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Schrödinger operators with δ-interactions supported on conical surfaces

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Abstract
We investigate the spectral properties of self-adjoint Schrödinger operators with attractive δ-interactions of constant strength $\alpha > 0$ supported on conical surfaces in $\mathbb{R}^3$. It is shown that the essential spectrum is given by $[-\alpha^2/4, +\infty)$ and that the discrete spectrum is infinite and accumulates to $-\alpha^2/4$. Furthermore, an asymptotic estimate of these eigenvalues is obtained.

Keywords: Schrödinger operator, delta potential, infinite discrete spectrum

1. Introduction

The purpose of this paper is to analyse the spectrum of the three–dimensional Schrödinger operator $-\Delta_{\alpha,C_\theta}$ with an attractive δ–interaction of constant strength $\alpha > 0$ supported on the conical surface

$$C_\theta := \left\{ (x, y, z) \in \mathbb{R}^3 : z := \cot(\theta)\sqrt{x^2 + y^2} \right\}, \quad \theta \in (0, \pi/2).$$

The Schrödinger operator $-\Delta_{\alpha,C_\theta}$ is defined via the first representation theorem [17, Theorem VI.2.1] as the unique self-adjoint operator in $L^2(\mathbb{R}^3)$ which is associated with the closed, densely defined, symmetric and semibounded quadratic form

$$\mathfrak{a}_{\alpha,C_\theta}[\psi] = \| \nabla\psi \|^2_{L^2\left(\mathbb{R}^3;\, \mathbb{C}^3\right)} - \alpha \int_{C_\theta} |\psi|^2 \,\mathrm{d}\sigma, \quad \mathrm{dom}\, \mathfrak{a}_{\alpha,C_\theta} = H^1\left(\mathbb{R}^3\right); \tag{1}$$

cf [1, 4]. In a short form the main result of this note is the following theorem.

Theorem. *For any $\theta \in (0, \pi/2)$ and $\alpha > 0$ the essential spectrum of the operator $-\Delta_{\alpha,C_\theta}$ is $[-\alpha^2/4, +\infty)$, the discrete spectrum is infinite and accumulates to $-\alpha^2/4$.*

In addition, we obtain an asymptotic estimate of the eigenvalues of $-\Delta_{\alpha,C_\theta}$ lying below $-\alpha^2/4$, see theorem 3.2. Roughly speaking, the infiniteness of the discrete spectrum is induced by global geometrical properties of the conical surface C_θ and is not related to the singularity at the tip or other local geometrical properties. In fact, the same effect remains present after a local deformation of C_θ; cf theorem 3.3.

Various relations between the geometry and the bound states of quantum systems have been studied intensively in recent decades (see, e.g. [19]) after it had been realized in [14] that curvature can give rise to an effective attractive interaction. In addition to systems with a hard-wall confinement the so-called 'leaky' structures attracted attention, see the review paper [11]. Their advantage is that they make it possible to take quantum tunnelling into account. The model discussed in this paper can describe, for instance, a structure composed of two semiconductors: a conical substrate of one material on the top of which we have a thin layer of the second one, covered by a bulk mass of the former.

The proof of our main result is based on standard techniques in spectral theory of self-adjoint operators: we construct singular sequences and use Neumann bracketing in the spirit of [13] to show the assertion on the essential spectrum; for the infiniteness of the discrete spectrum we employ variational principles. The same approach was applied in [25] in the context of Schrödinger operators with slowly decaying negative regular potentials, see also [23, §XIII.3]. Similar arguments were also used in [10, 15] for the closely related question of infiniteness of the discrete spectrum for the Dirichlet Laplacian in a conical layer, see also [7, 23, 18, 19, 21] for further progress in this problem. We also point out [6, 9, 12] for related spectral problems for Schrödinger operators with δ-potentials.

2. Essential spectrum of $-\Delta_{\alpha,C_\theta}$

In this section we show that the essential spectrum of the operator $-\Delta_{\alpha,C_\theta}$ is given by $[-\alpha^2/4, +\infty)$. The proof of the inclusion $\sigma_{\mathrm{ess}}(-\Delta_{\alpha,C_\theta}) \supseteq [-\alpha^2/4, +\infty)$ makes use of singular sequences and for the other inclusion a specially chosen Neumann bracketing is used. A similar type of argument was also employed in [1, 13] for δ and δ'-interactions on broken lines in the two-dimensional setting.

Theorem 2.1. *Let $-\Delta_{\alpha,C_\theta}$ be the self-adjoint operator in $L^2(\mathbb{R}^3)$ associated to the form (1) and let $\alpha > 0$ and $\theta \in (0, \pi/2)$. Then*

$$\sigma_{\mathrm{ess}}\left(-\Delta_{\alpha,C_\theta}\right) = [-\alpha^2/4, +\infty)$$

Remark 2.2. For completeness we mention that the above theorem is also valid in the case $\theta = \pi/2$, that is, the conical surface is a plane, and the statement can be shown directly via separation of variables.

Proof of theorem 2.1 Step 1. We verify the inclusion $\sigma_{ess}(-\Delta_{\alpha,C_\theta}) \supseteq [-\alpha^2/4, +\infty)$ by constructing singular sequences for the operator $-\Delta_{\alpha,C_\theta}$ for every point of the interval $[-\alpha^2/4, +\infty)$. Let us start by fixing a function $\chi_1 \in C_0^\infty(1, 2)$ such that

$$\| \chi_1 \|_{L^2(1,2)} = 1, \tag{2}$$

and a function $\chi_2 \in C_0^\infty(-\varepsilon, \varepsilon)$ with some fixed $\varepsilon \in (0, \tan\theta)$, which satisfies

$$0 \leqslant \chi_2 \leqslant 1 \quad \text{and} \quad \chi_2(t) = 1 \text{ for } |t| < \varepsilon/2. \tag{3}$$

Define for all $p \in \mathbb{R}$ and $n \in \mathbb{N}$ the functions $\omega_{n,p}: \mathbb{R}_+^2 \to \mathbb{C}$ as

$$\omega_{n,p}(s, t) := \frac{1}{\sqrt{n}}\left(\chi_1\left(\frac{s}{n}\right)\exp(\mathrm{i}ps)\right)\left(\chi_2\left(\frac{t}{n}\right)\exp\left(-\frac{\alpha}{2}|t|\right)\right) \in C\left(\mathbb{R}_+^2\right)$$

in the coordinate system (s, t) in figure 1. Here $\mathbb{R}_+^2$ denotes open right half-plane $\{(r, z) \in \mathbb{R}^2: r > 0\}$.

Note that because of the choice $\varepsilon \in (0, \tan\theta)$ we have $\operatorname{supp}\omega_{n,p} \subset \mathbb{R}_+^2$ for all $n \in \mathbb{N}$ and, moreover, the distances between the z-axis and the supports of $\omega_{n,p}$ satisfy

$$\rho_n := \inf\left\{r: (r, z) \in \operatorname{supp}\omega_{n,p}\right\} \to +\infty, \quad n \to \infty. \tag{4}$$

By dominated convergence, using (2) and (3), we get

$$\begin{aligned} \| \omega_{n,p} \|^2_{L^2(\mathbb{R}_+^2)} &= \left(\frac{1}{n}\int_n^{2n}\left|\chi_1\left(\frac{s}{n}\right)e^{\mathrm{i}ps}\right|^2 \mathrm{d}s\right)\left(\int_{-\varepsilon n}^{\varepsilon n}\left|\chi_2\left(\frac{t}{n}\right)\right|^2 e^{-\alpha|t|}\mathrm{d}t\right) \\ &= \int_{-\varepsilon n}^{\varepsilon n}\left|\chi_2\left(\frac{t}{n}\right)\right|^2 e^{-\alpha|t|}\mathrm{d}t \to \int_{-\infty}^{\infty} e^{-\alpha|t|}\mathrm{d}t = \frac{2}{\alpha}, \quad n \to \infty. \end{aligned} \tag{5}$$

