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Essential spectrum of Schrödinger operators with δ-interactions on the union of compact Lipschitz hypersurfaces

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In this note we prove that the essential spectrum of a Schrödinger operator with δ-potential supported on a finite number of compact Lipschitz hypersurfaces is given by $[0, +\infty)$. We emphasize that the union of a family of Lipschitz hypersurfaces is in general not Lipschitz.

1 Introduction

Let $\mathcal{S} := \{\Sigma_k\}_{k=1}^n$ be a family of $(d-1)$-dimensional Lipschitz manifolds, each of which separates the Euclidean space $\mathbb{R}^d$, $d \geq 2$, into a bounded part $\Omega_{k,\mathrm{i}}$ and an unbounded exterior part $\Omega_{k,\mathrm{e}}$. Let $\mathcal{A} := \{\alpha_k\}_{k=1}^n$ be a family of L^∞-functions $\alpha_k : \Sigma \to \mathbb{R}$ and define a sesquilinear form $\mathfrak{t}_{\mathcal{A},\mathcal{S}}$ by

$$\mathfrak{t}_{\mathcal{A},\mathcal{S}}[f,g] := (\nabla f, \nabla g)_{L^2(\mathbb{R}^d;\mathbb{C}^d)} - \sum_{k=1}^n (\alpha_k f|_{\Sigma_k}, g|_{\Sigma_k})_{L^2(\Sigma_k)}, \quad \mathrm{dom}\,\mathfrak{t}_{\mathcal{A},\mathcal{S}} = H^1(\mathbb{R}^d). \tag{1}$$

It can be shown that $\mathfrak{t}_{\mathcal{A},\mathcal{S}}$ is a closed, densely defined, symmetric sesquilinear form which is bounded from below and hence induces a self-adjoint operator $-\Delta_{\mathcal{A},\mathcal{S}}$ in $L^2(\mathbb{R}^d)$ via the first representation theorem, see [1, 2, 4, 5]. The main objective of this note is to prove the following result.

Theorem 1.1 $\sigma_{\mathrm{ess}}(-\Delta_{\mathcal{A},\mathcal{S}}) = [0, +\infty)$.

We note that Theorem 1.1 is slightly more general than [4, Theorem 4.2 (i)] since the δ-interaction is supported on the union of hypersurfaces which itself may not be locally the graph of a Lipschitz function. The proof of Theorem 1.1 is based on a compact perturbation argument for one hypersurface, variational principles and singular sequences.

2 Proof of Theorem 1.1 for one hypersurface

Let us introduce the self-adjoint free Laplacian $-\Delta_{\mathrm{free}}$ defined via the sesquilinear form

$$\mathfrak{t}_{\mathrm{free}}[f,g] := (\nabla f, \nabla g)_{L^2(\mathbb{R}^d;\mathbb{C}^d)}, \quad \mathrm{dom}\,\mathfrak{t}_{\mathrm{free}} = H^1(\mathbb{R}^d).$$

It is well-known that $\sigma_{\mathrm{ess}}(-\Delta_{\mathrm{free}}) = \sigma(-\Delta_{\mathrm{free}}) = [0, +\infty)$. Let $\mathcal{S} = \{\Sigma\}$ and $\mathcal{A} = \{\alpha\}$, where $\alpha : \Sigma \to \mathbb{R}$ is an L^∞-function, and denote the self-adjoint operator corresponding to the form (1) in this case by $-\Delta_{\alpha,\Sigma}$.

Theorem 2.1 *The resolvent difference*

$$(-\Delta_{\mathrm{free}} - \lambda)^{-1} - (-\Delta_{\alpha,\Sigma} - \lambda)^{-1} \tag{2}$$

is compact for all $\lambda \in \rho(-\Delta_{\mathrm{free}}) \cap \rho(-\Delta_{\alpha,\Sigma})$. *In particular,* $\sigma_{\mathrm{ess}}(-\Delta_{\alpha,\Sigma}) = [0, +\infty)$.

Proof. According to its definition, the operator $-\Delta_{\alpha,\Sigma}$ is semibounded from below. Hence we can fix a constant $a > 0$ such that $-\Delta_{\alpha,\Sigma} + a > 0$. We denote the resolvent difference in (2) with $\lambda = -a$ by W. Let $f, g \in L^2(\mathbb{R}^d)$ and set

$$u := (-\Delta_{\mathrm{free}} + a)^{-1} f, \qquad v := (-\Delta_{\alpha,\Sigma} + a)^{-1} g. \tag{3}$$

Using (3) and the definition of the operator W we obtain

$$\begin{aligned}(Wf, g)_{L^2(\mathbb{R}^d)} &= \big((-\Delta_{\mathrm{free}} + a)^{-1} f, g\big)_{L^2(\mathbb{R}^d)} - \big((-\Delta_{\alpha,\Sigma} + a)^{-1} f, g\big)_{L^2(\mathbb{R}^d)} \\ &= \big(u, g\big)_{L^2(\mathbb{R}^d)} - \big(f, (-\Delta_{\alpha,\Sigma} + a)^{-1} g\big)_{L^2(\mathbb{R}^d)} \\ &= \big(u, (-\Delta_{\alpha,\Sigma} + a)v\big)_{L^2(\mathbb{R}^d)} - \big((-\Delta_{\mathrm{free}} + a)u, v\big)_{L^2(\mathbb{R}^d)} \\ &= (u, -\Delta_{\alpha,\Sigma} v)_{L^2(\mathbb{R}^d)} - (-\Delta_{\mathrm{free}} u, v)_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{4}$$

