal{H} \supset \operatorname{dom}\big(\Gamma_1 |A_0 - \mu|^{-\beta}\big) \to \mathcal{G}$$

is bounded for some $\mu \in \rho(A_0)$ and some $\beta \in \big(0, \frac{1}{2}\big]$. Then for all $w_0 < \min \sigma(A_0)$ there exists $D > 0$ such that

$$\|M(\lambda)\| \le \frac{D}{\big(\operatorname{dist}(\lambda, \sigma(A_0))\big)^{1-2\beta}}$$

holds for all $\lambda < w_0$.

Appendix B. Proof of Theorem 4.5

In order to prove Theorem 4.5, we show that the quadratic forms corresponding to H_ε and A_α are close to each other in a suitable sense. We fix a sufficiently small $\beta > 0$ such that the map in (4.6) is bijective. Let $\mathfrak{a}_\alpha$ be the quadratic form associated to A_α introduced in (4.3) and define for $\varepsilon \in (0, \beta)$

(B.1) $$\mathfrak{h}_\varepsilon[f, g] := \big(\nabla_{\mathbf{A}} f, \nabla_{\mathbf{A}} g\big)_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \big(V_\varepsilon f, g\big)_{L^2(\mathbb{R}^2)}, \qquad \operatorname{dom}\mathfrak{h}_\varepsilon := \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2).$$

It is not difficult to see that $\mathfrak{h}_\varepsilon$ is a densely defined, closed, symmetric, and semibounded form which is associated to H_ε. In the first lemma we show that the forms $\mathfrak{h}_\varepsilon$ are uniformly bounded from below.

Lemma B.1. *Let $\varepsilon \in (0,\beta)$ and consider the form $\mathfrak{h}_\varepsilon$ in (B.1). Then there exists a constant $\lambda_1 \in \mathbb{R}$ such that $\mathfrak{h}_\varepsilon \geq \lambda_1$ for all $\varepsilon \in (0,\beta)$. In particular, $(-\infty,\lambda_1) \subset \rho(\mathsf{H}_\varepsilon)$ for all $\varepsilon \in (0,\beta)$.*

Proof. It follows from [7, Proposition 3.1][1] that there exists $\lambda_1 \in \mathbb{R}$ such that

$$\big(\nabla|f|,\nabla|f|\big)_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + (V_\varepsilon f,f)_{L^2(\mathbb{R}^2)} \geq \lambda_1 \|f\|^2_{L^2(\mathbb{R}^2)}$$

holds for all $f \in C_0^\infty(\mathbb{R}^2)$. Combining this with the diamagnetic inequality (2.6) one concludes that $\mathfrak{h}_\varepsilon[f] \geq \lambda_1\|f\|^2_{L^2(\mathbb{R}^2)}$ for all $f \in C_0^\infty(\mathbb{R}^2)$. Now the result follows from the fact that $C_0^\infty(\mathbb{R}^2)$ is dense in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. $\square$

Next, we verify that the form $\mathfrak{a}_0$ corresponding to Landau Hamiltonian A_0 is relatively bounded with respect to the form $\mathfrak{h}_\varepsilon$ with constants which are independent of ε.

Lemma B.2. *Let $V \in L^\infty(\mathbb{R}^2)$ be real and supported in Σ_β, let $\varepsilon \in (0,\beta)$, define the function V_ε as in (4.7), and let the quadratic form $\mathfrak{h}_\varepsilon$ be as in (B.1). Then there exist constants $c_1, c_2 > 0$ independent of ε such that*

$$\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} \leq c_1 \mathfrak{h}_\varepsilon[f] + c_2\|f\|^2_{L^2(\mathbb{R}^2)} \tag{B.2}$$

holds for all $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$.