We denote by $\omega_{n,p,\pm}$ the restrictions of $\omega_{n,p}$ onto the open subsets

$$S_+ = \left\{(r, z) \in \mathbb{R}_+^2: z > r\cot\theta\right\} \quad \text{and} \quad S_- = \left\{(r, z) \in \mathbb{R}_+^2: z < r\cot\theta\right\}$$

of $\mathbb{R}_+^2$. The partial derivatives of $\omega_{n,p,\pm}$ with respect to s and t are given by

$$\partial_s\omega_{n,p,\pm} = \frac{1}{\sqrt{n}}\left(\frac{1}{n}\chi_1'\left(\frac{s}{n}\right)e^{\mathrm{i}ps} + \mathrm{i}p\chi_1\left(\frac{s}{n}\right)e^{\mathrm{i}ps}\right)\left(\chi_2\left(\frac{t}{n}\right)e^{\pm\frac{\alpha}{2}t}\right),$$

$$\partial_t\omega_{n,p,\pm} = \frac{1}{\sqrt{n}}\left(\chi_1\left(\frac{s}{n}\right)e^{\mathrm{i}ps}\right)\left(\frac{1}{n}\chi_2'\left(\frac{t}{n}\right)e^{\pm\frac{\alpha}{2}t} \pm \frac{\alpha}{2}\chi_2\left(\frac{t}{n}\right)e^{\pm\frac{\alpha}{2}t}\right).$$

Similarly as in (5), using dominated convergence, we get

$$\| \nabla\omega_{n,p} \|^2_{L^2(\mathbb{R}_+^2;\, \mathbb{C}^2)} = \int_{\mathbb{R}_+^2}\left(\left|\partial_s\omega_{n,p}\right|^2 + \left|\partial_t\omega_{n,p}\right|^2\right)\mathrm{d}s\mathrm{d}t \to \left(p^2 + \frac{\alpha^2}{4}\right)\frac{2}{\alpha}, \quad n \to \infty. \tag{6}$$

Let us define the sequence of functions $\psi_{n,p}: \mathbb{R}^3 \to \mathbb{C}$ as

$$\psi_{n,p}(r, \varphi, z) := \frac{\omega_{n,p}(r, z)}{\sqrt{2\pi r}}, \quad n \in \mathbb{N}, \tag{7}$$

where the functions $\omega_{n,p}: \mathbb{R}_+^2 \to \mathbb{C}$ are interpreted as rotationally invariant functions on $\mathbb{R}^3$ in the cylindrical coordinate system (r, φ, z). The hypersurface C_θ separates the Euclidean space $\mathbb{R}^3$ into two unbounded Lipschitz domains Ω_+ and Ω_-, where

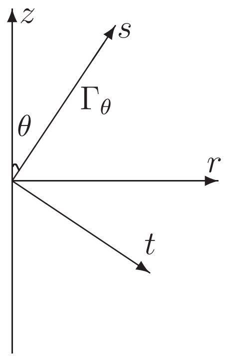

Figure 1. The right half-plane $\mathbb{R}^2_+$ with the coordinate system (r, z). The ray Γ_θ emerges from the origin and constitutes the angle $\theta \in (0, \pi/2)$ with the z-axis. The coordinate system (s,t) is associated with Γ_θ.

$$\Omega_+ = \left\{ (x, y, z) \in \mathbb{R}^3 \colon z > \cot(\theta)\sqrt{x^2 + y^2} \right\},$$
$$\Omega_- = \left\{ (x, y, z) \in \mathbb{R}^3 \colon z < \cot(\theta)\sqrt{x^2 + y^2} \right\}.$$

We use the notation $\psi_{n,p,\pm} := \psi_{n,p}|_{\Omega_\pm}$. Then $\psi_{n,p,\pm} \in C^\infty(\Omega_\pm)$ and from (5) we obtain

$$\| \psi_{n,p} \|^2_{L^2(\mathbb{R}^3)} = \| \omega_{n,p} \|^2_{L^2(\mathbb{R}^2_+)} \to \frac{2}{\alpha}, \quad n \to \infty. \tag{8}$$

We claim that $\psi_{n,p} \in \mathrm{dom}\,(-\Delta_{\alpha,C_\theta})$. For this we still need to check that the boundary conditions

$$\psi_{n,p,+}\big|_{C_\theta} = \psi_{n,p,-}\big|_{C_\theta} \quad \text{and} \quad \partial_{\nu_+}\psi_{n,p,+}\big|_{C_\theta} + \partial_{\nu_-}\psi_{n,p,-}\big|_{C_\theta} = \alpha\psi_{n,p}\big|_{C_\theta} \tag{9}$$

are satisfied; cf [1, Theorem 3.3(i)]. In fact, by the definition of $\omega_{n,p}$ we have $\omega_{n,p,+}|_{C_\theta} = \omega_{n,p,-}|_{C_\theta}$, where $\omega_{n,p,\pm}$ are interpreted as rotationally invariant functions on $\Omega_\pm$. This implies that the first condition (9) holds. Furthermore, one computes

$$\partial_{\nu_+}\omega_{n,p,+}\big|_{C_\theta} + \partial_{\nu_-}\omega_{n,p,-}\big|_{C_\theta} = \alpha\frac{1}{\sqrt{n}}\left(\chi_1\left(\frac{s}{n}\right)\exp\,(ips)\right) = \alpha\omega_{n,p}\big|_{C_\theta}. \tag{10}$$

The gradient of $\psi_{n,p,\pm}$ can be expressed as

$$\nabla\psi_{n,p,\pm} = \frac{1}{\sqrt{2\pi r}}\nabla\omega_{n,p,\pm} + \omega_{n,p,\pm}\nabla\left(\frac{1}{\sqrt{2\pi r}}\right),$$

where ∇ acts on the functions $(r, \varphi, z) \mapsto \omega_{n,p,\pm}(r, z)$ and $(r, \varphi, z) \mapsto \frac{1}{\sqrt{2\pi r}}$. Hence, we obtain

$$\partial_{\nu_+}\psi_{n,p,+}\Big|_{C_\theta} + \partial_{\nu_-}\psi_{n,p,-}\Big|_{C_\theta} = \left(\frac{1}{\sqrt{2\pi r}}\Big|_{C_\theta}\right)\left(\partial_{\nu_+}\omega_{n,p,+}\Big|_{C_\theta} + \partial_{\nu_-}\omega_{n,p,-}\Big|_{C_\theta}\right)$$
$$+\left(\omega_{n,p}\Big|_{C_\theta}\right)\left(\partial_{\nu_+}\left(\frac{1}{\sqrt{2\pi r}}\right)\Big|_{C_\theta} + \partial_{\nu_-}\left(\frac{1}{\sqrt{2\pi r}}\right)\Big|_{C_\theta}\right)$$
$$= \left(\frac{1}{\sqrt{2\pi r}}\Big|_{C_\theta}\right)\alpha\left(\omega_{n,p}\Big|_{C_\theta}\right) = \alpha\psi_{n,p}\Big|_{C_\theta},$$

where (10) was used in the second equality. Thus we have verified (9) and therefore $\psi_{n,p} \in \operatorname{dom}(-\Delta_{\alpha,C_\theta})$. Moreover, according to [1, Theorem 3.3(i)] we also have

$$-\Delta_{\alpha,C_\theta}\psi_{n,p} = \left(-\Delta\psi_{n,p,+}\right) \oplus \left(-\Delta\psi_{n,p,-}\right). \tag{11}$$