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This formula can be rewritten in a more suitable way. Observe that both functions u and v belong to $H^1(\mathbb{R}^d)$, which is the form domain of the operators $-\Delta_{\alpha,\Sigma}$ and $-\Delta_{\rm free}$. Hence, we can use the first representation theorem to rewrite (4) in the following form

$$(Wf,g)_{L^2(\mathbb{R}^d)} = -\big(u|_\Sigma, \alpha v|_\Sigma\big)_{L^2(\Sigma)}, \tag{5}$$

where we made use of the explicit formulae for $\mathfrak{t}_{\rm free}$ and $\mathfrak{t}_{\mathcal{A},\mathcal{S}}$. Introduce the operators $T_1, T_2 \colon L^2(\mathbb{R}^d) \to L^2(\Sigma)$ by

$$T_1 f := \big((-\Delta_{\rm free} + a)^{-1} f\big)\big|_\Sigma, \qquad T_2 g := -\alpha\Big[\big((-\Delta_{\alpha,\Sigma} + a)^{-1} g\big)\big|_\Sigma\Big].$$

It follows from the trace theorem for Sobolev functions [8, Theorem 3.37] that both operators T_1 and T_2 are everywhere defined in $L^2(\mathbb{R}^d)$ and bounded. Moreover, $\operatorname{ran} T_1 \subset H^{1/2}(\Sigma)$ and as Σ is a compact Lipschitz manifold the embedding of $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$ is compact, see, e.g., [7, Section 2] and the references therein. Therefore, we obtain in addition that T_1 is compact. Combining (3) with (5) and with the definition of the operators T_1 and T_2 we find

$$(Wf,g)_{L^2(\mathbb{R}^d)} = \big(T_1 f, T_2 g\big)_{L^2(\Sigma)}.$$

In fact, we have shown that $W = T_2^* T_1$ and the compactness of T_1 and boundedness of T_2 imply compactness of W. Note that by [3, Lemma 2.2] the resolvent difference in (2) is compact for all $\lambda \in \rho(-\Delta_{\rm free}) \cap \rho(-\Delta_{\alpha,\Sigma})$. □

3 Proof of Theorem 1.1 in the general case

We will make use of the following fact: Let A and B be self-adjoint operators which are semibounded from below and have the same form domain. Then the inequality

$$\min \sigma_{\rm ess}(A+B) \geq \min \sigma_{\rm ess}(A) + \min \sigma_{\rm ess}(B) \tag{6}$$

holds, where the sum $A+B$ should be understood in the form sense. In fact, this is a consequence of the min-max theorem [9, Theorem XIII.2] since the corresponding sequences for the operators A, B and $A+B$ satisfy the inequality $\lambda_{m+n-1}(A+B) \geq \lambda_m(A) + \lambda_n(B)$ and it remains to pass to the limit $m, n \to \infty$.

Obviously the equality $\mathfrak{t}_{\mathcal{A},S}[f,f] = \frac{1}{n}\sum_{k=1}^n \mathfrak{t}_{\{n\alpha_k\},\{\Sigma_k\}}[f,f]$ holds for all $f \in H^1(\mathbb{R}^d)$. Employing inequality (6) and Theorem 2.1 we arrive at

$$\min \sigma_{\rm ess}(-\Delta_{\mathcal{A},\mathcal{S}}) \geq \sum_{k=1}^n \min \sigma_{\rm ess}\big(-\tfrac{1}{n}\Delta_{\{n\alpha_k\},\{\Sigma_k\}}\big) = 0.$$

For the opposite inclusion we follow some ideas in the proof of [6, Proposition 5.1]. Pick a function $\varphi \in C_0^\infty([0,2))$ such that $\varphi(r) \geq 0$ and $\int_{\mathbb{R}^d} \varphi(|x|)^2 = 1$. Choose $p \in \mathbb{R}^d$ and $x_n \in \mathbb{R}^d$ such that the balls $B_{2n}(x_n)$ with the centers x_n and the radii $2n$ are mutually disjoint and do not intersect any hypersurface from the family $\mathcal{S}$. Then

$$\psi_{n,p}(x) := \frac{1}{n^{d/2}}\varphi\left(\frac{1}{n}|x - x_n|\right)e^{ipx}$$

with $n \in \mathbb{N}$ is a singular sequence for the operator $-\Delta_{\mathcal{A},S}$ corresponding to $|p|^2$. In fact, the sequence $\{\psi_{n,p}\}_n$ is a singular sequence for $|p|^2$ corresponding to the free Laplacian $-\Delta_{\rm free}$, but, since the supports of the functions $\psi_{n,p}$ do not intersect the support of the δ-interaction, this sequence is also a singular sequence for $-\Delta_{\mathcal{A},S}$ corresponding to the same value $|p|^2$. Hence, we get $\sigma_{\rm ess}(-\Delta_{\mathcal{A},S}) = [0,+\infty)$.

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