Proof. Fix $\delta > 0$ and let $v_\varepsilon := \sqrt{|V_\varepsilon|}$. Using the diamagnetic inequality (2.5) and a similar estimate as in [7, Proposition 3.1 (ii)][1] we deduce that there is a $\lambda_0 < 0$ depending on δ such that for all $\lambda \leq \lambda_0$ and all $g \in L^2(\mathbb{R}^2)$

$$\begin{aligned}
\|(\mathsf{A}_0-\lambda)^{-1/2} v_\varepsilon g\|^2_{L^2(\mathbb{R}^2)} &= \big(v_\varepsilon(\mathsf{A}_0-\lambda)^{-1} v_\varepsilon g, g\big)_{L^2(\mathbb{R}^2)} \\
&\leq \big\|v_\varepsilon(\mathsf{A}_0-\lambda)^{-1} v_\varepsilon g\big\|_{L^2(\mathbb{R}^2)} \cdot \|g\|_{L^2(\mathbb{R}^2)} \\
&\leq \big\|v_\varepsilon(-\Delta-\lambda)^{-1} v_\varepsilon |g|\big\|_{L^2(\mathbb{R}^2)} \cdot \|g\|_{L^2(\mathbb{R}^2)} \leq \delta\|g\|^2_{L^2(\mathbb{R}^2)}
\end{aligned}$$

is true. By taking adjoint we get that also $\|v_\varepsilon(\mathsf{A}_0-\lambda)^{-1/2} g\|^2_{L^2(\mathbb{R}^2)} \leq \delta\|g\|^2_{L^2(\mathbb{R}^2)}$ for all $g \in L^2(\mathbb{R}^2)$. This implies for $f \in \mathcal{H}^2_{\mathbf{A}}(\mathbb{R}^2)$

$$\begin{aligned}
\big|(V_\varepsilon f,f)_{L^2(\mathbb{R}^2)}\big| &\leq \|v_\varepsilon(\mathsf{A}_0-\lambda)^{-1/2}(\mathsf{A}_0-\lambda)^{1/2} f\|^2_{L^2(\mathbb{R}^2)} \\
&\leq \delta\|(\mathsf{A}_0-\lambda)^{1/2} f\|^2_{L^2(\mathbb{R}^2)} = \delta\big((\mathsf{A}_0-\lambda)f,f\big)_{L^2(\mathbb{R}^2)} \\
&= \delta\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} - \delta\lambda\|f\|^2_{L^2(\mathbb{R}^2)}
\end{aligned} \tag{B.3}$$

[1] Note that this result is formulated in [7] only for C^2-hypersurfaces but remains valid in the slightly less regular situation considered here. In fact, the key ingredient in the proof of [7, Proposition 3.1] that needs to be ensured for a regular, closed $C^{1,1}$-curve in $\mathbb{R}^2$ is [7, Hypothesis 2.3 (c)], which follows from [27, Theorem 5.1 and Theorem 5.7].

and since $\mathcal{H}^2_{\mathbf{A}}(\mathbb{R}^2)$ is dense in $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ this estimate extends to $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. Eventually, from (B.3) we conclude

$$\begin{aligned}\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} &= \mathfrak{h}_\varepsilon[f] - (V_\varepsilon f, f)_{L^2(\mathbb{R}^2)} \\ &\le \mathfrak{h}_\varepsilon[f] + \delta\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} - \delta\lambda\|f\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Choosing $\delta \in (0,1)$ this implies the claim (B.2). □

Let us denote by $\kappa = \dot\gamma_2\ddot\gamma_1 - \dot\gamma_1\ddot\gamma_2$ the signed curvature of Σ, where $\gamma = (\gamma_1, \gamma_2)\colon I \to \mathbb{R}^2$ is any natural parametrization of Σ ($|\dot\gamma| = 1$). In the following we will often make use of the transformation to tubular coordinates, which yields for $h \in L^1(\Sigma_\varepsilon)$ (see e.g. [7, Proposition 2.6] or [33])

$$\int_{\Sigma_\varepsilon} h(x)\mathsf{d}x = \int_\Sigma \int_{-\varepsilon}^{\varepsilon} h\big(x_\Sigma + t\nu(x_\Sigma)\big)\big(1 - t\kappa(x_\Sigma)\big)\mathsf{d}t\mathsf{d}\sigma(x_\Sigma). \tag{B.4}$$

In the next lemma we show a variant of the trace theorem which will be very useful for the proof of Theorem 4.5. For the sake of brevity, we use the following notation

$$\mathsf{j}(x_\Sigma, s) := x_\Sigma + s\nu(x_\Sigma) \qquad \text{and} \qquad \mathcal{J}(x_\Sigma, s) := 1 - s\kappa(x_\Sigma).$$