Using the expression for the three-dimensional Laplacian in cylindrical coordinates we find

$$-\Delta\psi_{n,p,\pm} = -\frac{1}{r}\partial_r\left(r\partial_r\psi_{n,p,\pm}\right) - \partial_z^2\psi_{n,p,\pm},$$

where the angular term is absent since the functions $\psi_{n,p,\pm}$ do not depend on φ. The above expression can be rewritten as

$$-\Delta\psi_{n,p,\pm} = -\partial_r^2\psi_{n,p,\pm} - \partial_z^2\psi_{n,p,\pm} - \frac{1}{r}\left(\partial_r\psi_{n,p,\pm}\right). \tag{12}$$

Next we compute the first and second order partial derivatives of $\psi_{n,p,\pm}$ with respect to r:

$$\begin{aligned}\partial_r\psi_{n,p,\pm} &= \frac{\partial_r\omega_{n,p,\pm}}{\sqrt{2\pi r}} - \frac{\omega_{n,p,\pm}}{2\sqrt{2\pi}\,r^{3/2}},\\ \partial_r^2\psi_{n,p,\pm} &= \frac{\partial_r^2\omega_{n,p,\pm}}{\sqrt{2\pi r}} - \frac{\partial_r\omega_{n,p,\pm}}{\sqrt{2\pi}\,r^{3/2}} + \frac{3}{4}\frac{\omega_{n,p,\pm}}{\sqrt{2\pi}\,r^{5/2}}.\end{aligned} \tag{13}$$

The last two summands in the expression for $\partial_r^2\psi_{n,p,\pm}$ can be estimated in L^2-norm as

$$\begin{aligned}\left\|\frac{\partial_r\omega_{n,p,\pm}}{\sqrt{2\pi}\,r^{3/2}}\right\|^2_{L^2(\mathbb{R}^3)} &\leqslant \frac{1}{\rho_n^2}\,\|\nabla\omega_{n,p}\|^2_{L^2(\mathbb{R}^2_+;\,\mathbb{C}^2)} \to 0, \quad n\to\infty,\\ \frac{9}{16}\left\|\frac{\omega_{n,p,\pm}}{\sqrt{2\pi}\,r^{5/2}}\right\|^2_{L^2(\mathbb{R}^3)} &\leqslant \frac{9}{16\rho_n^4}\,\|\omega_{n,p}\|^2_{L^2(\mathbb{R}^2_+)} \to 0, \quad n\to\infty,\end{aligned} \tag{14}$$

where we have used (4), (5) and (6). The second order partial derivatives of $\psi_{n,p,\pm}$ with respect to z are

$$\partial_z^2\psi_{n,p,\pm} = \frac{\partial_z^2\omega_{n,p,\pm}}{\sqrt{2\pi r}}. \tag{15}$$

Using (13), (14), (15) and the invariance of the Laplacian under rotation of the coordinate system we obtain that

$$-\partial_r^2\psi_{n,p,\pm} - \partial_z^2\psi_{n,p,\pm} = -\frac{1}{\sqrt{2\pi r}}\left(\partial_s^2\omega_{n,p,\pm} + \partial_t^2\omega_{n,p,\pm}\right) + o(1), \quad n\to\infty; \tag{16}$$

here and in the following we understand $o(1)$ in the strong sense with respect to the corresponding L^2-norm. With the help of (13) the norm of the last summand on the right hand side in (12) can be estimated as

$$\left\| \frac{\partial_r \psi_{n,p,\pm}}{r} \right\|_{L^2(\mathbb{R}^3)} \leqslant \left\| \frac{\partial_r \omega_{n,p,\pm}}{\sqrt{2\pi}\, r^{3/2}} \right\|_{L^2(\mathbb{R}^3)} + \left\| \frac{\omega_{n,p,\pm}}{2\sqrt{2\pi}\, r^{5/2}} \right\|_{L^2(\mathbb{R}^3)},$$

and from (14) we conclude

$$\left\| \frac{\partial_r \psi_{n,p,\pm}}{r} \right\|_{L^2(\mathbb{R}^3)} = o(1), \quad n \to \infty.$$

From (12), the latter result and (16) we obtain

$$-\Delta \psi_{n,p,\pm} = -\frac{1}{\sqrt{2\pi r}} \left(\partial_s^2 \omega_{n,p,\pm} + \partial_t^2 \omega_{n,p,\pm} \right) + o(1), \quad n \to \infty. \tag{17}$$

Again using dominated convergence we compute

$$\begin{aligned} \partial_s^2 \omega_{n,p,\pm} &= \frac{1}{\sqrt{n}} \left(\chi_2 \left(\frac{t}{n} \right) e^{\pm \frac{\alpha}{2} t} \right) \left(\frac{1}{n^2} \chi_1'' \left(\frac{s}{n} \right) e^{\mathrm{i}ps} + \frac{2\mathrm{i}p}{n} \chi_1' \left(\frac{s}{n} \right) e^{\mathrm{i}ps} - p^2 \chi_1 \left(\frac{s}{n} \right) e^{\mathrm{i}ps} \right) \\ &= -p^2 \omega_{n,p,\pm} + o(1), \quad n \to \infty, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \partial_t^2 \omega_{n,p,\pm} &= \frac{1}{\sqrt{n}} \left(\chi_1 \left(\frac{s}{n} \right) e^{\mathrm{i}ps} \right) \left(\frac{1}{n^2} \chi_2'' \left(\frac{t}{n} \right) e^{\pm \frac{\alpha}{2} t} \pm \frac{\alpha}{n} \chi_2' \left(\frac{t}{n} \right) e^{\pm \frac{\alpha}{2} t} + \frac{\alpha^2}{4} \chi_2 \left(\frac{t}{n} \right) e^{\pm \frac{\alpha}{2} t} \right) \\ &= \frac{\alpha^2}{4} \omega_{n,p,\pm} + o(1), \quad n \to \infty. \end{aligned} \tag{19}$$

Finally, employing (11), (17), the definition of $\psi_{n,p}$ in (7) and (18), (19) we arrive at

$$-\Delta_{\alpha, C_\theta} \psi_{n,p} = \left(-\frac{\alpha^2}{4} + p^2 \right) \psi_{n,p} + o(1), \quad n \to \infty. \tag{20}$$

Since the supports of $\psi_{2^k,p}$ and $\psi_{2^{k'},p}$, $k \neq k'$, are disjoint, the sequence $\{\psi_{2^k,p}\}_k$ converges weakly to zero. Moreover, by (8) we have $\liminf \|\psi_{2^k,p}\|_{L^2(\mathbb{R}^3)} > 0$ and hence (20) implies that $\{\psi_{2^k,p}\}_k$ is a singular sequence for the operator $-\Delta_{\alpha,C_\theta}$ corresponding to the point $-\alpha^2/4 + p^2$. Therefore, $-\alpha^2/4 + p^2 \in \sigma_{\mathrm{ess}}(-\Delta_{\alpha,C_\theta})$ for all $p \in \mathbb{R}$ (see, e.g. [3, Theorem 9.1.2] or [24, Proposition 8.11]) and it follows that $[-\alpha^2/4, +\infty) \subseteq \sigma_{\mathrm{ess}}(-\Delta_{\alpha,C_\theta})$.

Step 2. In this step we show the inclusion $\sigma_{\mathrm{ess}}(-\Delta_{\alpha,C_\theta}) \subseteq [-\alpha^2/4, +\infty)$ using form decomposition methods. For sufficiently large $n \in \mathbb{N}$ we define three subsets of the closed half-plane $\overline{\mathbb{R}^2_+} := \{(r, z) \in \mathbb{R}^2 : r \geqslant 0,\ z \in \mathbb{R}\}$

$$\begin{aligned} \pi_n^1 &:= \left\{ (r(s,t), z(s,t)) \in \overline{\mathbb{R}^2_+} : s > n, |t| < \sqrt{n} \right\} \subset \overline{\mathbb{R}^2_+}, \\ \pi_n^2 &:= \left\{ (r(s,t), z(s,t)) \in \overline{\mathbb{R}^2_+} : s < n, |t| < \sqrt{n} \right\} \subset \overline{\mathbb{R}^2_+}, \\ \pi_n^3 &:= \left\{ (r(s,t), z(s,t)) \in \overline{\mathbb{R}^2_+} : |t| > \sqrt{n} \right\} \subset \overline{\mathbb{R}^2_+}, \end{aligned}$$

as shown in figure 2.

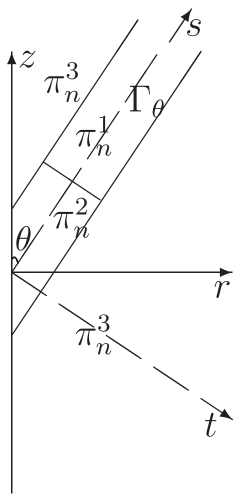

Figure 2. The subsets π_n^1, π_n^2 and π_n^3 of the closed half-plane $\overline{\mathbb{R}_+^2}$.

The ray Γ_θ, which emerges from the origin and constitutes the angle θ with z-axis, is decomposed into

$$\Gamma_{\theta,\,n}^1 := \{ (r(s,\,t),\, z(s,\,t)) \in \Gamma_\theta \colon s > n \},$$
$$\Gamma_{\theta,\,n}^2 := \{ (r(s,\,t),\, z(s,\,t)) \in \Gamma_\theta \colon s < n \}.$$

The splitting $\{\pi_n^k\}_{k=1}^3$ of $\overline{\mathbb{R}_+^2}$ induces the splitting of $\mathbb{R}^3$ into three domains

$$\Omega_n^k := \left\{ (r,\, \varphi,\, z) \colon (r,\, z) \in \pi_n^k,\ \varphi \in [0,\, 2\pi) \right\} \subset \mathbb{R}^3, \quad k = 1,\, 2,\, 3,$$

and the splitting of the conical surface C_θ into two parts

$$C_{\theta,\,n}^1 := \left\{ (r,\, \varphi,\, z) \colon (r,\, z) \in \Gamma_{\theta,\,n}^1,\ \varphi \in [0,\, 2\pi) \right\} \subset C_\theta,$$
$$C_{\theta,\,n}^2 := \left\{ (r,\, \varphi,\, z) \colon (r,\, z) \in \Gamma_{\theta,\,n}^2,\ \varphi \in [0,\, 2\pi) \right\} \subset C_\theta.$$

We agree to denote the restriction of $\psi \in L^2(\mathbb{R}^3)$ onto Ω_n^k with $k = 1, 2, 3$ by ψ_k.

Consider the quadratic form

$$\mathfrak{a}_{\alpha,C_\theta,n}[\psi] := \sum_{k=1}^{3} \| \nabla\psi_k \|^2_{L^2\left(\Omega_n^k;\, \mathbb{C}^3\right)} - \alpha \,\| \psi_1 |_{C_{\theta,n}^1} \|^2_{L^2\left(C_{\theta,n}^1\right)} - \alpha \,\| \psi_2|_{C_{\theta,n}^2} \|^2_{L^2\left(C_{\theta,n}^2\right)},$$
$$\mathrm{dom}\, \mathfrak{a}_{\alpha,C_\theta,n} := \bigoplus_{k=1}^{3} H^1\left(\Omega_n^k\right).$$

As in the proof of [1, Proposition 3.1] one verifies that the form $\mathfrak{a}_{\alpha,C_\theta,n}$ is closed, densely defined, symmetric and semibounded from below. Hence $\mathfrak{a}_{\alpha,C_\theta,n}$ induces a self-adjoint operator $-\Delta_{\alpha,C_\theta,n}$ in $L^2(\mathbb{R}^3)$ via the first representation theorem [17, Theorem VI.2.1]. The operator $-\Delta_{\alpha,C_\theta,n}$ can be decomposed into an orthogonal sum $\oplus_{k=1}^3 H_{n,k}$ of self-adjoint operators $H_{n,k}$ in $L^2(\Omega_n^k)$ with respect to the orthogonal decomposition $L^2(\mathbb{R}^3) = \oplus_{k=1}^3 L^2(\Omega_n^k)$, where $H_{n,1}$ and $H_{n,2}$ correspond to the quadratic forms

$$\mathfrak{a}_{n,1}[\psi_1] = \| \nabla\psi_1 \|^2_{L^2\left(\Omega_n^1;\, \mathbb{C}^3\right)} - \alpha \| \psi_1|_{C^1_{\theta,n}} \|^2_{L^2\left(C^1_{\theta,n}\right)}, \quad \operatorname{dom} \mathfrak{a}_{n,1} = H^1\left(\Omega_n^1\right),$$

$$\mathfrak{a}_{n,2}[\psi_2] = \| \nabla\psi_2 \|^2_{L^2\left(\Omega_n^2;\, \mathbb{C}^3\right)} - \alpha \| \psi_2|_{C^2_{\theta,n}} \|^2_{L^2\left(C^2_{\theta,n}\right)}, \quad \operatorname{dom} \mathfrak{a}_{n,2} = H^1\left(\Omega_n^2\right),$$

respectively, and $H_{n,3}$ corresponds to the quadratic form

$$\mathfrak{a}_{n,3}[\psi_3] = \| \nabla\psi_3 \|^2_{L^2\left(\Omega_n^3;\, \mathbb{C}^3\right)}, \ \operatorname{dom} \mathfrak{a}_{n,3} = H^1\left(\Omega_n^3\right).$$

Let us first estimate the spectrum of $H_{n,1}$. For this, note that $C^\infty(\Omega_n^1) \cap H^1(\Omega_n^1)$ is a core of $\mathfrak{a}_{n,1}$ and thus it suffices to use functions from this set in the estimates below (see, e.g. [8, Theorem 4.5.3]). For any $\psi_1 \in C^\infty(\Omega_n^1) \cap H^1(\Omega_n^1)$ normalized as $\| \psi_1 \|_{L^2(\Omega_n^1)} = 1$ we obtain

$$\mathfrak{a}_{n,1}[\psi_1] \geqslant \int_0^{2\pi} \left(\int_n^{+\infty} \int_{-\sqrt{n}}^{\sqrt{n}} r(s,\, t) \left| \partial_t \psi_1(s,\, t,\, \varphi) \right|^2 \mathrm{d}t\mathrm{d}s \right.$$
$$\left. -\alpha \int_n^{+\infty} r(s,\, 0) \left| \psi_1(s,\, 0,\, \varphi) \right|^2 \mathrm{d}s \right) \mathrm{d}\varphi,$$

where we have used the form of the gradient in cylindrical coordinates and the invariance of the gradient with respect to rotations of the coordinate system, and the non-negative terms corresponding to the partial derivatives of ψ_1 with respect to φ and s were estimated from below by zero. Note that for simple geometric reasons we have $r(s,\, t) \geqslant r(s,\, -\sqrt{n})$ for all $(s,\, t) \in \pi_n^1$. Using this observation we get

$$\mathfrak{a}_{n,1}[\psi_1] \geqslant \int_0^{2\pi} \int_n^{+\infty} r(s,\, -\sqrt{n}) \left(\int_{-\sqrt{n}}^{\sqrt{n}} \left| \partial_t \psi_1(s,\, t,\, \varphi) \right|^2 \mathrm{d}t \right.$$
$$\left. - \frac{\alpha r(s,\, 0)}{r(s,\, -\sqrt{n})} \left| \psi_1(s,\, 0,\, \varphi) \right|^2 \right) \mathrm{d}s\mathrm{d}\varphi. \tag{21}$$