Lemma B.3. *Let Σ be the boundary of the simply connected $C^{1,1}$-domain $\Omega_{\rm i}$ and let $\beta > 0$ be such that the mapping in (4.6) is bijective. Then there exists a constant $C > 0$ independent of $s \in (-\beta, \beta)$ such that*

$$\int_\Sigma \big|f(\mathsf{j}(x_\Sigma, s))\big|^2 \mathsf{d}\sigma(x_\Sigma) \le C\|f\|^2_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)}$$

holds for all $f \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$.

Proof. Throughout the proof $c > 0$ denotes a generic positive constant, which varies from line to line. It suffices to show the claim for functions in the dense subspace $C_0^\infty(\mathbb{R}^2)$ of $\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)$. For $f \in C_0^\infty(\mathbb{R}^2)$ the main theorem of calculus, the chain rule, and $\frac{\mathsf{d}}{\mathsf{d}t}\mathsf{j}(x_\Sigma, st) = s\nu(x_\Sigma)$ yield

$$\begin{aligned}\Big|\big|f(\mathsf{j}(x_\Sigma, s))\big|^2 - \big|f(\mathsf{j}(x_\Sigma, 0))\big|^2\Big| &= \left|\int_0^1 \frac{\mathsf{d}}{\mathsf{d}t}(|f|^2)(\mathsf{j}(x_\Sigma, st))\mathsf{d}t\right| \\ &\le \int_0^1 \big|\big\langle\nabla(|f|^2)(\mathsf{j}(x_\Sigma, st)), s\nu(x_\Sigma)\big\rangle\big|\,\mathsf{d}t \\ &\le 2|s|\int_0^1 \big||f|\cdot\nabla(|f|)(\mathsf{j}(x_\Sigma, st))\big|\mathsf{d}t \\ &\le |s|\int_0^1 \Big[\big|\nabla(|f|)(\mathsf{j}(x_\Sigma, st))\big|^2 + \big|f(\mathsf{j}(x_\Sigma, st))\big|^2\Big]\,\mathsf{d}t \\ &\le \int_0^\beta \Big[\big|\nabla(|f|)(\mathsf{j}(x_\Sigma, r))\big|^2 + \big|f(\mathsf{j}(x_\Sigma, r))\big|^2\Big]\,\mathsf{d}r,\end{aligned} \tag{B.5}$$

where the substitution $r = st$ was employed in the last step. Next, by applying Corollary 2.3 we obtain

$$I_1 := \int_\Sigma \big|f(x_\Sigma)\big|^2 \mathsf{d}\sigma(x_\Sigma) \le c\big(\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \|f\|^2_{L^2(\mathbb{R}^2)}\big). \tag{B.6}$$

Using that there is some $c > 0$ such that $1 \le c\mathcal{J}(x_\Sigma, r)$ for all sufficiently small $r \le \beta$, formula (B.4), the diamagnetic inequality (2.6), and (B.5) we get

$$\begin{aligned} I_2 &:= \left|\int_\Sigma \Big(\big|f(\mathsf{j}(x_\Sigma,s))\big|^2 - \big|f(\mathsf{j}(x_\Sigma,0))\big|^2\Big)\,\mathsf{d}\sigma(x_\Sigma)\right| \\ &\le \int_\Sigma \int_0^\beta \Big[|\nabla(|f|)(\mathsf{j}(x_\Sigma,r))|^2 + |f(\mathsf{j}(x_\Sigma,r))|^2\Big]\,\mathsf{d}r\mathsf{d}\sigma(x_\Sigma) \\ &\le c\int_\Sigma \int_0^\beta \Big[|\nabla(|f|)(\mathsf{j}(x_\Sigma,r))|^2 + |f(\mathsf{j}(x_\Sigma,r))|^2\Big]\mathcal{J}(x_\Sigma,r)\mathsf{d}r\mathsf{d}\sigma(x_\Sigma) \\ &\le c\big(\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \|f\|^2_{L^2(\mathbb{R}^2)}\big). \end{aligned} \tag{B.7}$$