Consider the closed, densely defined, symmetric and semibounded form

$$\mathfrak{b}[h] = \int_{-\sqrt{n}}^{\sqrt{n}} \left| h'(t) \right|^2 \mathrm{d}t - \beta |h(0)|^2, \quad \operatorname{dom} \mathfrak{b} = H^1((-\sqrt{n},\, \sqrt{n})),$$

and denote by $\mu(\beta,\, 2\sqrt{n}) < 0$ the lower bound of the spectrum of the associated 1-D Schrödinger operator on the interval $(-\sqrt{n},\, \sqrt{n})$ with Neumann boundary conditions at the endpoints and attractive δ-interaction of strength $\beta > 0$ located at 0. Then

$$\mathfrak{b}[h] \geqslant \mu(\beta,\, 2\sqrt{n}) \int_{-\sqrt{n}}^{\sqrt{n}} |h(t)|^2 \mathrm{d}t$$

holds for all $h \in H^1((-\sqrt{n},\, \sqrt{n}))$ and hence (21) can be further estimated as

$$\mathfrak{a}_{n,1}[\psi_1] \geqslant \int_0^{2\pi} \int_n^{+\infty} \mu\left(\frac{\alpha r(s,\, 0)}{r(s,\, -\sqrt{n})},\, 2\sqrt{n} \right) \int_{-\sqrt{n}}^{\sqrt{n}} r(s,\, -\sqrt{n}) \left| \psi_1(s,\, t,\, \varphi) \right|^2 \mathrm{d}t\mathrm{d}s\mathrm{d}\varphi. \tag{22}$$

By the definition of π_n^1 one has

$$r(s,\, -\sqrt{n}) = r(s,\, t)\left(1 + O\left(\frac{1}{\sqrt{n}} \right) \right), \quad n \to \infty, \tag{23}$$

for $(s, t) \in \pi_n^1$, where the remainder is uniform in s. Hence, we obtain from (22) and (23)

$$\mathfrak{a}_{n,1}[\psi_1] \geqslant \mu\left(\alpha\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), 2\sqrt{n}\right)\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), \quad n \to \infty, \tag{24}$$

where we used that

$$\int_0^{2\pi} \int_n^{+\infty} \int_{-\sqrt{n}}^{\sqrt{n}} r(s, t) \left|\psi_1(s, t, \varphi)\right|^2 \mathrm{d}t\mathrm{d}s\mathrm{d}\varphi = \| \psi_1 \|^2_{L^2\left(\Omega_n^1\right)} = 1.$$

According to [16, Proposition 2.5] the following estimate

$$\mu(\beta, 2\sqrt{n}) \geqslant -\frac{\beta^2}{4} - C\beta^2 \exp\left(-\tfrac{1}{2}\beta\sqrt{n}\right)$$

holds with some constant $C > 0$ and n sufficiently large. Hence,

$$\mu\left(\alpha\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), 2\sqrt{n}\right) \geqslant -\frac{\alpha^2}{4} + O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.$$

Plugging the above estimate into (24) we arrive at

$$\mathfrak{a}_{n,1}[\psi_1] \geqslant -\frac{\alpha^2}{4} + O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.$$

Hence, for any $\varepsilon > 0$ there exists a sufficiently large n for which

$$\inf \sigma\left(H_{n,1}\right) \geqslant -\frac{\alpha^2}{4} - \varepsilon. \tag{25}$$

As $H^1(\Omega_n^2)$ is compactly embedded into $L^2(\Omega_n^2)$ the essential spectrum of $H_{n,2}$ is empty. The operator $H_{n,3}$ is non-negative and hence $\sigma(H_{n,3}) \subseteq [0, +\infty)$. Due to the orthogonal decomposition $-\Delta_{\alpha,C_\theta,n} = \bigoplus_{k=1}^3 H_{n,k}$ the property (25) implies that for any $\varepsilon > 0$ there exists a sufficiently large n for which

$$\inf \sigma_{\mathrm{ess}}\left(-\Delta_{\alpha,C_\theta,n}\right) \geqslant -\frac{\alpha^2}{4} - \varepsilon. \tag{26}$$

Finally, we apply a Neumann bracketing argument. Notice that the ordering $\mathfrak{a}_{\alpha,C_\theta,n} \leqslant \mathfrak{a}_{\alpha,C_\theta}$ holds in the sense of quadratic forms; cf [17, §VI.5]. Hence by [3, Theorem 10.2.4]

$$\inf \sigma_{\mathrm{ess}}\left(-\Delta_{\alpha,C_\theta,n}\right) \leqslant \inf \sigma_{\mathrm{ess}}\left(-\Delta_{\alpha,C_\theta}\right). \tag{27}$$

In view of (27) the estimate (26) implies that for any $\varepsilon > 0$

$$\inf \sigma_{\mathrm{ess}}\left(-\Delta_{\alpha,C_\theta}\right) \geqslant -\frac{\alpha^2}{4} - \varepsilon$$

and thus passing to the limit $\varepsilon \to 0+$ we arrive at

$$\inf \sigma_{\mathrm{ess}}\left(-\Delta_{\alpha,C_\theta}\right) \geqslant -\frac{\alpha^2}{4},$$

which shows the inclusion $\sigma_{\mathrm{ess}}(-\Delta_{\alpha,C_\theta}) \subseteq [-\alpha^2/4, +\infty)$ and finishes the proof of theorem 2.1. □

3. Discrete spectrum of $-\Delta_{\alpha,C_\theta}$

In this section we show that the discrete spectrum of the self-adjoint operator $-\Delta_{\alpha,C_\theta}$ below the bottom $-\alpha^2/4$ of the essential spectrum is infinite for all angles $\theta \in (0, \pi/2)$ and we estimate the rate of the convergence of these eigenvalues to $-\alpha^2/4$ with the help of variational principles. The following lemma will be useful.

Lemma 3.1. *Let $\mathfrak{a}_{\alpha,C_\theta}$ be the form in (1). For $\omega \in H^1(\mathbb{R}^2_+)$ with compact support* $\operatorname{supp}\omega \subset \mathbb{R}^2_+$ *define the function $\psi(r,\varphi,z) := \frac{\omega(r,z)}{\sqrt{2\pi r}}$. Then $\psi \in H^1(\mathbb{R}^3)$ and*

$$\mathfrak{a}_{\alpha,C_\theta}[\psi] = \| \nabla\omega \|^2_{L^2(\mathbb{R}^2_+;\,\mathbb{C}^2)} - \int_{\mathbb{R}^2_+} \frac{1}{4r^2}|\omega(r,z)|^2 \mathrm{d}r\mathrm{d}z - \alpha \| \omega|_{\Gamma_\theta} \|^2_{L^2(\Gamma_\theta)}, \tag{28}$$

where Γ_θ is the ray in figure 1.

Proof. First of all observe that

$$\| \psi \|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}}\int_{\mathbb{R}_+}\int_0^{2\pi} \frac{|\omega(r,z)|^2}{2\pi r}\, r\, \mathrm{d}\varphi\mathrm{d}r\mathrm{d}z = \| \omega \|^2_{L^2(\mathbb{R}^2_+)} < \infty. \tag{29}$$

Moreover, we compute

$$\partial_r\psi = \frac{\partial_r\omega}{\sqrt{2\pi r}} - \frac{\omega}{2r\sqrt{2\pi r}} \quad \text{and} \quad \partial_z\psi = \frac{\partial_z\omega}{\sqrt{2\pi r}}, \tag{30}$$

and setting $\rho := \inf\{r\colon (r,z) \in \operatorname{supp}\omega\} > 0$ we obtain

$$\begin{aligned} \| \nabla\psi \|^2_{L^2(\mathbb{R}^3;\,\mathbb{C}^3)} &= \| \partial_r\psi \|^2_{L^2(\mathbb{R}^3)} + \| \partial_z\psi \|^2_{L^2(\mathbb{R}^3)} \\ &\leqslant 2\left\| \frac{\partial_r\omega}{\sqrt{2\pi r}} \right\|^2_{L^2(\mathbb{R}^3)} + 2\left\| \frac{\omega}{2r\sqrt{2\pi r}} \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \frac{\partial_z\omega}{\sqrt{2\pi r}} \right\|^2_{L^2(\mathbb{R}^3)} \\ &\leqslant 2\,\| \partial_r\omega \|^2_{L^2(\mathbb{R}^2_+)} + \frac{1}{2\rho^2}\| \omega \|^2_{L^2(\mathbb{R}^2_+)} + \| \partial_z\omega \|^2_{L^2(\mathbb{R}^2_+)} < \infty. \end{aligned} \tag{31}$$

Hence (29) and (31) imply $\psi \in H^1(\mathbb{R}^3)$. Next we substitute ψ in the form $\mathfrak{a}_{\alpha,C_\theta}$ in (1). It follows from the form of $\partial_z\psi$ in (30) and $\| \psi|_{C_\theta} \|^2_{L^2(C_\theta)} = \| \omega|_{\Gamma_\theta} \|^2_{L^2(\Gamma_\theta)}$ that

$$\begin{aligned} \mathfrak{a}_{\alpha,C_\theta}[\psi] &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} |\partial_r\psi|^2 2\pi r\mathrm{d}r\mathrm{d}z + \int_{\mathbb{R}}\int_{\mathbb{R}_+} |\partial_z\psi|^2 2\pi r\mathrm{d}r\mathrm{d}z - \alpha \| \psi|_{C_\theta} \|^2_{L^2(C_\theta)} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} |\partial_r\psi|^2 2\pi r\mathrm{d}r\mathrm{d}z + \int_{\mathbb{R}}\int_{\mathbb{R}_+} |\partial_z\omega|^2 \mathrm{d}r\mathrm{d}z - \alpha \| \omega|_{\Gamma_\theta} \|^2_{L^2(\Gamma_\theta)}. \end{aligned} \tag{32}$$

Denote the first integral by I_ψ. Making use of $\partial_r\psi$ in (30) we rewrite I_ψ as

$$I_\psi = \int_{\mathbb{R}}\int_{\mathbb{R}_+} |\partial_r\omega|^2 \mathrm{d}r\mathrm{d}z + \int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{1}{4r^2}|\omega|^2 \mathrm{d}r\mathrm{d}z - \int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{1}{r} \operatorname{Re}\left(\partial_r\omega\overline{\omega}\right)\mathrm{d}r\mathrm{d}z \tag{33}$$

and the last term can be further rewritten as

$$\int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{1}{r} \operatorname{Re}\left(\partial_r\omega\overline{\omega}\right)\mathrm{d}r\mathrm{d}z = \int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{1}{2r}\partial_r\left(|\omega|^2\right)\mathrm{d}r\mathrm{d}z = \int_{\mathbb{R}}\int_{\mathbb{R}_+} \frac{1}{2r^2}|\omega|^2 \mathrm{d}r\mathrm{d}z, \tag{34}$$

where we integrated by parts and used the fact that $\operatorname{supp}\omega$ is contained in the open half-plane $\mathbb{R}^2_+$. Hence, (33) and (34) imply

$$I_\psi = \int_{\mathbb{R}} \int_{\mathbb{R}_+} |\partial_r \omega|^2 \mathrm{d}r\mathrm{d}z - \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{1}{4r^2} |\omega|^2 \mathrm{d}r\mathrm{d}z.$$

Substituting this expression for the first integral in (32) we obtain (28). □

Now we are ready to formulate and prove our main result on the infiniteness of the discrete spectrum of $-\Delta_{\alpha,C_\theta}$ below the bottom of the essential spectrum for all $\alpha > 0$ and $\theta \in (0, \pi/2)$. Recall that $-\Delta_{\alpha,C_\theta}$ is bounded from below, and hence it also follows that the discrete spectrum has a single accumulation point, namely $-\alpha^2/4$. This result illustrates the typical phenomenon that curvature induces bound states. The peculiarity in this three-dimensional system is that the global geometry of the interaction support plays an important role. We point out that in the case $\theta = \pi/2$ the conical surface C_θ coincides with a plane, in which case it follows by separation of variables that the discrete spectrum is empty.

Theorem 3.2. *Let $-\Delta_{\alpha,C_\theta}$ be the self-adjoint operator in $L^2(\mathbb{R}^3)$ associated to the form (1) and let $\alpha > 0$ and $\theta \in (0, \pi/2)$. Then the discrete spectrum of $-\Delta_{\alpha,C_\theta}$ below $-\alpha^2/4$ is infinite, accumulates at $-\alpha^2/4$, and the eigenvalues $\lambda_k < -\alpha^2/4$ (enumerated in non-decreasing order with multiplicities taken into account) satisfy the estimate*

$$\lambda_k \leqslant -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^4}, \quad k \in \mathbb{N}, \tag{35}$$

where $\gamma(\theta) > 0$, $n_{k+1} := n_k^2 + n_k$ for $k \in \mathbb{N}$, and $n_1 = N$ with $N \in \mathbb{N}$ sufficiently large.

Proof. Let us pick a function $\chi_1 \in H_0^1(0, 1)$ with $\| \chi_1 \|_{L^2(0,1)} = 1$ such that

$$\| \chi_1' \|^2_{L^2(0,1)} < \frac{1}{4\sin^2\theta} \int_0^1 \frac{|\chi_1(t)|^2}{t^2} \mathrm{d}t \tag{36}$$

holds; [5, Lemma in §1]. Let us fix $\varepsilon > 0$ and choose $\chi_2 \in C_0^\infty(-\varepsilon, \varepsilon)$ such that $0 \leqslant \chi_2 \leqslant 1$ and $\chi_2(t) = 1$ for $|t| \leqslant \varepsilon/2$. In the coordinate system (s, t) in figure 1 we define the sequence of functions

$$\omega_n(s, t) := \frac{1}{n} \chi_1\left(\frac{s-n}{n^2}\right) \chi_2\left(\frac{t}{\sqrt{n}}\right) \exp\left(-\frac{\alpha}{2}|t|\right) \in H_0^1\left(\mathbb{R}_+^2\right), \tag{37}$$

where the support of ω_n satisfies

$$\operatorname{supp} \omega_n \subset [n, n+n^2] \times [-\varepsilon\sqrt{n}, \varepsilon\sqrt{n}], \quad n \in \mathbb{N}, \tag{38}$$

in the coordinate system (s, t).