Combining (B.6) and (B.7) we arrive at

$$\int_\Sigma \big|f(\mathsf{j}(x_\Sigma,s))\big|^2 \mathsf{d}\sigma(x_\Sigma) \le I_1 + I_2 \le C\big(\|\nabla_{\mathbf{A}} f\|^2_{L^2(\mathbb{R}^2;\mathbb{C}^2)} + \|f\|^2_{L^2(\mathbb{R}^2)}\big)$$

which is the claim of this lemma. ☐

Proof of Theorem 4.5. According to Lemma B.1 the operators H_ε, $\varepsilon \in (0,\beta)$, are uniformly bounded from below by $\lambda_1 \in \mathbb{R}$. Moreover, by Proposition 4.4 the operator A_α is semibounded. From now on we fix $\lambda_0 \in \rho(\mathsf{A}_\alpha) \cap (-\infty, \lambda_1)$ and we use the notations $\mathsf{R}_\varepsilon := (\mathsf{H}_\varepsilon - \lambda_0)^{-1}$ and $\mathsf{R}'_\alpha := (\mathsf{A}_\alpha - \lambda_0)^{-1}$. Note that $\|\mathsf{R}_\varepsilon\| \le (\lambda_1 - \lambda_0)^{-1}$ for $\varepsilon \in (0,\beta)$. We claim that there is a constant $c > 0$ such that

$$\big\|\mathsf{R}_\varepsilon - \mathsf{R}'_\alpha\big\| \le c\sqrt{\varepsilon}, \qquad \varepsilon \in (0,\beta). \tag{B.8}$$

In fact, note first that

$$\begin{aligned} \big\|\mathsf{R}_\varepsilon - \mathsf{R}'_\alpha\big\| &= \sup_{\|u\|,\|v\|=1} \Big|\big((\mathsf{R}_\varepsilon - \mathsf{R}'_\alpha)u, v\big)_{L^2(\mathbb{R}^2)}\Big| \\ &= \sup_{\|u\|,\|v\|=1} \Big|\big(\mathsf{R}_\varepsilon u, (\mathsf{A}_\alpha - \lambda_0)\mathsf{R}'_\alpha v\big)_{L^2(\mathbb{R}^2)} - \big((\mathsf{H}_\varepsilon - \lambda_0)\mathsf{R}_\varepsilon u, \mathsf{R}'_\alpha v\big)_{L^2(\mathbb{R}^2)}\Big| \\ &= \sup_{\|u\|,\|v\|=1} \Big|\mathfrak{a}_\alpha\big[\mathsf{R}_\varepsilon u, \mathsf{R}'_\alpha v\big] - \mathfrak{h}_\varepsilon\big[\mathsf{R}_\varepsilon u, \mathsf{R}'_\alpha v\big]\Big|. \end{aligned}$$

The estimate (B.8) follows if we prove

$$\big|\mathfrak{a}_\alpha[f,g] - \mathfrak{h}_\varepsilon[f,g]\big| \le c\sqrt{\varepsilon}\big(\|f\|^2_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)} + \|g\|^2_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)}\big), \quad f,g \in \mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2), \tag{B.9}$$

since with the choice $f = \mathsf{R}_\varepsilon u$ and $g = \mathsf{R}'_\alpha v$ the inequality (B.9) together with (4.5) and Lemma B.2 shows

$$\begin{aligned}\Big|\mathfrak{a}_\alpha\big[\mathsf{R}_\varepsilon u, \mathsf{R}'_\alpha v\big] - \mathfrak{h}_\varepsilon\big[\mathsf{R}_\varepsilon u, \mathsf{R}'_\alpha v\big]\Big|&\\ &\leq c\sqrt{\varepsilon}\big(\mathfrak{h}_\varepsilon[\mathsf{R}_\varepsilon u] + \|\mathsf{R}_\varepsilon u\|^2_{L^2(\mathbb{R}^2)} + \mathfrak{a}_\alpha[\mathsf{R}'_\alpha v] + \|\mathsf{R}'_\alpha v\|^2_{L^2(\mathbb{R}^2)}\big)\\ &= c\sqrt{\varepsilon}\big((\mathsf{R}_\varepsilon u, u)_{L^2(\mathbb{R}^2)} + (1+\lambda_0)\|\mathsf{R}_\varepsilon u\|^2_{L^2(\mathbb{R}^2)}\\ &\qquad\qquad + (\mathsf{R}'_\alpha v, v)_{L^2(\mathbb{R}^2)} + (1+\lambda_0)\|\mathsf{R}'_\alpha v\|^2_{L^2(\mathbb{R}^2)}\big)\\ &\leq c\sqrt{\varepsilon}\big(\|u\|^2_{L^2(\mathbb{R}^2)} + \|v\|^2_{L^2(\mathbb{R}^2)}\big),\end{aligned}$$