For sufficiently large $n \in \mathbb{N}$ the functions ω_n satisfy the conditions of lemma 3.1. The function ω_n can also be viewed as a function in r and z; cf figure 1. Then we define

$$\psi_n(r, \varphi, z) := \frac{\omega_n(r, z)}{\sqrt{2\pi r}}, \quad n \in \mathbb{N}. \tag{39}$$

Using lemma 3.1 we compute the values

$$\begin{aligned} S_n :&= \mathfrak{a}_{\alpha,C_\theta}[\psi_n] + \frac{\alpha^2}{4} \| \psi_n \|^2_{L^2(\mathbb{R}^3)} \\ &= \| \nabla\omega_n \|^2_{L^2(\mathbb{R}_+^2;\, \mathbb{C}^2)} - \int_{\mathbb{R}_+^2} \frac{1}{4r^2} |\omega_n|^2 \mathrm{d}r\mathrm{d}z - \alpha \| \omega_n|_{\Gamma_\theta} \|^2_{L^2(\Gamma_\theta)} + \frac{\alpha^2}{4} \| \omega_n \|^2_{L^2(\mathbb{R}_+^2)}. \end{aligned} \tag{40}$$

The choice of χ_1 in (36) together with a subtle treatment of the second term in (40) will finally lead to $S_n < 0$ for sufficiently large $n \in \mathbb{N}$. First of all it is not difficult to check the asymptotics

$$\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \left| \chi_2\left(\frac{t}{\sqrt{n}}\right)\right|^2 e^{-\alpha|t|}\mathrm{d}t = \frac{2}{\alpha} + O\left(e^{-c\sqrt{n}}\right), \quad n \to \infty, \tag{41}$$

$$\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \left| \chi_2'\left(\frac{t}{\sqrt{n}}\right)\right|^2 e^{-\alpha|t|}\mathrm{d}t = O\left(e^{-c\sqrt{n}}\right), \quad n \to \infty, \tag{42}$$

$$\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \chi_2\left(\frac{t}{\sqrt{n}}\right)\chi_2'\left(\frac{t}{\sqrt{n}}\right) e^{-\alpha|t|}\mathrm{d}t = O\left(e^{-c\sqrt{n}}\right), \quad n \to \infty, \tag{43}$$

with some constant $c > 0$. Using (41) we get

$$\begin{aligned}\frac{\alpha^2}{4} \parallel \omega_n \parallel^2_{L^2\left(\mathbb{R}^2_+\right)} &= \frac{\alpha^2}{4}\left(\frac{1}{n^2}\int_n^{n+n^2}\left|\chi_1\left(\frac{s-n}{n^2}\right)\right|^2 \mathrm{d}s\right)\left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\left|\chi_2\left(\frac{t}{\sqrt{n}}\right)\right|^2 e^{-\alpha|t|}\mathrm{d}t\right)\\ &= \frac{\alpha}{2} + O\left(e^{-c\sqrt{n}}\right), \quad n \to \infty,\end{aligned} \tag{44}$$

and

$$\begin{aligned}\parallel \partial_s\omega_n \parallel^2_{L^2\left(\mathbb{R}^2_+\right)} &= \left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\left|\chi_2\left(\frac{t}{\sqrt{n}}\right)\right|^2 e^{-\alpha|t|}\mathrm{d}t\right)\left(\frac{1}{n^4}\frac{1}{n^2}\int_n^{n+n^2}\left|\chi_1'\left(\frac{s-n}{n^2}\right)\right|^2 \mathrm{d}s\right)\\ &= \frac{2}{\alpha}\frac{1}{n^4}\parallel \chi_1' \parallel^2_{L^2(0,\,1)} + O\left(e^{-c\sqrt{n}}\right), \quad n \to \infty,\end{aligned}$$

and from (42) and (43) we obtain

$$\begin{aligned}\parallel \partial_t\omega_n \parallel^2_{L^2\left(\mathbb{R}^2_+\right)} &= \left(\frac{1}{n^2}\int_n^{n+n^2}\left|\chi_1\left(\frac{s-n}{n^2}\right)\right|^2 \mathrm{d}s\right)\\ &\quad\times\left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}}\left|\frac{1}{\sqrt{n}}\chi_2'\left(\frac{t}{\sqrt{n}}\right) - \frac{\alpha \mathrm{sign}\,(t)}{2}\chi_2\left(\frac{t}{\sqrt{n}}\right)\right|^2 e^{-\alpha|t|}\mathrm{d}t\right)\\ &= \frac{\alpha}{2} + O\left(e^{-c\sqrt{n}}\right), \quad n \to \infty,\end{aligned} \tag{45}$$

that is,

$$\parallel \nabla\omega_n \parallel^2_{L^2\left(\mathbb{R}^2_+;\,\mathbb{C}^2\right)} = \frac{2}{\alpha}\frac{1}{n^4}\parallel \chi_1' \parallel^2_{L^2(0,\,1)} + \frac{\alpha}{2} + O\left(e^{-c\sqrt{n}}\right), \quad n \to \infty. \tag{46}$$

It is straightforward to see that

$$\alpha \parallel \omega_n|_{\Gamma_\theta} \parallel^2_{L^2(\Gamma_\theta)} = \frac{\alpha}{n^2}\int_n^{n+n^2}\left|\chi_1\left(\frac{s-n}{n^2}\right)\right|^2 \mathrm{d}s = \alpha \parallel \chi_1 \parallel^2_{L^2(0,\,1)} = \alpha, \tag{47}$$

and hence it remains to estimate the term $\int_{\mathbb{R}^2_+}\frac{1}{4r^2}|\omega_n|^2$ in (40). For that we make the following splitting

$$\int_{\mathbb{R}_+^2} \frac{1}{4r^2} \left|\omega_n(r, z)\right|^2 \mathrm{d}r\mathrm{d}z = \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \int_n^{n+n^2} \frac{1}{4r(s, t)^2} \left|\omega_n(s, t)\right|^2 \mathrm{d}s\mathrm{d}t = I_n + J_n, \tag{48}$$

where

$$I_n := \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \int_n^{n+n^2} \frac{1}{4r(s, 0)^2} \left|\omega_n(s, t)\right|^2 \mathrm{d}s\mathrm{d}t \tag{49}$$

and

$$J_n := \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \int_n^{n+n^2} \left(\frac{1}{4r(s, t)^2} - \frac{1}{4r(s, 0)^2}\right) \left|\omega_n(s, t)\right|^2 \mathrm{d}s\mathrm{d}t.$$

The term J_n can be further rewritten as

$$J_n = \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \int_n^{n+n^2} \frac{(r(s, 0) - r(s, t))(r(s, 0) + r(s, t))}{4r(s, t)^2 r(s, 0)^2} \left|\omega_n(s, t)\right|^2 \mathrm{d}s\mathrm{d}t. \tag{50}$$

For geometric reasons we have $|r(s, 0) - r(s, t)| \leqslant a\sqrt{n}$ with some $0 < a \leqslant \varepsilon$ and $r(s, t) > bn$ with some $b > 0$ for all $(s, t) \in \operatorname{supp} \omega_n$. We first conclude from (50) that

$$\left|J_n\right| \leqslant a\sqrt{n} \int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \int_n^{n+n^2} \left|\frac{2}{4r(s, t)r(s, 0)^2} + \frac{r(s, 0) - r(s, t)}{4r(s, t)^2 r(s, 0)^2}\right| \left|\omega_n(s, t)\right|^2 \mathrm{d}s\mathrm{d}t$$

and hence

$$\left|J_n\right| \leqslant \left(\frac{2a}{b\sqrt{n}} + \frac{a^2}{b^2 n}\right) I_n \tag{51}$$

follows together with (49). For I_n we have

$$\begin{aligned} I_n &= \left(\frac{1}{n^2} \int_n^{n+n^2} \frac{1}{4\, s^2 \sin^2\theta} \left|\chi_1\left(\frac{s-n}{n^2}\right)\right|^2 \mathrm{d}s\right) \left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \left|\chi_2\left(\frac{t}{\sqrt{n}}\right)\right|^2 e^{-\alpha|t|} \mathrm{d}t\right) \\ &= \left(\frac{1}{n^4} \int_0^1 \frac{\left|\chi_1(u)\right|^2}{4\sin^2(\theta)(u + 1/n)^2} \mathrm{d}u\right) \left(\int_{-\varepsilon\sqrt{n}}^{\varepsilon\sqrt{n}} \left|\chi_2\left(\frac{t}{\sqrt{n}}\right)\right|^2 e^{-\alpha|t|} \mathrm{d}t\right), \end{aligned} \tag{52}$$

and the choice of χ_1 (see (36)) together with monotone convergence yields

$$\int_0^1 \frac{\left|\chi_1(u)\right|^2}{(u + 1/n)^2} \mathrm{d}u = \int_0^1 \frac{\left|\chi_1(u)\right|^2}{u^2} \mathrm{d}u + o(1), \quad n \to \infty.$$