where $\|\mathsf{R}_\varepsilon\| \leq (\lambda_1 - \lambda_0)^{-1}$ was used in the last estimate. Thanks to the polarization identity it suffices to prove (B.9) for $f = g$. Furthermore, it is sufficient to consider $f \in C_0^\infty(\mathbb{R}^2)$. By the definition of the forms $\mathfrak{a}_\alpha$ and $\mathfrak{h}_\varepsilon$, using $\operatorname{supp} V_\varepsilon \subset \Sigma_\varepsilon$, and (B.4) we find

$$\begin{aligned}\mathfrak{a}_\alpha[f] - \mathfrak{h}_\varepsilon[f] &= \int_\Sigma \alpha(x_\Sigma)|f(x_\Sigma)|^2 \mathsf{d}\sigma(x_\Sigma) - \int_{\mathbb{R}^2} V_\varepsilon(x)|f(x)|^2\mathsf{d}\sigma(x_\Sigma)\\ &= \int_\Sigma\int_{-\beta}^{\beta} V(\mathsf{j}(x_\Sigma,t))|f(x_\Sigma)|^2 \mathsf{d}t\mathsf{d}\sigma(x_\Sigma)\\ &\qquad - \frac{\beta}{\varepsilon}\int_\Sigma\int_{-\varepsilon}^{\varepsilon} V\big(\mathsf{j}(x_\Sigma, \tfrac{\beta s}{\varepsilon})\big)\big|f\big(\mathsf{j}(x_\Sigma,s)\big)\big|^2 \mathcal{J}(x_\Sigma,s)\mathsf{d}s\mathsf{d}\sigma(x_\Sigma),\end{aligned}$$

where in the last step the definitions of α and V_ε from (4.9) and (4.7) were substituted. Using the transformation $t = \frac{\beta}{\varepsilon}s$ in the last integral on the right hand side we find

$$\begin{aligned}\mathfrak{a}_\alpha[f] - \mathfrak{h}_\varepsilon[f] &= \frac{\varepsilon}{\beta}\int_\Sigma\int_{-\beta}^{\beta} V\big(\mathsf{j}(x_\Sigma,t)\big)\big|f\big(\mathsf{j}(x_\Sigma,\tfrac{\varepsilon t}{\beta})\big)\big|^2 t\kappa(x_\Sigma)\mathsf{d}t\mathsf{d}\sigma(x_\Sigma)\\ &\quad + \int_\Sigma\int_{-\beta}^{\beta} V\big(\mathsf{j}(x_\Sigma,t)\big)\big[|f(x_\Sigma)|^2 - \big|f\big(\mathsf{j}(x_\Sigma,\tfrac{\varepsilon t}{\beta})\big)\big|^2\big]\mathsf{d}t\mathsf{d}\sigma(x_\Sigma)\\ &:= I_1 + I_2.\end{aligned}\tag{B.10}$$

Since $\kappa, V \in L^\infty(\mathbb{R}^2)$ we obtain from Lemma B.3 for the first integral I_1 in (B.10) the estimate

$$|I_1| \leq c\varepsilon\|f\|^2_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)}.\tag{B.11}$$

In order to estimate the second integral I_2 in (B.10) we note first that by the main theorem of calculus

$$\begin{aligned}\Big|\big|f\big(\mathsf{j}(x_\Sigma,0)\big)\big|^2-\big|f\big(\mathsf{j}(x_\Sigma,\tfrac{\varepsilon t}{\beta})\big)\big|^2\Big| &= \left|\int_0^\varepsilon \frac{\mathsf{d}}{\mathsf{d}r}(|f|^2)\big(\mathsf{j}(x_\Sigma,\tfrac{rt}{\beta})\big)\mathsf{d}r\right| \\ &\leq \int_0^\varepsilon \Big|\Big\langle \nabla(|f|^2)\big(\mathsf{j}(x_\Sigma,\tfrac{rt}{\beta})\big),\tfrac{t}{\beta}\nu(x_\Sigma)\Big\rangle\Big|\mathsf{d}r \\ &\leq \frac{|t|}{\beta}\int_0^\varepsilon \Big|\nabla(|f|^2)\big(\mathsf{j}(x_\Sigma,\tfrac{rt}{\beta})\big)\Big|\mathsf{d}r \\ &\leq c\int_0^\varepsilon \big|\nabla(|f|)\big(\mathsf{j}(x_\Sigma,s)\big)\big|\cdot\big|f\big(\mathsf{j}(x_\Sigma,s)\big)\big|\mathsf{d}s,\end{aligned}$$

where the substitution $s=\frac{1}{\beta}rt$ was used in the last step. This and the Cauchy-Schwarz inequality lead to

$$\begin{aligned}|I_2|^2 &\leq c\left(\int_\Sigma\int_{-\beta}^{\beta}\int_0^\varepsilon \big|\nabla(|f|)\big(\mathsf{j}(x_\Sigma,s)\big)\big|\cdot\big|f\big(\mathsf{j}(x_\Sigma,s)\big)\big|\,)\mathsf{d}s\mathsf{d}t\mathsf{d}\sigma(x_\Sigma)\right)^2 \\ &\leq c\int_\Sigma\int_0^\varepsilon \big|\nabla(|f|)\big(\mathsf{j}(x_\Sigma,s)\big)\big|^2\mathsf{d}s\mathsf{d}\sigma(x_\Sigma)\cdot\int_\Sigma\int_0^\varepsilon \big|f\big(\mathsf{j}(x_\Sigma,s)\big)\big|^2\mathsf{d}s\mathsf{d}\sigma(x_\Sigma).\end{aligned}\tag{B.12}$$

Choose a constant c such that $1\leq c\mathcal{J}(x_\Sigma,s)$. Then using formula (B.4) and the diamagnetic inequality (2.6) we find that the first integral in the last equation can be estimated by

$$c\int_\Sigma\int_{-\varepsilon}^\varepsilon \big|\nabla(|f|)\big(\mathsf{j}(x_\Sigma,s)\big)\big|^2\mathcal{J}(x_\Sigma,s)\mathsf{d}s\mathsf{d}\sigma(x_\Sigma)\leq c\int_{\Sigma_\varepsilon}|\nabla_{\mathbf{A}}f|^2\mathsf{d}x\leq c\|f\|^2_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)}.$$

Moreover, the second integral on the right hand side of (B.12) can be estimated with Lemma B.3 by $c\varepsilon\|f\|^2_{\mathcal{H}^1_{\mathbf{A}}(\mathbb{R}^2)}$. Combining this with (B.11) and (B.10) we deduce (B.9) and hence (B.8).

Finally, we extend the result from (B.8) from $\lambda_0\in\rho(\mathsf{A}_\alpha)\cap(-\infty,\lambda_1)$ to all $\lambda\in\mathbb{C}\setminus\mathbb{R}$. For this we consider $\mathsf{D}_\varepsilon(\lambda):=(\mathsf{H}_\varepsilon-\lambda)^{-1}-(\mathsf{A}_\alpha-\lambda)^{-1}$. A simple computation shows

$$\mathsf{D}_\varepsilon(\lambda)=\big[1+(\lambda-\lambda_0)\big(\mathsf{A}_\alpha-\lambda\big)^{-1}\big]\cdot\mathsf{D}_\varepsilon(\lambda_0)\cdot\big[1+(\lambda-\lambda_0)\big(\mathsf{H}_\varepsilon-\lambda\big)^{-1}\big].$$

Hence the claimed convergence result is true for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$ and the order of convergence is $\sqrt{\varepsilon}$. This finishes the proof of Theorem 4.5. $\square$

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