Hence we conclude from (41) and (52) that

$$I_n = \frac{2}{\alpha} \frac{1}{n^4} \frac{1}{4\sin^2(\theta)} \int_0^1 \frac{\left|\chi_1(u)\right|^2}{u^2} \mathrm{d}u + o\left(\frac{1}{n^4}\right), \quad n \to \infty,$$

and from (51) we find

$$J_n = o\left(\frac{1}{n^4}\right), \quad n \to \infty.$$

It follows that (48) becomes

$$\int_{\mathbb{R}_+^2} \frac{1}{4r^2} |\omega_n(r, z)|^2 \mathrm{d}r\mathrm{d}z = \frac{2}{\alpha}\frac{1}{n^4}\frac{1}{4\sin^2(\theta)}\int_0^1 \frac{|\chi_1(u)|^2}{u^2}\mathrm{d}u + o\left(\frac{1}{n^4}\right), \tag{53}$$

as $n \to \infty$. Finally, (44), (46), (47) and (53) yield

$$S_n = \frac{2}{\alpha}\frac{1}{n^4}\left(\| \chi_1' \|^2_{L^2(0,1)} - \int_0^1 \frac{|\chi_1(u)|^2}{4\sin^2(\theta) u^2}\mathrm{d}u \right) + o\left(\frac{1}{n^4}\right), \quad n \to \infty, \tag{54}$$

for S_n in (40). In view of the above asymptotics and according to (36) there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have

$$S_n \leqslant -\frac{2\gamma(\theta)}{\alpha n^4} \tag{55}$$

for some constant $\gamma(\theta) > 0$. Let us consider a sequence $\{n_k\}_k$, where $n_1 := N$ and $n_{k+1} := n_k^2 + n_k$ for $k \in \mathbb{N}$. Then by (38) the measure of $\operatorname{supp}\omega_{n_k} \cap \operatorname{supp}\omega_{n_l}$ is zero for all $k, l \in \mathbb{N}$, $k \neq l$, and hence it follows from the definition (39) that the measure of $\operatorname{supp}\psi_{n_k} \cap \operatorname{supp}\psi_{n_l}$ is zero for all $k, l \in \mathbb{N}$, $k \neq l$, and, in particular, the functions ψ_{n_k} are orthogonal in $L^2(\mathbb{R}^3)$. The space

$$F_k := \operatorname{span}\left\{ \psi_{n_1}, \psi_{n_2}, ..., \psi_{n_k} \right\} \subset H^1\left(\mathbb{R}^3\right),$$

has dimension k and for an arbitrary $\psi = \sum_{l=1}^k a_l \psi_{n_l} \in F_k$, $a_l \in \mathbb{C}$, we get

$$\| \psi \|^2_{L^2(\mathbb{R}^3)} = \sum_{l=1}^k |a_l|^2 \| \psi_{n_l} \|^2_{L^2(\mathbb{R}^3)} = \sum_{l=1}^k |a_l|^2 \| \omega_{n_l} \|^2_{L^2(\mathbb{R}_+^2)} \leqslant \frac{2}{\alpha}\sum_{l=1}^k |a_l|^2, \tag{56}$$

where we have also used the estimate $\| \omega_{n_l} \|^2_{L^2(\mathbb{R}_+^2)} \leqslant \frac{2}{\alpha}$. Employing (55) we obtain

$$\mathfrak{a}_{\alpha,C_\theta}[\psi] + \frac{\alpha^2}{4} \| \psi \|^2_{L^2(\mathbb{R}^3)} = \sum_{l=1}^k |a_l|^2 S_{n_l} \leqslant -\frac{2\gamma(\theta)}{\alpha n_k^4}\sum_{l=1}^k |a_l|^2,$$

where we have again used that the mutual intersections of the supports of $\{\psi_{n_l}\}_{l=1}^k$ are of measure zero.

Combining the above estimate with (56) we get

$$\begin{aligned} \frac{\mathfrak{a}_{\alpha,C_\theta}[\psi]}{\| \psi \|^2_{L^2(\mathbb{R}^3)}} &= -\frac{\alpha^2}{4} + \frac{\mathfrak{a}_{\alpha,C_\theta}[\psi] + \left(\alpha^2/4\right) \| \psi \|^2_{L^2(\mathbb{R}^3)}}{\| \psi \|^2_{L^2(\mathbb{R}^3)}} \\ &\leqslant -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^4} < -\frac{\alpha^2}{4}. \end{aligned} \tag{57}$$

Hence, according to [3, Theorem 10.2.3] the operator $-\Delta_{\alpha,C_\theta}$ has at least k eigenvalues below the bottom of the essential spectrum $-\alpha^2/4$. The above construction works for any $k \in \mathbb{N}$, so that the operator $-\Delta_{\alpha,C_\theta}$ has infinitely many eigenvalues below $-\alpha^2/4$. The eigenvalue estimate (35) follows from [3, Theorem 10.2.3] and (57). □

Let $\theta \in (0, \pi/2)$ and C_θ be the conical surface as above. A hypersurface $\Sigma \subset \mathbb{R}^3$, which for some compact set $K \subset \mathbb{R}^3$ satisfies the condition $\Sigma \backslash K = C_\theta \backslash K$ and which splits the space $\mathbb{R}^3$ into two unbounded Lipschitz domains, is called *a local deformation* of C_θ; cf [1, Section 4.2]. Below we consider the self-adjoint Schrödinger operator $-\Delta_{\alpha,\Sigma}$ with an attractive

δ-interaction of constant strength $\alpha > 0$ supported on the Lipschitz hypersurface Σ. This Schrödinger operator is defined via the closed, densely defined, symmetric and semibounded quadratic form

$$\mathfrak{a}_{\alpha,\Sigma}[\psi] = \| \nabla\psi \|^2_{L^2(\mathbb{R}^3;\, \mathbb{C}^3)} - \alpha \int_\Sigma |\psi|^2 \, \mathrm{d}\sigma, \quad \mathrm{dom}\, \mathfrak{a}_{\alpha,\Sigma} = H^1\left(\mathbb{R}^3\right). \tag{58}$$

The assertion on the essential spectrum in the next theorem is a consequence of [1, Theorem 4.7]; the infiniteness of the discrete spectrum can be shown as in the proof of theorem 3.2 using the same functions ψ_n in (39) and $n \in \mathbb{N}$ sufficiently large.

Theorem 3.3. *Let $\theta \in (0, \pi/2)$ and $\alpha > 0$. Let Σ be a local deformation of the cone C_θ and let $-\Delta_{\alpha,\Sigma}$ be the self-adjoint operator in $L^2(\mathbb{R}^3)$ associated to (58). Then*

$$\sigma_{\mathrm{ess}}\left(-\Delta_{\alpha,\Sigma}\right) = \left[-\alpha^2/4, +\infty\right),$$

the discrete spectrum below $-\alpha^2/4$ is infinite, accumulates at $-\alpha^2/4$, and the eigenvalues $\lambda_k < -\alpha^2/4$ (enumerated in non-decreasing order with multiplicities taken into account) satisfy the estimate

$$\lambda_k \leqslant -\frac{\alpha^2}{4} - \frac{\gamma(\theta)}{n_k^4}, \quad k \in \mathbb{N},$$

where $\gamma(\theta) > 0$, $n_{k+1} := n_k^2 + n_k$ for $k \in \mathbb{N}$, and $n_1 = N$ with $N \in \mathbb{N}$ sufficiently large.

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